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Research Article

Partial Differential Equations in Zero-Sum Differential Game and Applications on Coronavirus

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In this paper, we studied a zero-sum game described by the partial differential equations as an application on Coronavirus. The game contains two players, player 1 is Coronavirus and player 2 is the population. We used ∞-Laplacian which is denoted by Δ_∞. We added the time variable to the partial differential equation to see the behaviour of the spreading of Coronavirus. We used analytical methods, the Homotopy Perturbation Method and New Iterative Method, for solving the partial differential equation. A comparison between the two methods to the residual error is made. We showed in the graph the decreasing of spreading for Coronavirus with increasing the area with the time.

1. Introduction

The partial differential equations are one of the important branches of mathematics as it serves many fields and it contains applications such as physical problems, fluid flow, elasticity, electrodynamics, and game theory [1, 2]. Blanc and Rossi in [3] illustrate the strong and important relationship between nonlinear second-order partial differential equations and the game theory; there are many applications in the game theory via nonlinear second-order partial differential equations.

A game is called zero-sum game if and only if the sum of all payoffs is zero when the game contains more than two players. A zero-sum game is one type of the game theory in which one player's gain is equivalent to another's loss in case of two players, i.e., the first player has a payoff and the second player has necessarily the negative payoff of the first player. It is easy to see that the second player gives this amount to the first player; therefore, the net change in benefit equals zero [4, 5]. There are many examples of zero-sum games, such as poker, gambling, chess, tennis, tug of wars, and financial markets. Megahed et al. used a min-max zero-sum differential game approach as an optimization method in counter terrorism [6].

Just as game theory is important in the economic aspect, it has an equally important role in the medical aspect, such as cancer and Covid-19. In [7], Kareva and Karev applied the study of single games to cancer. They demonstrated games between tumor and treatment depending on metabolism and development of resistance. Kabir and Tanimoto performed the behavioural dynamics of the economic shutdowns and immunization in COVID-19 pandemic [8].

Also, the COVID-19 pandemic is known as the Coronavirus pandemic. Coronavirus is a dangerous disease, and the entire world suffers from this virus till now. The novel virus was first identified from an outbreak in Wuhan, China, in December 2019. They failed to contain it, allowing the virus to spread to other areas of China and later worldwide. The World Health Organization (WHO) declared the outbreak a public health emergency of international concern on 30 January 2020 and a pandemic on 11 March 2020. As of 3 October 2022, the pandemic had caused more than 618 million cases and 6.54 million confirmed deaths, making it one of the deadliest in history. Watson et al. demonstrated a mathematical modelling for global impact of the first year of COVID-19 vaccination [9].

In this work, we are studying a differential game related to Coronavirus, and we consider a min-max differential game between Coronavirus and population. In our problem, Coronavirus is player 1; this virus tries adapting to the existing environment, and it wants to spread rapidly and resists vaccines and mutates. Population is player 2; they take the necessary precautionary measures and have to eat healthy and varied meals and not go to the crowded places to avoid the infection. Player 1 wants to maximize its outcome by rapid spreading and player 2 wants to minimize player 1's outcome by avoiding the infection.

We used ∞-Laplacian, which is denoted by Δ_∞. It is a nonlinear elliptic operator defined by $\Delta_\infty u = \sum_{i,j} u_{x_i} u_{x_j} u_{x_i x_j} = 0$. Blanc and Rossi in [3] apply the infinity Laplacian in one dimension; in this paper, we added the time variable to the equation for showing the behaviour of the spreading of the disease with the time.

In this paper, the rest of it is organized as follows: the description of the problem is explained in Section 2. Section 3 contains the analytical methods we used in solving the partial differential equation (the Homotopy Perturbation Method and the New Iterative Method). The purpose of Section 4 is to make sure that the approximate solutions of the two methods are accurate by finding the residual error to the two methods. Finally, the conclusion is given in Section 5.

2. Description of the Problem

A min-max game is a two-person zero-sum game where every player plays against the other and the total earnings of one of the both players are the losses of the other. Player 1 wants to maximize his objective, while the other player wants to minimize player 1's objective. In [10], Youness et al. determined the analytical and approximate solution to a min-max differential game with Cauchy initial value problem by using the Picard method and a suggested method. In the following subsection, we describe the problem.

