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Research Article

On a Family of Parameter-Based Bernstein Type Operators with Shape-Preserving Properties

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This article aims to introduce a new linear positive operator with a parameter. Our focus lies in analyzing the distinct characteristics and inherent properties exhibited by this operator. Additionally, we provide a proof of the convergence rate and present a revised version of the Voronovskaja theorem specifically tailored for this newly defined operator. Furthermore, we provide an upper bound for the error according to the modulus of continuity. Finally, the preservation of monotonicity and convexity by the operator is being investigated.

1. Introduction

The Bernstein polynomials are widely regarded as one of the most well-known algebraic polynomials in approximation theory, and they are defined as

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0,\ldots,n \,\&\, n \in \mathbb{N}. \tag{1}$$

These polynomials were first introduced by Bernstein in 1912 to provide the first constructive proof of Weierstrass' approximation theorem [1]. Bernstein employed the Bernstein operators to approximate a given function on the interval $[0,1]$ by a polynomial of degree n, as it is understood by its formula

$$\mathscr{B}_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1], \tag{2}$$

the structure gets advantage of weighted sum of the function values at equidistant points on the interval.

Many researchers have written books and papers dedicated to studying Bernstein polynomials and operators, with Lorentz's book being one of the most famous [2]. The significance of the Bernstein polynomials lies not only in their own properties but also in the fact that their form has inspired mathematicians to develop a wide range of other approximation operators. Since its introduction, many researchers have proposed modifications and generalizations of Bernstein operators to improve its approximation properties and extend its applications. Some of the most famous cases are the Schurer polynomials, Kantorovich polynomials, Stancu polynomials, q-Bernstein polynomials, Durrmeyer polynomials, Favard–Szász–Mirakyan operators, Baskakov operators, and numerous others [3–6]. Some of the recent advances could be traced in [7–11].

The construction and analysis of Bernstein-type operators aimed to achieve two primary objectives: preserving the form of various functions, such as polynomials and exponentials, and upholding their shape-preserving properties [12, 13].

In 2003, King [14] presented a sequence of Bernstein-type operators which preserve the function x^2. This study was extended by some authors [15–21].

Lately, attempts have been made to develop operators of the Bernstein type. In 2017, Chen et al. [22] proposed a new modification of Bernstein operators based on the so called α-Bernstein polynomials; this paper has garnered significant attention and inspired many researchers to further develop and extend their findings for other families of Bernstein-type operators, see [23–25] and the references therein. There are also other simpler modifications, like the one proposed by Usta in [26].

This paper introduces a fresh set of Bernstein-like operators that are based on a particular shape parameter. It has been verified that the new operators are linear positive operators that preserve linearity, monotonicity, and convexity. By taking advantage of Korovkin's theorem, we provide an alternative proof for Weierstrass' approximation theorem. Furthermore, we provide in-depth proofs regarding the convergence rate and the Voronovskaja-type asymptotic estimation formula for these operators.

Overall, our work contributes to the ongoing research on Bernstein-like operators and shows that further modifications and generalizations can lead to even more powerful approximation tools with wider applications.

The paper is structured as follows: Section 2 introduces the new Bernstein-type operator and examines its fundamental properties. In Section 3, we focus on the shape-preserving aspects. Finally, Section 4 concludes the paper by highlighting key findings of this study.

2. A Revised Version of the Bernstein Operator

In [27], the authors introduced a new set of Bernstein-like basis functions. Building on this set, we propose a unique variation of Bernstein-type operators and investigate their fundamental properties.

Definition 1 (Starting sq-basis) [27]. The starting sq-basis functions are defined on [0, 1] as

$$\begin{aligned} b_{2,0}(x) &= \frac{1}{2} - x + \varphi(x), \\ b_{2,1}(x) &= 1 - 2\varphi(x), \\ b_{2,2}(x) &= t - \frac{1}{2} + \varphi(x), \end{aligned} \tag{3}$$

where $\nu \in (0, 1]$ is a shape parameter and

$$\varphi(x) = \sqrt{(1-\nu)\left(x^2 - x\right) + \frac{1}{4}}. \tag{4}$$

Using the recursive relation of the classical Bernstein basis functions [28], we generate sq-basis functions of order n $(n \geq 3)$ as follows:

$$b_{n,i}(x) = (1-x)b_{n-1,i}(x) + x b_{n-1,i-1}(x), \quad x \in [0,1], \tag{5}$$

where $i = 0, 1, 2, \ldots, n$. For $i < 0$ or $i > n$, we set $b_{n,k}(x) = 0$.

