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Research Article

Study of Fractional-Order Boundary Value Problems Using Nondiscretization Numerical Method

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This paper is devoted to present a numerical scheme based on operational matrices to compute approximate solutions to fractional-order boundary value problems. For the mentioned operational matrices, we utilize fractional-order Bernoulli polynomials. Since fractional-order problems are usually difficult to treat for their corresponding analytical or exact solutions, therefore, we need sophisticated methods to find their best numerical solutions. The presented numerical scheme has the ability to reduce the proposed problem to the corresponding algebraic equations. The obtained algebraic equations are then solved by using the computational software MATLAB for the corresponding numerical results. The used method has the ability to save much more time and also is reliable to secure the proper amount of memory. Several examples are solved by using the considered method. Also, the solutions are compared with their exact solution graphically. In addition, the absolute errors for different scale values are presented graphically. In addition, we compare our results with the results of the shifted Legendre polynomials spectral method.

1. Introduction

Newton and Leibnitz in the seventeenth century established the theory of calculus. The notations of derivatives and integrals we use today have been introduced by Leibnitz. Later on, the concept of derivative and integral was extended from integer to any real order. First time in 1819, Lacroix gave the concept of noninteger order derivative [1]. The very first application was investigated by Abel in 1823 [1]. Therefore, Fourier, Liouville, Riemann, Grünwald, and Letnikov [2, 3] gave tremendous attention to the said area. Arbitrary order differentiation and integration provide a generalization to the classical order but fractional-order differentiation and integration do have not unique definitions. Therefore, various researchers have introduced various definitions of arbitrary order derivatives and integrations. Among all these, the definitions of Caputo and Riemann–Liouville are extensively applicable.

The researchers have investigated that almost every model of physics and biology consists of arbitrary order derivatives. Fractional-order differential equations (FODEs) are widely used in various fields of science and technology [4–6] and [7]. Furthermore, various properties of numerous materials such as hereditary properties are described by FODEs. Therefore, the significance of FODEs has appealed to researchers for further development in the theory. There are many areas in the theory of FODEs, one of them being the study of the numerical approach of fractional-order initial value problems (FOIVPs) and fractional-order boundary value problems (FOBVPs). It has been observed that the initial and boundary value problems with ordinary derivatives have been greatly worked on by the researchers but FOIVPs and FOBVPs are in their beginning stages and need proper further exploration. Here, we remark that the aforementioned area has significant applications in various other areas. For instance, the author of [8] studied nonlocal kinetics using fractional calculus applications. In addition, researchers [9] investigated a blood therapy model using the concept of new fractional differential operators very recently. The mentioned operators have been used to evaluate the diffusion process in [10]. Recently, two different dynamical systems of prey-predator and infectious diseases were studied using FODEs in [11].

The analytical solutions of most of the FODEs cannot be found because of the complexity of fractional order. Therefore, approximate analytical or numerical solutions are obtained for the said area. For this purpose, various analytical and numerical methods have emerged, for instance, eigenvector procedure [12], perturbation tools [13], iteration techniques [14], and transform methods [15]. Authors studied numerically discrete-time prey-predator model and SIR-type model by using concepts of fractional calculus in [16] and [17]. Authors [18] used the differential transform method to study some systems of FODEs. Authors used decomposition schemes to compute approximate solutions of some problems in [19]. The finite difference method has been applied in [20]. Also, the Tau method has been applied in [21] and [22]. Collocation techniques have also been utilized in [23]. Wavelet analysis [24] has been used for numerical results increasingly. Recently, authors studied the time-fractional model of generalized Couette flow of couple stress nanofluid with heat and mass transfer in [25]. Authors [26] studied fractional-order Caudrey–Dodd–Gibbon equations by using an integral transform. In addition, researchers [27] computed the approximate solution to wave-like equations. Additionally, authors [28] computed an approximate solution of the Noyes-field model for the time-fractional Belousov–Zhabotinsky reaction. In the same way, authors [29] have used the transform method to study fractional-order Swift–Hohenberg equations.