Consider the domain $\Omega \subseteq R$, with the initial time t_0, and the game starts at x_0 (initial place), if player 1 succeeded in spreading, then player 1 is the winner, so the game will continue from another place and so on. The winner will obtain $u(x_i, t)$ and player 2 will obtain $-u(x_i, t)$, u: $R \times [0, T] \longrightarrow R$.

Now, consider the nonlinear elliptic partial differential equation as follows:

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

with the boundary conditions

$$u(0,t) = 0,\ u(100,t) = 3.7 \times 10^{-44}, \quad t \in [0, 0.1], \tag{2}$$

and the initial condition

$$u(x,0) = e^{-x}, \quad 0 \le x \le 100, \tag{3}$$

where $u(x,t)$ refers to the disease spreading (the number of people infected with the disease). Here, x is the area where the disease wants to spread and also where people go to for keeping themselves safe from infection. The strategy in our problem is choosing the right place for both the players. The units of the area and the time of spreading, respectively, are Km^2 and a decade for $t = 1$.

3. Analytical Methods for Solving the Problem

In this section, we discuss the homotopy pertubation method and new iterative method for solving problems (1)–(3).

3.1. Homotopy Perturbation Method (HPM). To explain the basic ideas of the homotopy perturbation method [11, 12], we consider the following equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{4}$$

with the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right), \quad r \in \Gamma, \tag{5}$$

where A is the general differential operator, B is the boundary operator, $f(r)$ is the analytical function, and Γ is the boundary of the domain Ω. The operator A can be divided into two parts, L and N, where L is the linear part and N is the nonlinear part. Hence, (4) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{6}$$

The homotopy perturbation structure is shown as follows:

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0. \tag{7}$$

In (7), $p \in [0,1]$ is an embedding parameter and is the first approximation that satisfies the boundary conditions. We assume that the solution of (7) can be written as a power series of p:

$$v = v_0 + p v_1 + p^2 v_2 + \ldots, \tag{8}$$

and the best approximation will be

$$u = \lim_{p \longrightarrow 1} v = v_0 + v_1 + v_2 + \ldots. \tag{9}$$

In [13], Hemeda found the exact solutions for linear (nonlinear) ordinary (partial) differential equations of fractional order applied in fluid mechanics by using the modified homotopy perturbation method.

Consider the nonlinear homogeneous partial differential equation as follows:

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2}, \tag{10}$$

with the boundary conditions

$$u(0,t) = 0,\ u(100,t) = 3.7 \times 10^{-44}, \quad t \in [0, 0.1], \tag{11}$$

and the initial condition

$$u(x,0) = e^{-x}, \quad 0 \le x \le 100. \tag{12}$$

Suppose that $u_0^* = u(x,0) = e^{-x}$, by substituting (10) into (7), we have

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u_0^*}{\partial t} - p\frac{\partial u_0^*}{\partial t} + p\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial}{\partial t}\left(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots\right) &= \frac{\partial u_0^*}{\partial t} - p\frac{\partial u_0^*}{\partial t} + p\left[\frac{\partial}{\partial x}\left(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots\right)\right]^2 \\ &\quad \cdot \frac{\partial^2}{\partial x^2}\left(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots\right). \end{aligned} \tag{13}$$

We have a system of $(n+1)$ equations which is simultaneously solved, and n is the order of p in (8). Assuming $n = 3$, after comparing the coefficients of p to the previous equation, the system will be written as follows:

$$\begin{aligned} p^0&: \frac{\partial u_0}{\partial t} = \frac{\partial u_0^*}{\partial t}, \\ p^1&: \frac{\partial u_1}{\partial t} = -\frac{\partial u_0^*}{\partial t} + \left(\frac{\partial u_0}{\partial x}\right)^2 \frac{\partial^2 u_0}{\partial x^2}, \\ p^2&: \frac{\partial u_2}{\partial t} = \left(\frac{\partial u_0}{\partial x}\right)^2 \frac{\partial^2 u_1}{\partial x^2} + 2\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{\partial x^2}, \\ p^3&: \frac{\partial u_3}{\partial t} = \left(\frac{\partial u_0}{\partial x}\right)^2 \frac{\partial^2 u_2}{\partial x^2} + \left(\frac{\partial u_1}{\partial x}\right)^2 \frac{\partial^2 u_0}{\partial x^2} + 2\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2} + 2\frac{\partial u_0}{\partial x}\frac{\partial u_2}{\partial x}\frac{\partial^2 u_0}{\partial x^2}. \end{aligned} \tag{14}$$