Definition 2 (sq-Bernstein operator). The sq-Bernstein operator for $f(x)$ on $[0,1]$ can be defined for $n \in \mathbb{N}$ and any $\nu \in [0,1]$ as follows:

$$R_{n,\nu}(f;x) = \sum_{i=0}^{n} f_i b_{n,i}(x), \tag{6}$$

where $\{b_{n,i}(x)\}_{i=0}^{n}$ $(n \geq 3)$ are the sq-basis functions defined in (3) and (5) and $f_i = f(i/n)$ for $i = 0, 1, \ldots, n$.

The sq-Bernstein operator is a type of approximation operator that maps the given function $f(x)$ defined on $[0,1]$ to $R_{n,\nu}(f;x)$ defined in (6). The parameter ν controls the level of smoothness of the approximation. In comparison with the classical Bernstein operator, the sq-Bernstein operator provides greater flexibility for controlling the shape of the curve being approximated. By fine-adjusting the shape parameter, one can exert a better influence over the curve's curvature, degree of smoothness, and other geometric attributes. Furthermore, the sq-Bernstein operator surpasses the constraints of polynomial approximation and accommodates more intricate curved representations.

There are several properties and results for these operators that we will discuss.

Lemma 3 (*End point interpolation*). *The sq-Bernstein operator applied to the function $f(x)$ guarantees interpolation of $f(x)$ at the endpoints of [0, 1], i.e.,*

$$R_{n,\nu}(f;0) = f(0), R_{n,\nu}(f;1) = f(1). \tag{7}$$

Lemma 4 (*Linearity*). *The sq-Bernstein operator satisfies linearity, that is,*

$$R_{n,\nu}(\lambda f_1 + \mu f_2; x) = \lambda R_{n,\nu}(f_1;x) + \mu R_{n,\nu}(f_2;x), \tag{8}$$

for any functions $f_1(x)$ and $f_2(x)$ defined on the interval $[0,1]$, as well as any real values λ and μ.

Proof. From the definition of the sq-Bernstein operator, for any functions $f_1(x)$ and $f_2(x)$ and any real values λ and μ, we have

$$\begin{aligned} R_{n,\nu}(\lambda f_1 + \mu f_2; x) &= \sum_{i=0}^{n} (\lambda f_1 + \mu f_2)\left(\frac{i}{n}\right) b_{n,i}(x) \\ &= \lambda \sum_{i=0}^{n} f_1\left(\frac{i}{n}\right) b_{n,i}(x) + \mu \sum_{i=0}^{n} f_2\left(\frac{i}{n}\right) b_{n,i}(x) \\ &= \lambda R_{n,\nu}(f_1;x) + \mu R_{n,\nu}(f_2;x). \end{aligned} \tag{9}$$

□

Lemma 5 (*Positivity*). *The sq-Bernstein operators for $0 < \nu \leq 1$ form a collection of positive operators. This means that for $x \in [0,1]$, we have*

$$\text{if } f(x) \geq 0, \text{then } R_{n,\nu}(f;x) \geq 0, \nu \in (0,1]. \tag{10}$$

Proof. It is an obvious result of the nonnegativity of the sq-basis functions (see [27]).

The above lemma leads to the following direct result: □

Corollary 6 (*Monotonicity*). *For all $x \in [0,1]$ and $\nu \in (0, 1]$, if $f(x) \geq g(x)$, then $R_{n,\nu}(f;x) \geq R_{n,\nu}(g;x)$ for the sq-Bernstein operators.*

From the above result, we have

Corollary 7 (*Boundedness preservation*). *If the function $f(x)$ satisfies the inequality $m \leq f(x) \leq M$ for all x in the interval $[0,1]$, then it can be concluded that $R_{n,\nu}(f;x)$ also satisfies the inequality $m \leq R_{n,\nu}(f;x) \leq M$ for x in the same interval $[0,1]$ and $\nu \in (0, 1]$.*

