Research Article

Study of Fractional-Order Boundary Value Problems Using Nondiscretization Numerical Method

Zareen A. Khan, Sajjad Ahmad, Salman Zeb, and Muhammad Yousaf

1Department of Mathematical Sciences, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
2Department of Mathematics, University of Malakand, Chakdara, Dir Lower 18800, Khyber Pakhtunkhwa, Pakistan

Correspondence should be addressed to Muhammad Yousaf; musaf2004@yahoo.com

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This paper is devoted to present a numerical scheme based on operational matrices to compute approximate solutions to fractional-order boundary value problems. For the mentioned operational matrices, we utilize fractional-order Bernoulli polynomials. Since fractional-order problems are usually difficult to treat for their corresponding analytical or exact solutions, therefore, we need sophisticated methods to find their best numerical solutions. The presented numerical scheme has the ability to reduce the proposed problem to the corresponding algebraic equations. The obtained algebraic equations are then solved by using the computational software MATLAB for the corresponding numerical results. The used method has the ability to save much more time and also is reliable to secure the proper amount of memory. Several examples are solved by using the considered method. Also, the solutions are compared with their exact solution graphically. In addition, the absolute errors for different scale values are presented graphically. In addition, we compare our results with the results of the shifted Legendre polynomials spectral method.

1. Introduction

Newton and Leibnitz in the seventeenth century established the theory of calculus. The notations of derivatives and integrals we use today have been introduced by Leibnitz. Later on, the concept of derivative and integral was extended from integer to any real order. First time in 1819, Lacroix gave the concept of noninteger order derivative [1]. The very first application was investigated by Abel in 1823 [1]. Therefore, Fourier, Liouville, Riemann, Grünwald, and Letnikov [2, 3] gave tremendous attention to the said area. Arbitary order differentiation and integration provide a generalization to the classical order but fractional-order differentiation and integration do have not unique definitions. Therefore, various researchers have introduced various definitions of arbitrary order derivatives and integrations. Among all these, the definitions of Caputo and Riemann–Liouville are extensively applicable.

The researchers have investigated that almost every model of physics and biology consists of arbitrary order derivatives. Fractional-order differential equations (FODEs) are widely used in various fields of science and technology [4–6] and [7]. Furthermore, various properties of numerous materials such as hereditary properties are described by FODEs. Therefore, the significance of FODEs has appealed to researchers for further development in the theory. There are many areas in the theory of FODEs, one of them being the study of the numerical approach of fractional-order initial value problems (FOIVPs) and fractional-order boundary value problems (FOBVPs). It has been observed that the initial and boundary value problems with ordinary derivatives have been greatly worked on by the researchers but FOIVPs and FOBVPs are in their beginning stages and need proper further exploration. Here, we remark that the aforementioned area has significant applications in various other areas. For instance, the author of [8] studied nonlocal...
polynomials. By using these OMs, we convert the considered operational matrices by using Bernoulli computational software like MATLAB. The analytical solutions of most of the FODEs cannot be found because of the complexity of fractional order. Therefore, approximate analytical or numerical solutions are obtained for the said area. For this purpose, various analytical and numerical methods have emerged, for instance, eigenvector procedure [12], perturbation tools [13], iteration techniques [14], and transform methods [15]. Authors studied numerically discrete-time prey-predator model and SIR-type model by using concepts of fractional calculus in [16] and [17]. Authors [18] used the differential transform method to study some systems of FODEs. Authors used decomposition schemes to compute approximate solutions of some problems in [19]. The finite difference method has been applied in [20]. Also, the Tau method has been applied in [21] and [22]. Collocation techniques have also been utilized in [23]. Wavelet analysis [24] has been used for numerical results increasingly. Recently, authors studied the time-fractional model of generalized Couette flow of couple stress nanofluid with heat and mass transfer in [25]. Authors [26] studied fractional-order Caudrey–Dodd–Gibbon stress nanofluid with heat and mass transfer in [25]. Authors [27] computed the approximate solution to wave-like equations. Additionally, authors [28] computed an approximate solution of the Noyes-field model for the time-fractional Belousov–Zhabotinsky reaction. In the same way, authors [29] have used the transform method to study fractional-order Swift–Hohenberg equations.

Here, we remark that spectral methods based on the operational matrices (OMs) for various orthogonal polynomials such as Jacobi, Legendre, and Laguerre have been developed for solving FODEs numerically [30]. The said OMs have been constructed from the mentioned polynomials by the techniques of discretization which occupies extra memory and consumes much time. Furthermore, the aforesaid methods have been applied to FOIVPs in a variety of cases. However, FOBVPs are very rarely investigated. To deal with boundary value problems, some extra operations are needed. For this purpose, we establish a numerical algorithm based on OMs for Bernoulli functions which are not orthogonal. We construct the said OMs through Bernoulli functions without discretization to save memory and time. By using these OMs, we establish a numerical algorithm to find the numerical solution for the considered problems. A new operational matrix corresponding to boundary conditions has also been obtained.

Inspired by the above discussion, we consider the given class of FOBVPs [31] under the fractional order described by $1 < \nu \leq 2, 0 < \nu_1 \leq 1$ as

\[
\begin{align*}
D^\nu U(x) &= k_1 D^\nu U(x) + k_2 D^\nu V(x) + k_3 U(x) + k_4 V(x) + f(x), \quad x \in [0, 1], \\
U(x)|_{x=0} &= u_0, \quad U(x)|_{x=1} = u_1, \\
V(x)|_{x=0} &= v_0, \quad V(x)|_{x=1} = v_1,
\end{align*}
\]

where $k_1, k_2$ are real constants and $f \in C(J, R)$ is a source function with $J = [0, 1]$.