By solving these differential equations, we have

$$\begin{aligned} u_0(x,t) &= e^{-x}, \\ u_1(x,t) &= e^{-3x}t, \\ u_2(x,t) &= \frac{15}{2}e^{-5x}t^2, \\ u_3(x,t) &= \frac{217}{2}e^{-7x}t^3. \end{aligned} \tag{15}$$

The approximate solution of equations (10)–(12) can be obtained by putting

$$u(x,t) = \sum_{n=0}^{\infty} u_n, \tag{16}$$

so we have

$$u(x,t) = e^{-x} + e^{-3x}t + \frac{15}{2}e^{-5x}t^2 + \frac{217}{2}e^{-7x}t^3, \tag{17}$$

where $u(x,t)$ is the spreading of Coronavirus at $t \in [0, 0.1]$ and $x \in [0, 100]$.

To explain the effect of the spreading of Coronavirus on the population with respect to the time and the area, we have to show Figures 1–5.

In Figure 1, we show the relation between the spreading of Coronavirus and the region where we study the spreading of the disease in it. We found from the figure that the spreading decreases with increasing the region. The larger the area, the less crowding and gatherings, so the spreading will decrease with the adoption of precautionary measures such as wearing a mask, not being in crowded places, and using disinfectant in a continuous way to reduce the spreading of infection.

In Figure 2, we put $x = 20$ and drew the relation between the spreading of the disease and the time of spreading, and we found that the spreading is constant with passing the time. This means that there are deaths, new cases resulting from the infection, and cases whose immunity resists the disease.

In Figure 3, at $x = 50$, we show the relation between the spreading of the disease and the time of spreading, and we found that the spreading is constant with time. There is a good chance in the graph that the spreading decreased significantly with increasing the region; this means that the deaths and the cases decreased.

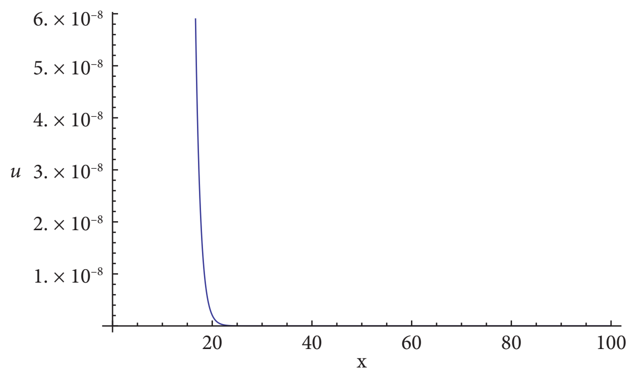

Figure 1: The relation between u and x.

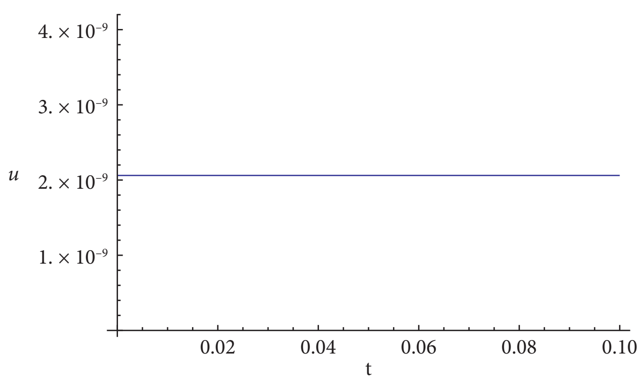

Figure 2: The relation between u and t when $x = 20$.