Theorem 8. *The sq-Bernstein operators fulfill equalities*

$$R_{n,\nu}(1;x) = 1, \quad \forall \nu \in (0,1], \tag{11}$$

$$R_{n,\nu}(x;x) = x, \quad \forall \nu \in (0,1]. \tag{12}$$

Proof. We use induction for verifying this case. Assuming (3) yields $\sum_{i=0}^{2} b_{2,i}(x) = 1$ and $\sum_{i=0}^{2} i/2 b_{2,i}(x) = x$, which verifies the base case. Now, for the induction step, we suppose that the result holds true for $m \in \mathbb{N}^{\geq 2}$. By utilizing the recurrence relation (5), we can derive the following conclusions:

$$\begin{aligned}
&\sum_{i=0}^{m+1} b_{m+1,i}(x) = (1-x)\sum_{i=0}^{m+1} b_{m,i}(x) + x\sum_{i=0}^{m+1} b_{m,i-1}(x) = 1 - x + x = 1,\\
&\sum_{i=0}^{m+1} \frac{i}{m+1} b_{m+1,i}(x) = \frac{m}{m+1}\sum_{i=0}^{m+1}\frac{i}{m}\left[(1-x)b_{m,i}(x) + x b_{m,i-1}(x)\right]\\
&\quad = \frac{m}{m+1}(1-x)\sum_{i=0}^{m}\frac{i}{m} b_{m,i}(x) + \frac{m}{m+1}x\sum_{i=0}^{m}\frac{i+1}{m} b_{m,i}(x)\Bigg]\\
&\quad = \frac{m}{m+1}(1-x)x + \frac{m}{m+1}x\left(x + \frac{1}{m}\right) = x.
\end{aligned} \tag{13}$$

□

Lemma 9 (*Preserving linearity*). *The sq-Bernstein operators for $0 < \nu \leq 1$ accurately approximate linear functions, i.e.,*

$$R_{n,\nu}(\lambda x + \mu; x) = \lambda x + \mu, \lambda, \quad \mu \in \mathbb{R}. \tag{14}$$

Proof. By combining equations (11) and (12) with the linearity property (8), it is an evident fact.

Our next step involves analyzing the sq-Bernstein operators applied to the functions $f(x) = x^2, x^3, x^4$. □

Lemma 10. *The sq-Bernstein operators satisfy the following equalities:*

$$R_{n,\nu}\left(x^2;x\right) = \frac{1}{n^2}\left[(n+1)(n-2)x^2 + (n+2)x - 1 + 2\varphi(x)\right], \tag{15}$$

$$\begin{aligned}
R_{n,\nu}\left(x^3;x\right) = \frac{1}{n^3}&\left[(n+2)(n-2)(n-3)x^3 + 3(n+3)(n-2)x^2\right.\\
&\left.+ 2(6-n)x + 6(n-2)x\varphi(x) - 3 + 6\varphi(x)\right],
\end{aligned} \tag{16}$$

$$\begin{aligned}
R_{n,\nu}\left(x^4;x\right) = \frac{1}{n^4}&\left[(n+3)(n-2)(n-3)(n-4)x^4 + 6(n+4)(n-2)(n-3)x^3\right.\\
&+ [(n+61)(n-2) + 12(n-2)(n-3)\varphi]x^2\\
&\left.+ [-17(n-3) - 1 + 36(n-2)\varphi]x - 7 + 14\varphi\right].
\end{aligned} \tag{17}$$

Proof. To prove these relations, we will use mathematical induction. The case $n = 2$ is simple, and the induction steps are straightforward.

The most effective criteria for determining the convergence of a positive linear operator to the identity operator is provided by the Bohman–Korovkin theorem, as follows: □

Lemma 11 (see [29, 30]). *Consider a sequence of linear positive operators, denoted as $\{L_n f; n = 0, 1, \ldots\}$, operating on the interval $[a,b]$ from the space $C[a,b]$ to itself. If this sequence of operators converges uniformly to $f(x)$ for the functions $f(x) = 1, x, x^2$ on $[a,b]$, then for any function $f(x)$ belonging to $C[a,b]$ and any x within the interval $[a,b]$,* *the corresponding sequence of functions* $\{L_n(f;x)\}$ *will also converge uniformly to* $f(x)$*; i.e.,*

$$\lim_{n\longrightarrow\infty} L_n(f;x) = f(x), \text{uniformly}. \tag{18}$$

Based on Lemma 11, we can now present the key finding of this paper, showcasing the convergence of the sequence comprised of sq-Bernstein operators.

Theorem 12. *For any function* $f(x)$ *that is continuous on the interval [0, 1], the sq-Bernstein operators* $\{R_{n,\nu}(f(x);x)\}$ *will uniformly converge to* $f(x)$ *for any* $\nu \in (0,1)$.

Proof. From equation (15), we can deduce that the sq-Bernstein operators converge uniformly to x^2 for any $\nu \in (0,1)$. Considering this fact along with equations (11) and (12), the convergence is a straightforward result of the Bohman–Korovkin theorem.