In addition, we also extend the aforementioned scheme for the coupled system of FOBVPs as described by

\[
\begin{align*}
D^\nu U(x) &= m_1 D^\nu U(x) + m_2 D^\nu V(x) + m_3 U(x) + m_4 V(x) + g(x), \quad x \in [0, 1], \\
U(x)|_{x=0} &= u_0, \quad U(x)|_{x=1} = u_1, \\
V(x)|_{x=0} &= v_0, \quad V(x)|_{x=1} = v_1,
\end{align*}
\]

with BCs given by

\[
\begin{align*}
U(x)|_{x=0} &= u_0, \quad U(x)|_{x=1} = u_1, \\
V(x)|_{x=0} &= v_0, \quad V(x)|_{x=1} = v_1, \quad 1 < \nu, \\
0 < \nu_1 \leq 1, \quad \omega_1, \omega_2 \leq 1.
\end{align*}
\]

where the fractional orders are described by $1 < \nu, \omega \leq 2, 0 < \nu_1, \nu_2, \omega_1, \omega_2 \leq 1$. Furthermore, $f, g: [0, 1] \rightarrow R$ are linear continuous source functions, and $u_0, u_1, v_0, v_1 \in R, k_i, m_i (i = 1, 2)$ are any real constants. Here, we first establish the mentioned operational matrices by using Bernoulli polynomials. By using these OMs, we convert the considered system of FODEs to Sylvester-type equations given by

\[
\begin{align*}
A_1 X + XB_1 &= C_1, \\
A_2 X + XB_2 &= C_2,
\end{align*}
\]

where the notions $A_1, A_2, B_1, B_2, C_1, C_2$ represent matrices and $X$ is an unknown matrix to be computed. We solve the obtained equation (4) for unknown matrix $X$ by using computational software like MATLAB. Here, we demonstrate some novelty of our work. Boundary value problems (BVPs) constitute a very important branch of applied analysis because many engineering and physical problems to understand real-world procedures or phenomena are devoted to BVPs. The concerned branch when studied under the fractional-order derivatives and integrals sense further enhances this field of study, as many real-world processes involve short or long nature memory effects which cannot be detected through applications of ordinary derivatives. Hence, with the help of fractional-order derivatives, we can explain long- and short-memory effects more clearly for best use in the area of engineering and physical sciences. Furthermore, finding the exact solutions to fractional-order boundary value problems is a very tedious job and cannot be achieved in a simple way. Therefore, various numerical techniques have been developed to treat such problems. Among the available numerical methods, spectral numerical schemes based on OMs are powerful techniques. However, the operational matrices when established by using the discretization technique
utilize more memory and time-consuming process. Therefore, omitting the discretization and obtaining the required matrices are our goal. In this work, we have utilized Bernoulli polynomials a powerful tool to establish operational matrices of fractional-order integration and differentiation. The concerned matrices convert the considered FODE to an algebraic equation of Sylvester-type matrix equation. The obtained matrix equation is then solved by the use of the Gauss elimination procedure using computational software like MATLAB or Mathematica. In this way, we obtain the numerical scheme which needs no discretization as earlier used in research. Furthermore, the proposed polynomials have the ability to produce more accurate results than wavelet and other different methods. Here, it is interesting that in the last few years, spectral methods have been used very well for different problems of FODEs. For some more frequent works, we refer to [32–36] and [37]. Here, researchers have used optimal control and Bernstein and Legendre polynomials to study various problems of FODEs. Result devoted to stability has also been deduced by following the methodology of [38].

The rest of the work of this article is organized as follows. In Section 2, we provide some necessary definitions from the fundamental theory of calculus and FOBPs required in the subsequent development. The function approximation procedure and the OMs of fractional integration and differentiation are given in Section 3. Also, in the same Section 3, we give new OM corresponding to BCs. Section 4 is devoted to the establishment of numerical algorithms for the solutions of FOBVPS and coupled system FOBVPS. Part 5 is related to investigating the convergency of the proposed method. In Section 6, the proposed method is applied to various examples, and the concerned numerical results are presented through graphs. Also, the exact and numerical solutions are compared in the same Section 6. The last Section 7 is devoted to the conclusion of this article 7.

2. Preliminaries

Here, we give some important definitions from the basic fractional calculus theory and Bernoulli polynomials necessary for the subsequent development.

Definition 1. The Riemann–Liouville’s arbitrary order integral operator is defined by [32]:

\[
\int_{\alpha}^{\beta} \gamma(t) = \frac{1}{\Gamma (\nu)} \int_{\alpha}^{\beta} (t-s)^{\nu-1} \gamma(s)ds, \quad t > 0.
\]

For \( k_1, k_2 \in R \) and any arbitrary order \( \nu, \nu_1, \nu_2 \) and \( a > -1 \), the following properties are satisfied [32, 33]:

(i) \( \Gamma (k_1 \gamma_1(t) + k_2 \gamma_2(t)) = k_1 \Gamma (\gamma_1(t)) + k_2 \Gamma (\gamma_2(t)) \)

(ii) \( \Gamma (\nu + 1) = \Gamma (\nu) \Gamma (\gamma(t)) \)

(iii) \( \Gamma (\nu + 1) = \Gamma (\nu) \Gamma (\gamma(t)) \)

(iv) \( \Gamma (a + 1) = (a + 1) \Gamma (a) \Gamma (\gamma(t)) \)

Definition 2. The Caputo’s arbitrary order derivative is defined by [32]

\[
D^\nu y(t) = \frac{1}{\Gamma (n - \nu)} \int_{0}^{t} (t-s)^{n-\nu-1} y^{(n)}(s)ds, \quad n - 1 < \nu \leq n, \quad t > 0,
\]

where \( n = [\nu] + 1 \).

The given properties are satisfied by Caputo’s arbitrary order derivative (details can be seen in [32, 33]).

(i) \( D^\nu \gamma(t) = \gamma(t) \)

(ii) \( D^\nu \gamma(t) = \gamma(t) - \sum_{i=0}^{n-1} \gamma(i)(0)i!i! \)

(iii) \( D^\nu t^a = \begin{cases} 0, & a < n, \\ \Gamma (a + 1)\Gamma (a + 1 - \nu)t^{a-n}, & \text{elsewhere} \end{cases} \)

(iv) \( D^\nu c = 0 \)

(v) \( D^\nu (k_1 \gamma_1(t) + k_2 \gamma_2(t)) = k_1 D^\nu \gamma_1(t) + k_2 D^\nu \gamma_2(t) \)

where \( c, k_1, k_2 \) are constants.