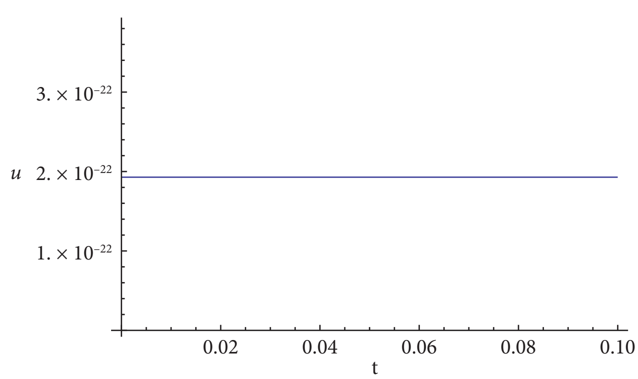

Figure 3: The relation between u and t when $x = 50$.

The benefit of vaccines and their effect on the virus is shown in Figure 4 because the spreading has almost vanished, as shown in the figure. By increasing the area, the spreading became very weak. Even in the existence of the disease, the immunity became stronger to attack it and the disease became like a normal flu.

In Figure 5, we show the relation between the spreading of the disease, the area where the disease spreads, and the time period of spreading. We found that the spreading gradually decreases with passing time and increasing the area, which means that precautionary measures and vaccinations have reduced the disease and its stability at

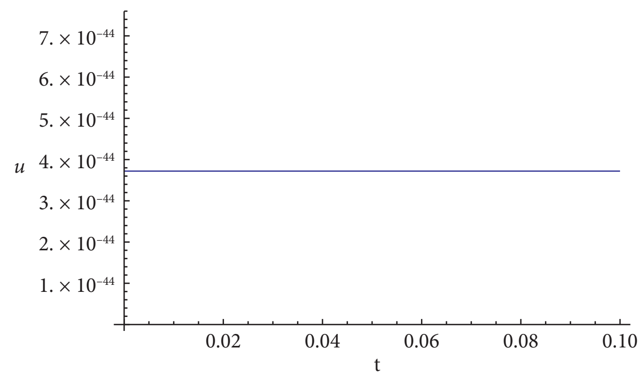

Figure 4: The relation between u and t when $x = 100$.

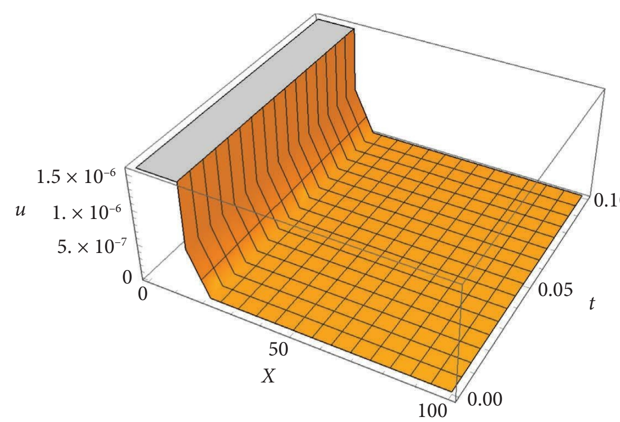

Figure 5: The relation between u, t, and x.

certain times due to vaccinations. Despite the presence of the disease, there is no danger from it because it has become like a normal flu.

The results in Figures 2–4 are shown in Table 1 in details at three values of x and different times. We found from Table 1 that the spreading of the disease decreased with increasing the area x.

3.2. New Iterative Method (NIM). For the main idea of the new iterative method [14–17], we consider the following general functional equation:

$$u(x) = f(x) + N(u(x)), \tag{18}$$

where N is a nonlinear operator from a Banach space $B \longrightarrow B$ and f is a known function. We want to find a solution of (10) using the series form

$$u(x) = \sum_{i=0}^{\infty} u_i(x). \tag{19}$$

The nonlinear operator N can be written as follows:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right]. \tag{20}$$

Table 1: Comparison table among Figures 2–4.