The following lemma is a prerequisite for the Voronovskaja theorem. □

Lemma 13. *Suppose*

$$M_k(x) = \sum_{i=0}^{n} (i - nx)^k b_{n,i}(x). \tag{19}$$

We have the following equalities:

$$(i)\ M_1(x) = 0, \tag{20}$$

$$(ii)\ M_2(x) = (n+2)x(1-x) - 1 + 2\varphi(x), \tag{21}$$

$$\begin{aligned}(iii)\ M_3(x) &= 2(n+6)x^3 - 6(n+3)x^2 + (n+12)x - 12x\varphi(x) + 6\varphi(x) - 3,\\ (iv)\ M_4(x) &= 3(n+6)(n-4)x^4 - 6(n+6)(n-4)x^3 + \left(3n^2 + 11n - 122\right)x^2\\ &\quad + 60(2-n)\varphi(x)x^2 + 5(10-n)x + 12(n-6)\varphi(x)x + 14\varphi(x) - 7.\end{aligned} \tag{22}$$

Proof. The necessary outcomes can be obtained by utilizing the binomial expansion of $(i-nx)^k$, $k = 1,2,3$, along with Theorem 8 and Lemma 10.

Once we have established the convergence of the newly introduced Bernstein-type operators, the next crucial consideration is how quickly these operators approximate the function $f(x)$. Voronovskaya (1932) answered this question for the Bernstein operator. In the next theorem, we present a new variant of Voronovskaja's result [31] for our newly defined operator in (6) and we introduce the asymptotic error for the sq-Bernstein operators. □

Theorem 14. *Let* $f(x)$ *be a bounded function on* $[0,1]$, *then for any* $x \in [0,1]$, *at which* $f''(x)$ *exists, we have*

$$\lim_{t\longrightarrow x} n\left[R_{n,\nu}(f;x) - f(x)\right] = \frac{1}{2}x(1-x)f''(x), \tag{23}$$

where $0 < \nu \leq 1$.

Proof. Employing Taylor's expansion for $i \leq n$

$$\begin{aligned}f(t) &= f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x)\\ &\quad + r(t,x)(t-x)^2,\end{aligned} \tag{24}$$

where $\lim_{t\longrightarrow x} r(t,x) = 0$, at $t = i/n$ results in

$$\begin{aligned}f\left(\frac{i}{n}\right) &= f(x) + \left(\frac{i}{n} - x\right)f'(x)\\ &\quad + \frac{1}{2}\left(\frac{i}{n} - x\right)^2 f''(x) + r\left(\frac{i}{n}, x\right)\left(\frac{i}{n} - x\right)^2.\end{aligned} \tag{25}$$

Consequently,

$$\begin{aligned}n\left[R_{n,\nu}(f;x) - f(x)\right] &= n\sum_{i=0}^{n} b_{n,i}(x)\left[f\left(\frac{i}{n}\right) - f(x)\right]\\ &= n\sum_{i=0}^{n} b_{n,i}(x)\left[\left(\frac{i}{n} - x\right)f'(x) + \frac{1}{2}\left(\frac{i}{n} - x\right)^2 f''(x) + r\left(\frac{i}{n}, x\right)\left(\frac{i}{n} - x\right)^2\right]\\ &= M_1(x)f'(x) + \frac{1}{2n}M_2(x)f''(x) + n\sum_{i=0}^{n} r\left(\frac{i}{n}, x\right)\left(\frac{i}{n} - x\right)^2 b_{n,i}(x).\end{aligned} \tag{26}$$

Then, according to (20) and (21), one can write

$$n\left[R_{n,\nu}(f;x)-f(x)\right]=\left(\frac{1}{2}+\frac{1}{n}\right)x(1-x)f''(x)+\left(\frac{2\varphi(x)-1}{2n}\right)f''(x)+K_n(x), \quad (27)$$

where

$$\begin{aligned} K_n(x) &= n\sum_{i=0}^{n} r\left(\frac{i}{n},x\right)\left(\frac{i}{n}-x\right)^2 b_{n,i}(x) \\ &= nR_{n,\nu}\left(r(t,x)(t-x)^2;x\right). \end{aligned} \quad (28)$$

To complete the proof, it is necessary to demonstrate that

$$\lim_{n\longrightarrow\infty} nR_{n,\nu}\left(r(t,x)(t-x)^2;x\right)=0. \quad (29)$$

By utilizing the Cauchy–Schwarz inequality, it is straightforward to infer that

$$nR_{n,\nu}\left(r(t,x)(t-x)^2;x\right)\le\sqrt{R_{n,\nu}\left(r^2(t,x);x\right)}\sqrt{n^2R_{n,\nu}\left((t-x)^4;x\right)}. \quad (30)$$

Using Korovkin's theorem, we get

$$\lim_{n\longrightarrow\infty} R_{n,\nu}\left(r^2(t,x);x\right)=r^2(x,x)=0. \quad (31)$$

Since $r^2(x,x)=0$ and continuity of the function $r^2(.,x)$ in $(0,1)$, along with the fact that $R_{n,\nu}((t-x)^4;x)$ does not increase faster than $O(n^{-2})$, we can conclude that equation (29) holds, which completes the proof.