2.1. The Fractional-Order Bernoulli Polynomials. In this section, we define FOBPs and provide some of their properties.

The FOBPs \( B_n^a(x) \) are defined on \([0, 1]\) in [35] as

\[
B_n^a(x) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j}^a x^j, \quad x \in [0, 1],
\]

where \( B_j^a = B_j^a(0) = B_j^a, \quad j = 0, 1, 2, \ldots, n \) are Bernoulli constants. Thus, the first four FOBPs are

\[
B_0^a(x) = 1, \quad B_1^a(x) = x^a - \frac{1}{2^a}, \quad B_2^a(x) = x^{2a} - x^a + \frac{1}{6}, \quad B_3^a(x) = x^{3a} - \frac{3}{2}x^{2a} + \frac{1}{2}x^a.
\]

The fractional-order Bernoulli polynomials of order \( na \) in the determinant form are defined by
\[ B_0^\alpha(x) = 1, \]

\[
\begin{bmatrix}
1 & x & x^2 & x^3 & \cdots & x^{(n-1)a} & x^{na} \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & 2 & 3 & \cdots & n-1 & n \\
0 & 0 & 0 & 0 & \cdots & (n-1) & (n) \\
0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\end{bmatrix}
\]

\[
B_0^\alpha(x) = \frac{(-1)^n}{(n-1)!}, \quad B_n^\alpha(x) = \frac{m!n!}{(m+n)!}B_{m+n}, \quad m, n \geq 1. \tag{9}
\]

\[
\int_0^1 B_0^\alpha(x)B_n^\alpha(x)x^{a-1} \ dx = \frac{1}{\alpha}(-1)^{n-1}\frac{m!n!}{(m+n)!}B_{m+n}, \quad m, n \geq 1. \tag{10}
\]

### 3. Function Approximation

In this section, we give the procedure for how to approximate a function in terms of FOBPs. Here, we define \( I = \{ y' : 0 \leq y \leq 1 \} \) and \( L^2(I) = \{ g : I \rightarrow Rg \text{ is measurable, and } \|g\|_2 < \infty \} \), where

\[
\|g\|_2 = \left( \int_0^1 |g(x)|^2 \ dx \right)^{1/2}. \tag{11}
\]

Now, suppose that

\[
\{ B_0^\alpha(x), B_1^\alpha(x), B_2^\alpha(x), \ldots, B_m^\alpha(x) \} \subset L^2(I), \quad m \in W, \tag{12}
\]

is the set of FOBPs and

\[
Z_m = \text{Span}\{ B_0^\alpha(x), B_1^\alpha(x), B_2^\alpha(x), \ldots, B_m^\alpha(x) \}. \tag{13}
\]

Here, \( Z_m \) is finite-dimensional and closed vector space. So, every element \( g \) of \( L^2(I) \) has a unique best approximation \( g_0 \) out of \( Z_m \), such that

\[
\forall z \in Z_m, \| g - g_0 \| \leq \| g - z \|, \tag{14}
\]

which can be written as

\[
\langle g - g_0, z \rangle = 0, \quad \forall z \in Z_m, \tag{15}
\]

where \( \langle \cdot, \cdot \rangle \) stands for inner product. Now, as \( g_0 \in Z_m \), the unique coefficients \( c_0, c_1, c_2, \ldots, c_m \) are exist such that

\[
f(x) = f_0(x) = \sum_{i=0}^{m} c_i B_i^\alpha(x) = C_{M+1}^T B_{M+1}^\alpha(x), \tag{16}
\]

where \( M = m + 1 \), \( C_{M+1} \) is the coefficients matrix and \( B_{M+1}^\alpha(x) \) is fractional-order Bernoulli functions vector given by

\[
C_{M+1} = \begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_m
\end{bmatrix},
\]

\[
B_{M+1}^\alpha(x) = \begin{bmatrix}
B_0^\alpha(x) \\
B_1^\alpha(x) \\
\vdots \\
B_m^\alpha(x)
\end{bmatrix}.
\]

For evaluating \( C_{M+1} \), consider

\[
f_j = \langle f, B_j^\alpha \rangle = \int_0^1 f(x)B_j^\alpha(x)x^{a-1} \ dx. \tag{18}
\]

Using (11), we get
where
\[ d_{ij}^a = \int_0^1 B_i^a (x) B_j^a (x) x^{\alpha - 1} \, dx, \quad i = j = 0, 1, 2, \ldots, m. \] (20)

This implies that
\[ f_j = \sum_{r=0}^m c_i d_{ij}^a, \quad j = 0, 1, 2, \ldots, m, \] (21)
which gives
\[ F^T_{M \times 1} = C^T_{M \times 1} D^a_{M \times M} \] (22)
where
\[ F^T_{M \times 1} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_m \end{bmatrix}, \quad D^a_{M \times M} = \begin{bmatrix} d_{00}^a & d_{01}^a & \cdots & d_{0m}^a \\ d_{10}^a & d_{11}^a & \cdots & d_{1m}^a \\ \vdots & \vdots & \ddots & \vdots \\ d_{m0}^a & d_{m1}^a & \cdots & d_{mm}^a \end{bmatrix} \] (23)
where
\[ d_{ij}^a = \int_0^1 B_i^a (x) B_j^a (x) x^{\alpha - 1} \, dx \]
\[ = \frac{1}{\alpha} (-1)^{j-1} \frac{j!}{(j + 1)!} B_{ij}, \quad i, j \geq 1. \] (24)

3.1. Derivation of OMs. Here, in this section, we derive Bernoulli-type OMs of fractional-order integration and differentiation. Moreover, we construct a new OM corresponding to some boundary value.

**Theorem 3.** Let \( B^a_{M \times 1} (x) \) be the fractional-order Bernoulli functions vector, then
\[ \Gamma^a B^a_{M \times 1} (x) = P_{M \times M} B^a_{M \times 1} (x), \] (25)
where \( P_{M \times M} \) is given by
\[ \Gamma^a B^a_{j} (x) = \sum_{r=0}^m \omega_{j,r} p_{j,r}^a B^a_{M \times 1} (x), \quad i = 0, 1, 2, \ldots, m. \] (33)

Hence, we have
Consider the arbitrary order differential operator in (11), one can easily prove the lemma.

**Theorem 5.** Let $B_{M \times 1}^{\alpha} (x)$ be the FOBPs vector, then

$$D^\alpha B_{M \times 1}^{\alpha} (x) = D_{M \times M}^{\alpha} B_{M \times 1}^{\alpha} (x),$$

where $D_{M \times M}^{\alpha}$ is given by

$$D_{M \times M}^{\alpha} = \begin{bmatrix}
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0 \\
  \sum_{r=[y/a]} \Omega^{(\alpha)}_{r,0}, \sum_{r=[y/a]} \Omega^{(\alpha)}_{r,1}, \cdots, \sum_{r=[y/a]} \Omega^{(\alpha)}_{r,m} \\
  \vdots & \vdots & \ddots & \vdots \\
  \sum_{r=[y/a]} \Omega^{(\alpha)}_{m,0}, \sum_{r=[y/a]} \Omega^{(\alpha)}_{m,1}, \cdots, \sum_{r=[y/a]} \Omega^{(\alpha)}_{m,m} 
\end{bmatrix},$$

Proof. By applying the properties (iii – v) of Caputo arbitrary order differential operator in (11), one can easily prove the lemma.