	$x = 20$	$x = 50$	$x = 100$
$t = 0.02$	$u = 2.06 \times 10^{-9}$	$u = 1.93 \times 10^{-22}$	$u = 3.72 \times 10^{-44}$
$t = 0.04$	$u = 2.06 \times 10^{-9}$	$u = 1.93 \times 10^{-22}$	$u = 3.72 \times 10^{-44}$
$t = 0.06$	$u = 2.06 \times 10^{-9}$	$u = 1.93 \times 10^{-22}$	$u = 3.72 \times 10^{-44}$
$t = 0.08$	$u = 2.06 \times 10^{-9}$	$u = 1.93 \times 10^{-22}$	$u = 3.72 \times 10^{-44}$
$t = 0.1$	$u = 2.06 \times 10^{-9}$	$u = 1.93 \times 10^{-22}$	$u = 3.72 \times 10^{-44}$

From (19) and (20), (18) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right]. \tag{21}$$

We can obtain the solution $u(t)$ from the following recurrence relation:

$$\begin{aligned} u_0 &= f,\\ u_1 &= N(u_0),\\ u_{r+1} &= N\left(\sum_{i=0}^{r} u_i\right) - N\left(\sum_{i=0}^{r-1} u_i\right), \quad r = 1, 2, \ldots. \end{aligned} \tag{22}$$

The approximations will be written as follows:

$$\begin{aligned} u_0 &= f,\\ u_1 &= N(u_0),\\ u_2 &= N(u_0 + u_1) - N(u_0),\\ u_3 &= N(u_0 + u_1 + u_2) - N(u_0 + u_1),\\ &\vdots \end{aligned} \tag{23}$$

and so on.

Then, $\sum_{i=1}^{r+1} u_i = N(\sum_{i=0}^{r} u_i)$, $r = 0, 1, 2, \ldots$, and

$$\sum_{i=0}^{\infty} u_i = f + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{24}$$

The nth term approximate solution of (18) is given by $\sum_{i=0}^{n-1} u_i$.

The convergence of NIM has been proved in [16, 17].

Consider the nonlinear homogeneous partial differential equations (10)–(12).

According to equation (18), we found that equations (10)–(12) are equivalent to the integral equation:

$$u(x,t) = u(x,0) + \int_0^t \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2}\, dt. \tag{25}$$

By applying NIM to equation (25), we obtain

$$\begin{aligned} u_0 &= f = u(x,0) = e^{-x},\\ u_1 &= N(u_0) = \int_0^t \left(\frac{\partial u_0}{\partial x}\right)^2 \frac{\partial^2 u_0}{\partial x^2}\, dt = te^{-3x},\\ u_2 &= N(u_0+u_1) - N(u_0) = \int_0^t \left(\frac{\partial (u_0+u_1)}{\partial x}\right)^2 \frac{\partial^2 (u_0+u_1)}{\partial x^2}\, dt - \int_0^t \left(\frac{\partial u_0}{\partial x}\right)^2 \frac{\partial^2 u_0}{\partial x^2}\, dt = \frac{15}{2}e^{-5x}t^2 + 21e^{-7x}t^3 + \frac{81}{4}e^{-9x}t^4,\\ u_3 &= N(u_0+u_1+u_2) - N(u_0+u_1) = \int_0^t \left(\frac{\partial (u_0+u_1+u_2)}{\partial x}\right)^2 \frac{\partial^2 (u_0+u_1+u_2)}{\partial x^2}\, dt - \int_0^t \left(\frac{\partial u_0}{\partial x}\right)^2 \frac{\partial^2 u_0}{\partial x^2}\, dt\\ &= \frac{175}{2}e^{-7x}t^3 + 837e^{-9x}t^4 + \frac{30888}{5}e^{-11x}t^5 + \frac{306111}{8}e^{-13x}t^6 + \frac{10508535}{56}e^{-15x}t^7 + \frac{12448539}{16}e^{-17x}t^8 + \frac{21289899}{8}e^{-19x}t^9\\ &\quad + \frac{1064378259}{160}e^{-21x}t^{10} + \frac{3815280549}{352}e^{-23x}t^{11} + \frac{651015225}{64}e^{-25x}t^{12} + \frac{3486784401}{832}e^{-27x}t^{13}. \end{aligned} \tag{26}$$

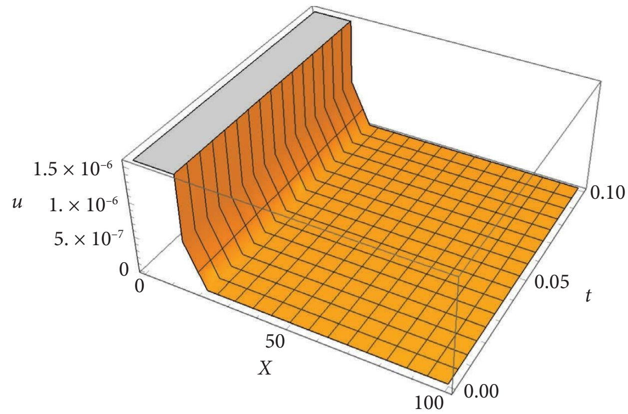

Figure 6: The relation between u, x, and t.