Now, we can analyze the rate of convergence of sq-Bernstein operators with respect to the modulus of continuity (ω), and by getting advantage of the characteristics of the modulus of continuity stated in [32], we can demonstrate the principal outcome regarding the upper bound of the approximation error. The error is measured by the uniform norm, which is defined on the interval [0, 1] as follows:

$$\left\|R_{n,\nu}(f;x)-f(x)\right\|=\max_{0\le x\le 1}\left|R_{n,\nu}(f;x)-f(x)\right|. \quad (32)$$

□

Theorem 15. *For any value $0<\nu\le 1$, if the function $f(x)$ is bounded on the interval $[0,1]$, then*

$$\left\|R_{n,\nu}(f;x)-f(x)\right\|\le\frac{3}{2}\omega\left(\frac{\sqrt{n+6}}{n}\right). \quad (33)$$

Proof. According to (8) and (11), for $0<\nu\le 1$, one has

$$\begin{aligned} \left\|f(x)-R_{n,\nu}(f;x)\right\| &= \left|\sum_{i=0}^{n} b_{n,i}(x)\left[f(x)-f\left(\frac{i}{n}\right)\right]\right| \\ &\le \sum_{i=0}^{n}\left|f(x)-f\left(\frac{i}{n}\right)\right| b_{n,i}(x) \\ &\le \sum_{i=0}^{n}\omega\left(\left|x-\frac{i}{n}\right|\right) b_{n,i}(x). \end{aligned} \quad (34)$$

Now, considering properties of modulus of continuity, one can deduce

$$\begin{aligned} \omega\left(\left|x-\frac{i}{n}\right|\right) &= \omega\left(\frac{n}{\sqrt{n+6}}\left|x-\frac{i}{n}\right|\frac{\sqrt{n+6}}{n}\right) \\ &\le \left(1+\frac{n}{\sqrt{n+6}}\left|x-\frac{i}{n}\right|\right)\omega\left(\frac{\sqrt{n+6}}{n}\right), \end{aligned} \quad (35)$$

which results in

$$\begin{aligned}\left\|f(x)-R_{n,\nu}(f;x)\right\| &= \sum_{i=0}^{n}\left(1+\frac{n}{\sqrt{n+6}}\left|x-\frac{i}{n}\right|\right)\omega\left(\frac{\sqrt{n+6}}{n}\right)b_{n,i}(x)\\ &\leq \omega\left(\frac{\sqrt{n+6}}{n}\right)\left(1+\frac{n}{\sqrt{n+6}}\sum_{i=0}^{n}\left|x-\frac{i}{n}\right|b_{n,i}(x)\right).\end{aligned} \tag{36}$$

Next, we get advantage of the Cauchy–Schwarz's inequality to have

$$\begin{aligned}\sum_{i=0}^{n}\left|x-\frac{i}{n}\right|b_{n,i}(x) &= \sum_{i=0}^{n}\left|x-\frac{i}{n}\right|\sqrt{b_{n,i}(x)}\sqrt{b_{n,i}(x)}\\ &\leq \left[\sum_{i=0}^{n}\left(x-\frac{i}{n}\right)^{2}b_{n,i}(x)\right]^{1/2}\left[\sum_{i=0}^{n}b_{n,i}(x)\right]^{1/2}\\ &= \left[\sum_{i=0}^{n}\left(x-\frac{i}{n}\right)^{2}b_{n,i}(x)\right]^{1/2}.\end{aligned} \tag{37}$$

For the last term, we can have an upper bound according to (21) as follows:

$$\sum_{i=0}^{n}\left(x-\frac{i}{n}\right)^{2}b_{n,i}(x)=\frac{1}{n^{2}}M_{2}(x)=\frac{(n+2)x(1-x)-1+2\varphi(x)}{n^{2}}\leq\frac{n+6}{4n^{2}},\quad x\in[0,1], \tag{38}$$

which leads us to the final result

$$\begin{aligned}\left\|f(x)-R_{n,\nu}(f;x)\right\| &\leq \omega\left(\frac{\sqrt{n+6}}{n}\right)\left(1+\frac{n}{\sqrt{n+6}}.\frac{\sqrt{n+6}}{2n}\right)\\ &= \frac{3}{2}\omega\left(\frac{\sqrt{n+6}}{n}\right),\end{aligned} \tag{39}$$

and this completes the proof.