**Lemma 4.** Let $B_i^\alpha (x)$ be the FOBP, then

$$D^\alpha B_i^\alpha (x) = 0, \quad i = 0, 1, \ldots, \left\lfloor \frac{\omega}{\alpha} \right\rfloor - 1, \omega > 0.$$
where
\[ \Omega_{i,j,r}^{(v,a)} = \eta_{r,i}^{(v,a)} \phi_{r,j}. \tag{43} \]

Equation (42) implies that
\[ D_{B_r^{x}}(x) = \sum_{r=0}^{\lfloor y/a \rfloor} \Omega_{i,j,r}^{(v,a)} \phi_{r,j}. \tag{44} \]

Also, using Lemma 4, we get from (44)
\[ D_{B_r^{x}}(x) = [0, 0, 0, \ldots, 0] B_{M+1}^r (x), \quad i = 0, 1, \ldots, \left\lfloor \frac{y}{a} \right\rfloor - 1. \tag{45} \]

**Theorem 6.** Let \( B_{M+1}^r (x) \) be a FOBPs matrix and let \( \phi(x) \) be any function given \( \phi(x) = x^l, l = 0, 1, 2, \ldots, I \in R \) and \( U(x) = C_{M+1}^T B_{M+1}^r (x) \), then
\[ \phi(x) U(x) = C_{M+1}^T Q_{M+1}^{(a,\phi)} B_{M+1}^{x} (x), \tag{47} \]

where \( Q_{M+1}^{(a,\phi)} \) is given by
\[ Q_{M+1}^{(a,\phi)} = \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,m} \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{m,0} & \Phi_{m,1} & \cdots & \Phi_{m,m} \end{bmatrix}. \tag{48} \]

**Proof.** Consider
\[ \partial_{t}^a B_r^{x}(x) = \frac{1}{\Gamma(a)} \int_{0}^{1} (1-t)^{a-1} B_r^{x}(t)dt \]
\[ = \frac{1}{\Gamma(a)} \int_{0}^{1} (1-t)^{a-1} \sum_{i=0}^{m} \binom{m}{i} B_{m-i}^{x} t^i dt \]
\[ = \frac{1}{\Gamma(a)} \sum_{i=0}^{m} \binom{m}{i} B_{m-i}^{x} \int_{0}^{1} (1-t)^{a-1} t^i dt. \tag{49} \]

Using the well-known property of the Beta function given by
\[ \beta(a, b) = \int_{0}^{1} y^{a-1} (1-y)^{b-1} dy = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{50} \]
in equation (47), we have
\begin{equation} 
0 \Gamma_x B^\alpha_x (x) = \sum_{i=0}^{m} \binom{m}{i} B^\alpha_{m-i} \Gamma^{(j+1)} (x+1) = \Delta_1, \tag{51} \end{equation}

which implies
\begin{equation} 
\phi(x) \Gamma^{(j+1)} (x+1) = \Delta_1 \phi(x), \quad \Delta_1 \phi(x) = \sum_{j=0}^{m} \Phi_{i,j} B^\alpha_{m-i} (x). \tag{52} \end{equation}

Putting the values of \(i\) and \(j\) for the Caputo differential operator, we get from (54)
\begin{equation} 
U(x) = C^{T}_{M+\alpha} P_{M+\alpha} B^\alpha_{M+\alpha} (1) + u_0 + (u_1 - u_0) x - x C^{T}_{M+\alpha} P_{M+\alpha} B^\alpha_{M+\alpha} (1). \tag{57} \end{equation}

Using the above established OMs, and after simplification, we get from (57)
\begin{equation} 
U(x) = C^{T}_{M+\alpha} (P_{M+\alpha} - Q_{M+\alpha}) B^\alpha_{M+\alpha} (x) + F^{T}_{M+\alpha} B^\alpha_{M+\alpha} (x), \tag{58} \end{equation}

\begin{equation} 
D^{\gamma} U(x) = C^{T}_{M+\alpha} (P_{M+\alpha} - Q_{M+\alpha}) D^{(\gamma)} B^\alpha_{M+\alpha} (x) + F^{T}_{M+\alpha} D^{(\gamma)} B^\alpha_{M+\alpha} (x). \tag{59} \end{equation}

Putting (54), (58), and (59) in (1) yields
\begin{equation} 
C^{T}_{M+\alpha} B^\alpha_{M+\alpha} (x) = k_1 C_{M+\alpha} (P_{M+\alpha} - Q_{M+\alpha}) D^{(\gamma)} B^\alpha_{M+\alpha} (x) + k_1 F_{M+\alpha} B^\alpha_{M+\alpha} (x) + k_2 C_{M+\alpha} (P_{M+\alpha} - Q_{M+\alpha}) B^\alpha_{M+\alpha} (x) + k_2 F_{M+\alpha} B^\alpha_{M+\alpha} (x), \tag{60} \end{equation}

where \(C_{M+\alpha} B^\alpha_{M+\alpha} (x) = f(x)\). After simplification, we obtain
\begin{equation} 
\left( C^{T}_{M+\alpha} - C_{M+\alpha} \right) \left( D^{(\gamma)} B^\alpha_{M+\alpha} + k_1 D_{M+\alpha} + k_2 I_{M+\alpha} \right) - L_{M+\alpha} B^\alpha_{M+\alpha} (x) = 0, \tag{61} \end{equation}

where \(k_1 F_{M+\alpha} D^{(\gamma)} B^\alpha_{M+\alpha} + k_2 F_{M+\alpha} B^\alpha_{M+\alpha} = L_{M+\alpha}\). Equation (61) implies that

4.1. Numerical Scheme for Scaler Problem (1). Here, we develop a numerical scheme for a single problem given in (1). To obtain the solution in terms of FOBPs, we assume that
\begin{equation} 
\mathbf{D}U(x) = C^{T}_{M+\alpha} B^\alpha_{M+\alpha} (x). \tag{54} \end{equation}

Applying \(
\mathbf{D}^{\gamma}\) and property (ii) for the Caputo differential operator, we get from (54)
\begin{equation} 
U(x) = C^{T}_{M+\alpha} P_{M+\alpha} B^\alpha_{M+\alpha} (x) + c_0 + c_1 x, \tag{55} \end{equation}

Using the given boundary conditions, we have \(c_0 = u_0\) and \(c_1 = u_1 - u_0 - C^{T}_{M+\alpha} P_{M+\alpha} B^\alpha_{M+\alpha} (1). \tag{56} \)

Putting the values of \(c_0\) and \(c_1\) in (55), one can get
c^T_{M\times 1} - c^T_{M\times 1}(P^{(\gamma,a)}_{M\times M} - Q^{(\gamma,\phi)}_{M\times M} D^{(\gamma,a)}_{M\times M} + k_2 I_{M\times M})

- L_{M\times M} = 0.