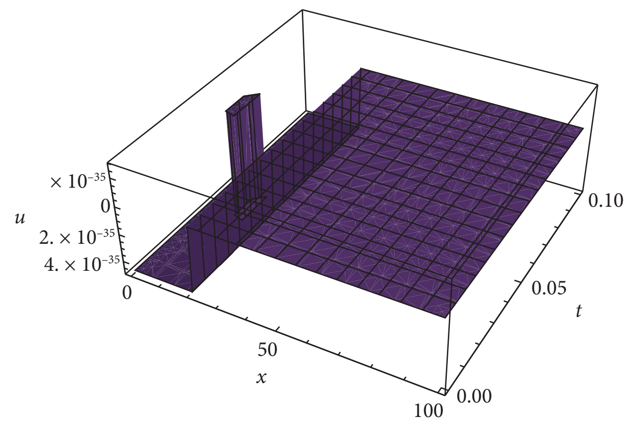

Figure 7: Residual error of the approximate solution of HPM.

The 3-term approximate solution for equations (10)–(12) is

$$
\begin{aligned}
u(x,t) = \sum_{i=0}^{3} u_i &= e^{-x} + e^{-3x}t + \frac{15}{2}e^{-5x}t^2 + \frac{217}{2}e^{-7x}t^3 + \frac{3429}{4}e^{-9x}t^4 + \frac{30888}{5}e^{-11x}t^5 \\
&+ \frac{306111}{8}e^{-13x}t^6 + \frac{10508535}{56}e^{-15x}t^7 + \frac{12448539}{16}e^{-17x}t^8 \\
&+ \frac{21289899}{8}e^{-19x}t^9 + \frac{1064378259}{160}e^{-21x}t^{10} + \frac{3815280549}{352}e^{-23x}t^{11} \\
&+ \frac{651015225}{64}e^{-25x}t^{12} + \frac{3486784401}{832}e^{-27x}t^{13}.
\end{aligned}
\tag{27}
$$

In Figure 6, we explain the relation between the spreading of the Coronavirus u, the area where the disease spreads in x, and the time period of spreading t. We found that the spreading decreases with increasing the area and the time.

4. Results

In Figures 7 and 8, we show the residual error of the approximate solutions of the HPM and NIM.

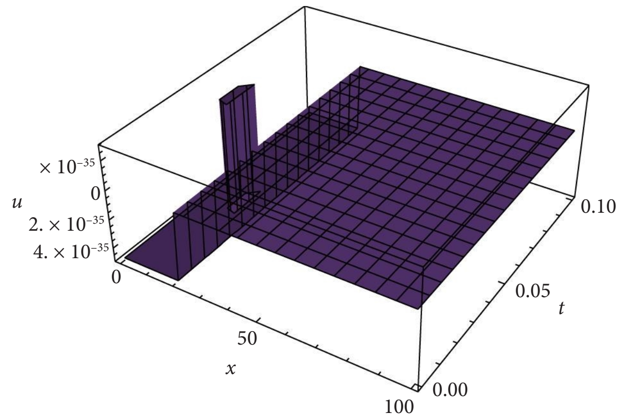

FIGURE 8: Residual error of the approximate solution of NIM.

TABLE 2: Comparison table between the residual error of HPM and the residual error of NIM.

t	0.02	0.04	0.06	0.08	0.1
x	20	40	60	80	100
Re of HPM	0	9.27302×10^{-69}	1.19851×10^{-94}	$-7.74518 \times 10^{-121}$	$-1.00104 \times 10^{-146}$
Re of NIM	0	9.27302×10^{-69}	1.19851×10^{-94}	$-7.74518 \times 10^{-121}$	$-1.00104 \times 10^{-146}$

After calculating the residual error for the two methods HPM and NIM, we found that it is too small and the residual error for the two methods is the same as shown in Table 2.