According to Theorem 15, we provided an upper bound for the error $f(x)-R_{n,\nu}(f;x)$ in terms of the modulus of continuity. In addition, according to properties of the modulus of continuity and continuity of the function $f(x)$ on [0, 1], we have

$$\lim_{n\longrightarrow\infty}\omega\left(\frac{\sqrt{n+6}}{n}\right)=0. \tag{40}$$

In another manner, Theorem 12 can be proven. ☐

Remark 16. Similar to the above proof process, another upper bound for the error can be presented in terms of the parameter ν as follows:

$$\left\|f(x)-R_{n,\nu}(f;x)\right\|\leq\frac{3}{2}\omega\left(\frac{\sqrt{n-2+4\sqrt{2-\nu}}}{n}\right). \tag{41}$$

3. Shape-Preserving Properties

In this section, we will demonstrate that the sq-Bernstein operators preserve certain geometric properties such as monotonicity and convexity.

3.1. Monotonicity Preservation. For verifying the monotony-preserving property of the sq-Bernstein operators, we first state the following lemma:

Lemma 17. *If $A=\{\alpha_i\}_{i=0}^{n}$ is a monotone set of real values, i.e., $\alpha_0\leq\alpha_1\leq\cdots\leq\alpha_n$, then the function $\sum_{i=0}^{n}\alpha_i b_{n,i}(x)$, as a linear combination of the elements of A, is monotonically increasing.*

Proof. We use induction to verify the result. The base of induction is $n=2$, so for any set of real values $\alpha_0\leq\alpha_1\leq\alpha_2$, we show that the function $G(x,\nu)=\sum_{i=0}^{2}\alpha_i b_{2,i}(x)$ is monotonically increasing on [0, 1], and by rewriting $G(x,\nu)$ in a suitable way, one has

$$G(x,\nu)=\sum_{i=0}^{2}\alpha_i b_{2,i}(x)=(\alpha_2-\alpha_1)b_{2,2}(x)+(\alpha_1-\alpha_0)\left(\sum_{i=1}^{2}b_{2,i}(x)\right)+\alpha_0\left(\sum_{i=0}^{2}b_{2,i}(x)\right). \tag{42}$$

We need to show that $G(x,\nu)$ has a nonnegative derivative with respect to x:

$$\frac{d}{dx}G(x,\nu)=(\alpha_2-\alpha_1)\frac{d}{dx}b_{2,2}(x)+(\alpha_1-\alpha_0)\frac{d}{dx}\left(\sum_{i=1}^{2}b_{2,i}(x)\right)+\alpha_0\frac{d}{dx}\left(\sum_{i=0}^{2}b_{2,i}(x)\right). \tag{43}$$

As the coefficients are nonnegative in (43), it is sufficient to verify the nonnegativity of the corresponding functions which we verify term by term.

(i) From the partition of unity of the sq-basis functions, one has

$$\sum_{j=0}^{2}b_{2,j}(x)=1\Longrightarrow\frac{d}{dx}\left(\sum_{j=0}^{2}b_{2,j}(x)\right)=0. \tag{44}$$

(ii) The second term could be treated as follows:

$$\begin{aligned}\sum_{j=1}^{2}b_{2,j}(x)&=b_{2,1}(x)+b_{2,2}(x)\\&=\frac{1}{2}+x-\varphi(x)\Rightarrow\frac{d}{dx}\left(\sum_{j=1}^{2}b_{2,j}(x)\right)\\&=1-\frac{d}{dx}\varphi(x).\end{aligned} \tag{45}$$

From the convexity of the function $\varphi(x)$ in [0, 1], we conclude that the function $-d/dx\varphi(x)$ takes its minimum at $x=1$, so one has $\sum_{j=1}^{2}d/dx\, b_{2,j}(x)|_{x=1}=\nu$ which has a nonnegative value.

(iii) For the first term, we have

$$\begin{aligned}b_{2,2}(x)&=x-\frac{1}{2}+\varphi(x)\Longrightarrow\frac{d}{dx}\left(b_{2,2}(x)\right)\\&=1+\frac{d}{dx}\varphi(x),\end{aligned} \tag{46}$$

which by taking advantage of the convexity of $\varphi(x)$, one observes that $d/dx\varphi(x)$ has its minimum at $x=0$. Since $d/dx b_{2,2}(x)|_{x=0}=\nu$, the nonnegativity of the first term is verified.