(62)

This is a Sylvester-type equation. Calculating $C_{M\times 1}$ from (23) and substituting back in (20), one can have the desired approximate solution of the given FOBVP.

4.2. Numerical Scheme for Coupled System (2) with BCs (3).

Here, we develop a numerical scheme for the computation of an approximate solution for a coupled system (2) with BCs (3). In this regard, we assume that

\[
\begin{align*}
D^\gamma U(x) &= C^T_{M\times 1} B_{M\times 1}^a(x), \\
D^\omega V(x) &= E^T_{M\times 1} B_{M\times 1}^a(x).
\end{align*}
\]

(63)

Applying $\Gamma^\gamma, \Gamma^\omega$ and property (ii) for the Caputo differential operator, we get

\[
\begin{align*}
U(x) &= C^T_{M\times 1} P_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(x) + c_0 + c_1 x, \\
V(x) &= E^T_{M\times 1} P_{M\times M}^{(\omega,a)} B_{M\times 1}^a(x) + d_0 + d_1 x.
\end{align*}
\]

(64)

Using the given BCs, we have

\[
\begin{align*}
c_0 &= u_0, c_1 = u_1 - u_0 - C^T_{M\times 1} P_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(1), \\
d_0 &= v_0, d_1 &= v_1 - v_0 - E^T_{M\times 1} P_{M\times M}^{(\omega,a)} B_{M\times 1}^a(1).
\end{align*}
\]

(65)

Inserting (65) in (64), we obtain

\[
\begin{align*}
U(x) &= C^T_{M\times 1} P_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(x) + u_0 + (u_1 - u_0)x - xC^T_{M\times 1} P_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(1), \\
V(x) &= E^T_{M\times 1} P_{M\times M}^{(\omega,a)} B_{M\times 1}^a(x) + v_0 + (v_1 - v_0)x - xE^T_{M\times 1} P_{M\times M}^{(\omega,a)} B_{M\times 1}^a(1).
\end{align*}
\]

(66)

After simplification of (66), we get

\[
\begin{align*}
U(x) &= C^T_{M\times 1} P_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(x) + u_0 + (u_1 - u_0)x - xC^T_{M\times 1} P_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(1), \\
V(x) &= E^T_{M\times 1} P_{M\times M}^{(\omega,a)} B_{M\times 1}^a(x) + v_0 + (v_1 - v_0)x - xE^T_{M\times 1} P_{M\times M}^{(\omega,a)} B_{M\times 1}^a(1),
\end{align*}
\]

(67)

where $F^T_{M\times 1} B_{M\times 1}^a(1) = u_0 + (u_1 - u_0)x, f^T_{M\times 1} B_{M\times 1}^a(x) = v_0 + (v_1 - v_0)x$ and $\phi = x$. Now, by applying $D^\gamma, D^\omega, D^\alpha\gamma$, and $D^\alpha\omega$ to (67), we have

\[
\begin{align*}
D^\gamma U(x) &= C^T_{M\times 1} P_{M\times M}^{(\gamma,a)} D_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(x) + F^T_{M\times 1} D_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(1), \\
D^\omega V(x) &= E^T_{M\times 1} P_{M\times M}^{(\omega,a)} D_{M\times M}^{(\omega,a)} B_{M\times 1}^a(x) + f^T_{M\times 1} D_{M\times M}^{(\omega,a)} B_{M\times 1}^a(1),
\end{align*}
\]

(68)

Putting (63), (66), and (68) in coupled system (2) yields

\[
\begin{align*}
C^T_{M\times 1} B_{M\times 1}^a(1) &= \begin{bmatrix} k_1 C^T_{M\times 1} H_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(1) \\
&+ k_2 E^T_{M\times 1} H_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(1) \\
&+ k_3 C^T_{M\times 1} M_{M\times M}^{(\gamma,a)} B_{M\times 1}^a(1)\end{bmatrix}, \\
E^T_{M\times 1} B_{M\times 1}^a(1) &= \begin{bmatrix} m_1 C^T_{M\times 1} H_{M\times M}^{(\omega,a)} B_{M\times 1}^a(1) \\
&+ m_2 E^T_{M\times 1} H_{M\times M}^{(\omega,a)} B_{M\times 1}^a(1) \\
&+ m_3 C^T_{M\times 1} M_{M\times M}^{(\omega,a)} B_{M\times 1}^a(1)\end{bmatrix},
\end{align*}
\]

(69)

where
\[ H^{(r,a)}_{M \times M} = (p^{(r,a)}_{M \times M} - Q^{(r,a)}_{M \times M}) D^{(r,a)}_{M \times M}, \quad i = 1, 2, \]
\[ H^{(s,a)}_{M \times M} = (p^{(s,a)}_{M \times M} - Q^{(s,a)}_{M \times M}) D^{(s,a)}_{M \times M}, \quad i = 1, 2, \]
\[ M^{(r,a)}_{M \times M} = f^{(r,a)}_{M \times M} - Q^{(r,a)}_{M \times M}, \]
\[ M^{(s,a)}_{M \times M} = f^{(s,a)}_{M \times M} - Q^{(s,a)}_{M \times M}, \]
\[ \vec{F}^1_{M \times 1} = k_1 F^T_{M \times 1} D^{(r,a)}_{M \times M} + k_2 H^T_{M \times 1} D^{(s,a)}_{M \times M} + k_3 F^T_{M \times 1} + k_4 M^T_{M \times M} = W^T_{M \times 1}, \]
\[ \vec{F}^2_{M \times 1} = m_1 F^T_{M \times 1} f^{(r,a)}_{M \times M} + m_2 F^T_{M \times 1} f^{(s,a)}_{M \times M} + m_3 F^T_{M \times 1} + m_4 M^T_{M \times M} = N^T_{M \times 1}, \]
\[ W^T_{M \times 1} B^a_{M \times 1}(x) = f(x), \]
\[ N^T_{M \times 1} B^a_{M \times 1}(x) = g(x). \]