In Table 2, we present the residual error for the two approximate solutions solved by HPM and NIM.

5. Conclusion

In this paper, we were concerned with a zero-sum game which is applied on Coronavirus, which is described by the partial differential equations, and we solved this problem by using HPM and NIM. Also, we made a comparison between the two methods with respect to the residual error, and we found that the residual error is very small and this is good for solution accuracy and this means that the two methods are successful for finding the solution. Finally, we showed in the figures the decreasing of spreading for Coronavirus with increasing the area by passing the time. The graphs showed the importance of spacing between people to reduce the chances of infection, as well as all precautionary measures such as wearing a mask and spraying sterilizers. Here, the important and main role of vaccines comes, which is considered to be a very large percentage that made the presence of the virus not scary because it made the virus look like a normal flu, which reduced the risk to most people.

Data Availability

No new data were collected or generated for this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] Y. Sibuya and P. F. Hsieh, *Basic Theory Differential Equations*, Springer-Verlag, New York, NY, USA, 1999.

[2] S. J. Farlow, *Partial Differential Equations for Scientists and Engineers*, John Wiley Sons, Hoboken, NJ, USA, 1982.

[3] P. Blanc and J. D. Rossi, "Game theory and partial differential equations," *De Gruyter series in Nonlinear Analysis and Applications*, Walter de Gruyter GmbH and Co KG, Berlin, Germany, 2019.

[4] A. Friedman, *Differential Games*, John Wiley Sons, Hoboken, NJ, USA, 1971.

[5] G. Owen, *Game Theory*, Academic Press, INC, Cambridge, MA, USA, 1982.

[6] A. E. M. A. Megahed, E. A. Youness, and H. K. Arafat, "Optimization method in counter terrorism: min-max zero-sum differential game approach," *Numerical Algebra, Control and Optimization*, vol. 13, no. 2, pp. 299–313, 2023.

[7] I. Kareva and G. Karev, "Natural selection between two games with applications to game theoretical models of cancer," *Bulletin of Mathematical Biology*, vol. 81, no. 7, pp. 2117–2132, 2019.

[8] K. M. A. Kabir and J. Tanimoto, "Evolutionary game theory modelling to represent the behavioural dynamics of economic shutdowns and shield immunity in the COVID-19 pandemic," *Royal Society Open Science*, vol. 7, no. 9, p. 201095, 2020.

[9] O. J. Watson, G. Barnsley, J. Toor, J. Azra, A. Hogan, and P. Winskill, "Global impact of the first year of covid-19 vaccination: a mathematical modelling study," *The Lancet Infectious Diseases*, vol. 22, pp. 1–22, 2022.

[10] E. A. Youness, A. E. M. A. Megahed, E. E. Eladdad, and H. F. A. Madkour, "Min-max differential game with partial differential equation," *AIMS Mathematics*, vol. 7, no. 8, pp. 13777–13789, 2022.

[11] J. H. He, "Application of homotopy perturbation method to nonlinear wave equations," *Chaos, Solitons and Fractals*, vol. 26, no. 3, pp. 695–700, 2005.

[12] H. Jafari, M. Zabihi, and M. Saidy, "Application of homotopy-perturbation method for solving gas dynamics equation," *Applied Mathematical Sciences*, vol. 2, no. 48, pp. 2393–2396, 2008.

[13] A. A. Hemeda, "Modified homotopy perturbation method for solving fractional DifferentialEquations," *Journal of Applied Mathematics*, vol. 2014, Article ID 594245, 9 pages, 2014.

[14] J. H. He, "A new iteration method for solving algebraic equations," *Applied Mathematics and Computation*, vol. 135, no. 1, pp. 81–84, 2003.

[15] M. A. Noor, K. I. Noor, E. A. Said, and M. Waseem, "Some new iterative methods for nonlinear equations," *Mathematical Problems in Engineering*, vol. 2010, Article ID 198943, 12 pages, 2010.

[16] A. A. Hemeda, "New iterative method: an application for solving fractional physical differential equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 617010, 9 pages, 2013.

[17] V. Daftardar-Gejji and H. Jafari, "An iterative method for solving non-linear functional equations," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 753–763, 2006.