As the induction hypothesis, we assume that for any set of monotone real values $\alpha_0\le\alpha_1\le\cdots\le\alpha_n$, the function $\sum_{i=0}^{n}\alpha_i b_{n,i}(x)$ is monotonically increasing.

Now, by considering the set of increasing values $\beta_0\le\beta_1\le\cdots\le\beta_{n+1}$, we need to show that $\sum_{i=0}^{n+1}\beta_i b_{n+1,i}(x)$ is monotonically increasing.

$$\begin{aligned}G(x,\nu)&=\sum_{i=0}^{n+1}\beta_i b_{n+1,i}(x)=\sum_{i=0}^{n+1}\beta_i\left[(1-x)b_{n,i}(x)+xb_{n,i-1}(x)\right],\\\Longrightarrow\frac{d}{dx}G(x,\nu)&=\sum_{i=0}^{n+1}\beta_i\frac{d}{dx}\left[(1-x)b_{n,i}(x)+xb_{n,i-1}(x)\right]\\&=\sum_{i=0}^{n+1}\beta_i\left[-b_{n,i}(x)+(1-x)\frac{d}{dx}b_{n,i}(x)+b_{n,i-1}(x)+x\frac{d}{dx}b_{n,i-1}(x)\right]\\&=\sum_{i=0}^{n}(\beta_{i+1}-\beta_i)b_{n,i}(x)+(1-x)\sum_{i=0}^{n}\beta_i\frac{d}{dx}b_{n,i}(x)+x\sum_{i=0}^{n}\beta_{i+1}\frac{d}{dx}b_{n,i}(x).\end{aligned} \tag{47}$$

According to the induction hypothesis, we have $\sum_{i=0}^{n}\beta_i d/dx b_{n,i}(x),\sum_{i=0}^{n}\beta_{i+1}d/dx b_{n,i}(x)\ge0$ and $\beta_{i+1}-\beta_i$, x, $1-x\ge0$, so $d/dx G(x,\nu)\ge0$. This completes the proof.

The preceding lemma is more than needed and illustrates a more general result, and we employ a special case to prove the monotonicity preservation of the sq-Bernstein bases. □

Theorem 18. *3.2. If $f(x)$ is a continuous and monotonically increasing (resp. decreasing) function on [0, 1], then its sq-Bernstein operators are also increasing (resp. decreasing).*

Proof. Let $f(x)$ be a monotonically increasing function, so one has $f(0/n)\le f(1/n)\le\cdots\le f(n/n)$. Now, Lemma 17 could be employed to verify that the function $R_{n,\nu}(f;x)=\sum_{i=0}^{n}f(i/n)b_{n,i}(x)$ is monotonically increasing.

The decreasing case could be verified in a similar manner. □

3.2. Convexity Preservation. The sq-Bernstein operators preserve convexity, and this is verified in the following theorem.

Theorem 19. *If $f(x)$ is a convex function in $C[0,1]$, then all its sq-Bernstein operators are convex.*

Proof. We can employ the method of induction to demonstrate the validity of the result. Our base case for induction will be when $n = 2$. Consider a set of real values $\alpha_0, \alpha_1, \alpha_2$ and assume that they form a convex data set, satisfying the condition $\alpha_2 - 2\alpha_1 + \alpha_0 \geq 0$. We aim to prove that the function $S(x, \nu) = \sum_{i=0}^{2} \alpha_i b_{2,i}(x)$ is convex on the interval [0, 1]. We show that the function's second derivative is nonnegative over the interval [0, 1]:

$$\frac{d^2}{dx^2} S(x, \nu) = \sum_{i=0}^{2} \alpha_i \frac{d^2}{dx^2} b_{2,i}(x) = (\alpha_2 - 2\alpha_1 + \alpha_0) \frac{d^2}{dx^2} \varphi(x). \tag{48}$$

Given that $\varphi(x)$ is a convex function, we can establish that $d^2/\mathrm{dx}^2 \varphi(x) \geq 0$. Consequently, based on this fact and $\alpha_2 - 2\alpha_1 + \alpha_0 \geq 0$, we can conclude that $d^2/\mathrm{dx}^2 S(x, \nu) \geq 0$.

Assuming that a set of real values $\{\alpha_i\}_{i=0}^{n}$ is convex, we can consider the expression $\sum_{i=0}^{n} \alpha_i b_{n,i}(x)$. As the induction hypothesis, we assume this expression to be convex.