By taking transpose and rearranging the terms of (69), we have

\[ \begin{bmatrix} C^T_{M \times 1} & E^T_{M \times 1} \end{bmatrix} \begin{bmatrix} B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \\ O_M & B^a_{M \times 1}(x) \end{bmatrix} = \begin{bmatrix} C^T_{M \times 1} & E^T_{M \times 1} \end{bmatrix} \begin{bmatrix} k_1 H^{(r,a)}_{M \times M} & m_2 H^{(a,s)}_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \\ O_M & m_2 H^{(a,s)}_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \end{bmatrix} \]
\[ + \begin{bmatrix} C^T_{M \times 1} & E^T_{M \times 1} \end{bmatrix} \begin{bmatrix} O_{M \times M} & m_1 H^{(s,a)}_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \\ k_2 H^{(r,a)}_{M \times M} & O_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \end{bmatrix} \]
\[ + \begin{bmatrix} C^T_{M \times 1} & E^T_{M \times 1} \end{bmatrix} \begin{bmatrix} O_{M \times M} & m_1 H^{(s,a)}_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \\ O_{M \times M} & m_2 H^{(a,s)}_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \end{bmatrix} \]
\[ + \begin{bmatrix} C^T_{M \times 1} & E^T_{M \times 1} \end{bmatrix} \begin{bmatrix} O_{M \times M} & O_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \\ k_2 H^{(r,a)}_{M \times M} & k_2 H^{(r,a)}_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \end{bmatrix} \]
\[ = \begin{bmatrix} C^T_{M \times 1} & E^T_{M \times 1} \end{bmatrix} \begin{bmatrix} k_1 H^{(r,a)}_{M \times M} + k_2 H^{(r,a)}_{M \times M} & m_2 H^{(a,s)}_{M \times M} & m_1 H^{(s,a)}_{M \times M} + m_2 H^{(a,s)}_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \\ O_{M \times M} & m_2 H^{(a,s)}_{M \times M} & B^a_{M \times 1}(x) & O_M & B^a_{M \times 1}(x) \end{bmatrix} \]
\[ \begin{bmatrix} F^1_{M \times 1} & F^2_{M \times 1} \end{bmatrix} = 0. \]

This is a system of Sylvester-type equations. Calculating the matrices \(C^T_{M \times 1}\) and \(E^T_{M \times 1}\) from (72) and substituting back in (67), one can obtain the desired solution.
5. Convergence Analysis

Here, we discuss the convergence of the proposed method developed above. For this purpose, we provide the following theorems to show that the error matrices $E_{i}^{(\nu, \alpha)}$ and $E_{D}^{(\nu, \alpha)}$ getting smaller and smaller as we increase the number of FOBPs.

**Theorem 7.** Let $V \subseteq H$ be a closed subspace of $H$. Space $H$ with finite dimension and $S = \{ v_{1}, v_{2}, v_{3}, \ldots, v_{n} \}$ form basis for $V$. Let $h \in H$ and $h$ have the unique best approximation $v'$ in $V$.

Then,

$$
\| h - v' \|_2^2 = \frac{G(h, v_{1}, v_{2}, \ldots, v_{n})}{G(v_{1}, v_{2}, v_{3}, \ldots, v_{n})},
$$

where

$$
G(h, v_{1}, v_{2}, \ldots, v_{n}) = \begin{vmatrix}
\langle h, h \rangle & \langle h, v_{1} \rangle & \cdots & \langle h, v_{n} \rangle \\
\langle v_{1}, h \rangle & \langle v_{1}, v_{1} \rangle & \cdots & \langle v_{1}, v_{n} \rangle \\
\vdots & \vdots & \cdots & \vdots \\
\langle v_{n}, h \rangle & \langle v_{n}, v_{1} \rangle & \cdots & \langle v_{n}, v_{n} \rangle 
\end{vmatrix}.
$$

(74)

**Proof.** The proof is given in [30].

**Theorem 8.** Let $D_{0}^{\alpha} g$ be a continuous real-valued function defined on $[0, 1]$, where $l = 0, 1, 2, \ldots, n$ and $Z_{n} = \text{Span} \{ B_{0}^{p}(x), B_{1}^{p}(x), \ldots, B_{n}^{p}(x) \}$. Let $g$ is approximated by $g_{n} \in Z_{n}$ as

$$
g(x) = g_{n}(x) = \sum_{i=0}^{n} c_{i} B_{i}^{p}(x) = C_{M_{1}, l}^{T} B_{M_{1}, l}^{p}(x),
$$

(75)

$$
K_{n}(g) = \int_{0}^{1} [g(x) - g_{n}(x)]^{2}dx.
$$

Then, we have

$$
\lim_{n \to \infty} K_{n}(g) = 0.
$$

(76)

**Proof.** The proof is given in [35].

**Theorem 9.** The error matrix $E_{i}^{(\nu, \alpha)}$ of OM $P^{(\nu, \alpha)}$ defined by

$$
E_{i}^{(\nu, \alpha)} = B_{0}^{p}(x) - \Gamma B_{M_{1}, l}^{p}(x), E_{i}^{(\nu, \alpha)} = \begin{bmatrix}
E_{1}^{(\nu, \alpha)} \\
\vdots \\
E_{n}^{(\nu, \alpha)}
\end{bmatrix},
$$

(77)

is bounded by the following inequality:

$$
\| E_{i}^{(\nu, \alpha)} \|_{2} \leq \sum_{r=0}^{i} \left( \frac{i}{r} \right) \frac{\Gamma (ar + 1)}{(ar + 1 + \nu)} B_{r}^{p}(x),
$$

(78)

where

$$
x^{ar+\nu} = \sum_{j=0}^{m} B_{r}^{p}(x).
$$

(79)

**Proof.** It has been proved in [35].

**Theorem 10.** The error matrix $E_{D}^{(\nu, \alpha)}$ of the OM $D^{(\nu, \alpha)}$ given by

$$
\| E_{D}^{(\nu, \alpha)} \|_{2} \leq \sum_{r=\lceil \nu \rceil}^{i} \left( \frac{i}{r} \right) \frac{\Gamma (ar + 1)}{(ar + 1 + \nu)} B_{r}^{p}(x),
$$

(80)

where

$$
E_{D}^{(\nu, \alpha)} = D_{0}^{\alpha} g_{n}(x) - D_{0}^{\alpha} g(x), E_{D}^{(\nu, \alpha)} = \begin{bmatrix}
E_{D_{1}}^{(\nu, \alpha)} \\
\vdots \\
E_{D_{n}}^{(\nu, \alpha)}
\end{bmatrix}.
$$

(81)
\[ x^{\alpha - r} = \sum_{j=0}^{m} g_r(x) B_j^u(x), \]  
\[ \| e_{p_n}^{(x,a)} \|_2 = 0, \quad 0 \leq i \leq \left[ \frac{\gamma}{a} \right] - 1. \]  

**Proof.** It has been proved in [35].

From the above theorems, we come to the conclusion that as we increase the number of FOBPs, the error matrices \( E_1^{\alpha} \) and \( E_D^{\alpha} \) approach to zero.