Now, let us consider a new set of convex values $\{\beta_i\}_{i=0}^{n+1}$. Our goal is to demonstrate that the function $\sum_{i=0}^{n+1} \beta_i b_{n+1,i}(x)$ is also convex. To do so, we consider

$$S(x, \nu) = \sum_{i=0}^{n+1} \beta_i b_{n+1,i}(x) = \sum_{i=0}^{n+1} \beta_i \left[(1 - x) b_{n,i}(x) + x b_{n,i-1}(x) \right], \tag{49}$$

and show that the function $d/\mathrm{dx} S(x, \nu)$ is increasing.

$$\begin{aligned} \frac{d}{\mathrm{dx}} S(x, \nu) &= \sum_{i=0}^{n+1} \beta_i \frac{d}{\mathrm{dx}} \left[(1 - x) b_{n,i}(x) + x b_{n,i-1}(x) \right] \\ &= \sum_{i=0}^{n+1} \beta_i \left[-b_{n,i}(x) + (1 - x) \frac{d}{\mathrm{dx}} b_{n,i}(x) + b_{n,i-1}(x) + x \frac{d}{\mathrm{dx}} b_{n,i-1}(x) \right] \\ &= \sum_{i=0}^{n} (\beta_{i+1} - \beta_i) b_{n,i}(x) + (1 - x) \sum_{i=0}^{n} \beta_i \frac{d}{\mathrm{dx}} b_{n,i}(x) + x \sum_{i=0}^{n} \beta_{i+1} \frac{d}{\mathrm{dx}} b_{n,i}(x). \end{aligned} \tag{50}$$

By assuming the induction hypothesis, since $\{\beta_i\}_{i=0}^{n+1}$ represents convex data, it follows that both $\sum_{i=0}^{n} \beta_i b_{n,i}(x)$ and $\sum_{i=0}^{n} \beta_{i+1} b_{n,i}(x)$ are convex functions. This implies that their respective derivatives, $\sum_{i=0}^{n} \beta_i d/\mathrm{dx} b_{n,i}(x)$ and $\sum_{i=0}^{n} \beta_{i+1} d/\mathrm{dx} b_{n,i}(x)$, are increasing functions.

Since $\{\beta_i\}_{i=0}^{n+1}$ represents convex data, we can observe that $\beta_{i+1} - 2\beta_i + \beta_{i-1} \geq 0, i = 1, \ldots, n - 1$. This inequality implies that $\beta_{i+1} - \beta_i \geq \beta_i - \beta_{i-1}, i = 1, \ldots, n - 1$. In other words, the differences $\beta_i - \beta_{i-1}, i = 1, \ldots, n - 1$ form an increasing sequence. Utilizing Lemma 17, we can ascertain that the function $\sum_{i=0}^{n} (\beta_{i+1} - \beta_i) b_{n,i}(x)$ is an increasing function.

By utilizing the derived findings and considering $x \in [0, 1]$, it can be deduced that $d/\mathrm{dx} S(x, \nu)$ is an increasing function. Consequently, this implies that $d^2/\mathrm{dx}^2 S(x, \nu) \geq 0$, indicating that $S(x, \nu)$ is a convex function.

We observe that if f is a convex function, then $f(0/n), f(1/n), \ldots, f(n/n)$ forms a set of convex data. Based on the obtained results, we can conclude that $R_{n,\nu}(f; x) = \sum_{i=0}^{n} f(i/n) b_{n,i}(x)$ is also a convex function. □

4. Conclusion

We have introduced a novel class of linear positive operators characterized by shape parameters. These operators not only share several properties with the Bernstein operators but also possess the ability to preserve important shape characteristics such as monotonicity and convexity of the underlying data. The operators converge uniformly for any value of the parameter, and an upper bound for the approximation error has been provided based on the modulus of continuity. By altering the parameter value, it becomes possible to adjust the shape of the resulting approximate curve produced by the operator.

The sq-Bernstein operators may provide an appropriate basis for solving functional equations from different disciplines. This certainly could be a proposition for future studies. Moreover, the structure used in equation (3) may be generalized by using some novel auxiliary parameter-based functions, $\varphi(t)$, from different families of functions such as trigonometrics, exponentials, etc. This will result in an operator that is not only linear and preserves the properties of the Bernstein operator but also preserves the monotonicity and convexity of the data. The operator defined by these bases surpasses the constraints of polynomial approximation and accommodates more intricate curved representations.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The two authors contributed equally to this work.

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