Furthermore, here, we deduce some results regarding stability following [38]. Let \( U(x) \in C[0,1] \) be exact solution, and \( U_M(x) \) be the approximate solution of the proposed problem, then the error bounded is computed as

\[ \delta(U) = \| U(x) - U_M(x) \| \leq \frac{B_M^2 U_M}{(1 + M + 1)^2} = \epsilon, \]  
where \( B_M, U_M \) are the maximum values of \( B_M(x), U_M(x) \) in \([0,1]\), respectively. Also, if \( M \to \infty \), then \( \epsilon \to 0 \). Furthermore, the given remark holds. \( \square \)

**Remark 11.** For \( K > 0 \), the relation

\[ \tilde{U}(B_M^u(x))^T \leq KU(x), \text{ for all } x \in [0,1], \]  
holds.

**Theorem 12.** Let \( U(x) \) be the exact solution to the problem under our consideration, and \( U_M(x) = \tilde{U}(B_M^u(x))^T \) be the approximate solution of the proposed problem, where \( \tilde{U} \) is the Bernoulli coefficient matrix that is determined by solving the algebraic equation (62), then using Remark 11, the method is stable if \( K < 1 \).

**Proof.** Consider the exact solution as \( U(x) \) and \( \tilde{U}(x) \) be any approximate solution to the proposed problem, we have using the above remark

\[ \| U(x) - \tilde{U}(x) \| = \| U(x) - U_M(x) + U_M(x) - \tilde{U}(x) \| \leq \| U(x) - U_M(x) \| + \| U_M(x) - \tilde{U}(x) \| \leq \epsilon + \| \tilde{U}(B_M^u(x))^T - \tilde{U}(x) \| \leq \epsilon + K\| U(x) - \tilde{U}(x) \|. \]

Taking maxima over \([0,1]\) of both sides and rearranging the terms, (85) implies that

\[ \| U(x) - \tilde{U}(x) \| \leq \frac{\epsilon}{1 - K}. \]

Hence, the solution is stable. Furthermore, if there exists a nondecreasing function say \( \varphi : (0, 1) \to (0, \infty) \), such that \( \varphi(x) = \epsilon \), where \( \varphi(0) = 0 \), then from relation (87), we conclude that

\[ \| U(x) - \tilde{U}(x) \| \leq \frac{\varphi(e)}{1 - K}. \]

which further yields that the solution of the proposed linear problem of fractional order is generalized stable. \( \square \)

### 6. Numerical Examples

Here, we solve some examples through the scheme we developed in Section 4 and present the concerned numerical results through graphs. Also, we compare the exact and numerical solutions of the said examples.

**Example 13.** Consider the general FOBVP [36].

\[ \mathbf{D}^\alpha U(x) = 4\mathbf{D}^\alpha U(x) + 6U(x) + f(x), \]
\[ U(0) = 1, U(1) = -8, \quad k_1, k_2 \in R, \]  
where \( 1 < \gamma \leq 2, 0 < \gamma_1 \leq 1 \).

Let us assume that the given problem has an exact solution \( u = 2 \) and \( \beta = 1 \) which is given by

\[ U(x) = \frac{x}{x^2 - 1} - \frac{1}{(1 - x)^2}. \]

Additionally, \( f(x) \) can be approximated as

\[ f(x) = 2(x - 3)(x - 2)^2(1 - 10x + 3x) + 4(1 - x)^2(3 - 2x^2). \]

In Figure 1, we compare the actual and approximate solutions at different fractional orders, and the corresponding absolute errors are also presented graphically. We see that as the fractional order approaches the corresponding integer value 2, the concerned curve tends to the corresponding integer order curve. The concerned error also reduces as the order increases. In Figure 2, we provide a graphical presentation of numerical solutions at various scale levels. Here, we see that as the value of \( M \) is increasing, the corresponding absolute error also decreases. Hence, the mentioned method is also scale-oriented.

**Example 14.** Here, we take another FOBVP given by

\[ \mathbf{D}^\alpha U(x) = 5\mathbf{D}^\alpha U(x) + 6U(x) + f(x), \]
\[ U(x)|_{x=0} = 1, U(x)|_{x=1} = 4, \]  
where \( 1 < \gamma \leq 2, 0 < \gamma_1 \leq 1 \). We solve this problem for various fractional orders. Let us assume that the given problem has an exact solution at \( \gamma = 2 \) and \( \gamma_1 = 1 \) which is given by

\[ U(x) = x^6 - x^5 + x^4 + x^3 + x + 1, \]

with the source term given by

\[ f(x) = -6x^6 - 24x^5 + 49x^4 - 46x^3 - 3x^2 - 11. \]

In Figure 3, we compare the exact and approximate solutions at different fractional orders, and the corresponding absolute errors are also presented graphically. We see that as the fractional order approaches the corresponding integer value 2, the concerned curve tends to the corresponding integer order curve. The concerned error also reduces as the order increases. In Figure 4, we provide
a graphical presentation of numerical solutions at various scale levels. We provide a graphical presentation of numerical solutions at various scale levels. Here, we see that as the value of $M$ is increasing, the corresponding absolute error also decreases. Hence, the mentioned method is also scale-oriented.

Example 15. Consider the system of FOBVPs [36]

\[ D^{\alpha}U(x) = 2D^{\omega_3}U(x) + 2D^{\omega_2}V(x) + 3U(x) + 5V(x) + f(x), \]

\[ D^{\omega}V(x) = 2D^{\omega_3}U(x) + 2D^{\omega_2}V(x) + 4U(x) + 5V(x) + g(x), \]

with BCs given by

\[ U(x)|_{x=0} = 100, U(x)|_{x=1} = 100, \]
\[ V(x)|_{x=0} = 50, V(x)|_{x=1} = 50. \]

We solve this problem under the set of parameters

\[ S_3 = \{ \nu = \omega = 2, \nu_1 = \omega_1 = 1, \nu_2 = \omega_2 = 1 \}. \]

Let us assume that this problem has the exact solution given below under the given parameters
Then, \( f(x) \) and \( g(x) \) are given by

\[
U(x) = \frac{x^2}{(1-x)^2} + 100, \\
V(x) = \frac{x}{(1-x)^3} + 50.
\]

Then, \( f(x) \) and \( g(x) \) are given by

\[
f(x) = \frac{4}{(2x-2)^2} \frac{x^3}{(1-x)^2} x + \frac{2}{(x-1)^2} x + \frac{5}{(x-1)^3} x \left( \frac{2}{(x-1)^3} x + \frac{2}{(x-1)^2} + 2(x-1)^3 - 2x^2(x-2) - \frac{3}{(x-1)^2} x^2 + 2x^2 - 550, \\
g(x) = \frac{2}{(x-1)^2} x - 3x(2x-2) + \frac{5}{(x-1)^3} x - \frac{6}{(x-1)^2} + \frac{2}{(2x-2)^3} x^2 - \frac{4}{(x-1)^2} x^2 - 650.
\]
Here, we give a plot at $M = 6$ and different values of fractional order $\omega$. In Figure 5, we compare the actual and approximate solutions at different fractional orders, and the corresponding absolute errors are also presented graphically. We see that as the fractional order approaches the corresponding integer value 2, the concerned curves tend to the corresponding integer order curves. The concerned error also reduces as the order increases. In Figures 6 and 7, we provide plots of approximate solutions at various scale levels. Also, we provide a graphical presentation of numerical solutions at various scale levels. Here, we see that as the value of $M$ is increasing, the corresponding absolute error also decreases. Hence, the mentioned method is also scale-oriented. In Figure 7, we provide plots of absolute errors at different scale levels. Here, we compare the numerical results for Example 15 with those given in [36] in Table 1. We see that the existing method produces slightly good results than the numerical results obtained in [36] for the given problem by using the shifted Legendre polynomials spectral method. In addition, here in Table 2, we compare the CPU time of our proposed method with that of [36].

**Example 16.** Consider the system of FOBVPs

\[
\begin{align*}
D^\nu Y(x) &= D^\nu Z(x) + 3Y(x) + 2Z(x) + h(x), \\
D^\nu Z(x) &= 4D^\nu Y(x) + 3D^\nu Z(x) + 2Y(x) + Z(x) + k(x),
\end{align*}
\]

with BCs given by

![Graphical presentation of approximate solutions at different values of $\nu$, $\omega$, and $\nu_i = 0.8$, $\omega_i = 0.8$, $i = 1, 2$ for $M = 6$.](image1.png)

![Graphical presentation of approximate solutions at $\nu = 2$ and $\nu_i = 1$ and different values of $M$.](image2.png)
Let us assume that this problem has the following exact solution under the set of parameters.

\[ S_4 = \{ y = 1.8, y_1 = 0.6, y_2 = 0.5, \eta = 1.6, \eta_1 = 0.6, \eta_2 = 0.5 \} , \]

\[ Y(x) = x^6 - x^5 + 2, \]

\[ Z(x) = x^5 - x^4 + 1. \]

(100)

Let us assume that this problem has the following exact solution under the set of parameters.
Figure 8: Actual and numerical solutions comparison at $M=6$.

Figure 9: Absolute error at both solutions of the given system at $M=6$.

Figure 10: Graphical presentation of numerical solutions at various scale levels of $M$. 
Then, \( h(x) \) and \( k(x) \) are given by

\[
\begin{align*}
    h(x) &= \frac{2229536516744740625x^{16/5}(10x - 7)}{1008806516530991104} - \frac{508176796463981x^{7/2}(10x - 9)}{22166154415964160} \\
    &\quad - \frac{198274579688234375x^{22/5}(10x - 9)}{6632113401256476672} - 2x^4 + 5x^5 - 3x^6 - 4, \\
    k(x) &= \frac{89181460669789625x^{11/5}(25x - 16)}{144115188075855872} - \frac{198274579688234375x^{22/5}(10x - 9)}{1658028350314119168} \\
    &\quad + x^4 + x^5 - 2x^6 - 5. \\
\end{align*}
\]

(102)

Here, we compare the actual and numerical solutions and the corresponding absolute errors in Figures 8 and 9, respectively. We compare the actual and approximate solutions at different fractional orders, and the corresponding absolute errors are also presented graphically. We see that as the fractional orders approach the corresponding integer value 2, the concerned curves tend to the corresponding integer order curves. The concerned error also reduces as the order increases. Furthermore, we provide a graphical presentation of approximate solutions at various scale levels and the corresponding absolute errors in Figures 10 and 11, respectively. In addition, we provide a graphical presentation of numerical solutions at various scale levels. Here, we see that as the value of \( M \) is increasing, the corresponding absolute error also decreases. Hence, the mentioned method is also scale-oriented.

7. Conclusion and Discussion

In this article, we have investigated different classes of FODEs under boundary conditions for numerical solutions. We have used FOBPs along with some important properties to construct some operational matrices corresponding to fractional-order derivative and integration. Based on these matrices, we have converted the proposed problems to a system of Sylvester-type operational matrices. Upon using MATLAB, we solved several examples to demonstrate the procedure. From the numerical investigation, we see that the method is powerful and can be used to investigate various FODEs for numerical solutions. The efficiency of the method can be further improved by enlarging the scale level. The greater the scale level, the greater the accuracy, and vice versa. Also, we have computed the maximum absolute error by using different scale levels. In some examples, we have also investigated the problem by using various fractional orders to simulate the results. From the numerical results, we see that the operational method based on FOBPs can also be used as a powerful spectral method to handle FODEs for numerical purposes. We have compared our numerical results with those given in [36] by using the shifted Legendre polynomials spectral method. Our results are good than those given in the mentioned reference. The CPU time of the proposed method has been compared with the CPU time of
the shifted Legendre spectral method. The CPU time of the proposed method is better than the spectral numerical method based on Legendre polynomials. In the future, we will extend the mentioned OMs method to variable order problems. Also, we will study FODEs involving nonsingular type derivatives by using our proposed method.

Data Availability
The data used to support the findings of this study are included within the article.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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