Here, we remark that spectral methods based on the operational matrices (OMs) for various orthogonal polynomials such as Jacobi, Legendre, and Laguerre have been developed for solving FODEs numerically [30]. The said OMs have been constructed from the mentioned polynomials by the techniques of discretization which occupies extra memory and consumes much time. Furthermore, the aforesaid methods have been applied to FOIVPs in a variety of cases. However, FOBVPs are very rarely investigated. To deal with boundary value problems, some extra operations are needed. For this purpose, we establish a numerical algorithm based on OMs for Bernoulli functions which are not orthogonal. We construct the said OMs through Bernoulli functions without discretization to save memory and time. By using these OMs, we establish a numerical algorithm to find the numerical solution for the considered problems. A new operational matrix corresponding to boundary conditions has also been obtained.

Inspired by the above discussion, we consider the given class of FOBVPs [31] under the fractional order described by $1 < \nu \leq 2,\ 0 < \nu_1 \leq 1$ as

$$\begin{cases} \mathbf{D}^{\nu} U(x) = k_1 \mathbf{D}^{\nu_1} U(x) + k_2 U(x) + f(x), k_1, k_2 \in R, x \in [0,1], \\ U(x)|_{x=0} = u_0, U(x)|_{x=1} = u_1, u_0, u_1 \in R, \end{cases} \tag{1}$$

where k_1, k_2 are real constants and $f \in C(\mathcal{J}, R)$ is a source function with $\mathcal{J} = [0,1]$.

In addition, we also extend the aforementioned scheme for the coupled system of FOBVPs as described by

$$\begin{cases} \mathbf{D}^{\nu} U(x) = k_1 \mathbf{D}^{\nu_1} U(x) + k_2 \mathbf{D}^{\nu_2} V(x) + k_3 U(x) + k_4 V(x) + f(x), x \in [0,1], \\ \mathbf{D}^{\omega} V(x) = m_1 \mathbf{D}^{\omega_1} U(x) + m_2 \mathbf{D}^{\omega_2} V(x) + m_3 U(x) + m_4 V(x) + g(x), x \in [0,1], \end{cases} \tag{2}$$

with BCs given by

$$\begin{aligned} U(x)|_{x=0} &= u_0, U(x)|_{x=1} = u_1, \\ V(x)|_{x=0} &= v_0, V(x)|_{x=1} = v_1. \end{aligned} \tag{3}$$

where the fractional orders are described by $1 < \nu, \omega \leq 2, 0 < \nu_1, \nu_2, \omega_1, \omega_2 \leq 1$. Furthermore, $f, g\colon [0,1] \longrightarrow R$ are linear continuous source functions, and $u_0, u_1, v_0, v_1 \in R$, $k_i, m_i\ (i = 1, 2)$ are any real constants. Here, we first establish the mentioned operational matrices by using Bernoulli polynomials. By using these OMs, we convert the considered system of FODEs to Sylvester-type equations given by

$$\begin{aligned} A_1 X + X B_1 &= C_1, \\ A_2 X + X B_2 &= C_2, \end{aligned} \tag{4}$$

where the notions $A_1, A_2, B_1, B_2, C_1, C_2$ represent matrices and X is an unknown matrix to be computed. We solve the obtained equation (4) for unknown matrix X by using computational software like MATLAB.

Here, we demonstrate some novelty of our work. Boundary value problems (BVPs) constitute a very important branch of applied analysis because many engineering and physical problems to understand real-world procedures or phenomena are devoted to BVPs. The concerned branch when studied under the fractional-order derivatives and integrals sense further enhances this field of study, as many real-world processes involve short or long nature memory effects which cannot be detected through applications of ordinary derivatives. Hence, with the help of fractional-order derivatives, we can explain long- and short-memory effects more clearly for best use in the area of engineering and physical sciences. Furthermore, finding the exact solutions to fractional-order boundary value problems is a very tedious job and cannot be achieved in a simple way. Therefore, various numerical techniques have been developed to treat such problems. Among the available numerical methods, spectral numerical schemes based on OMs are powerful techniques. However, the operational matrices when established by using the discretization technique utilize more memory and time-consuming process. Therefore, omitting the discretization and obtaining the required matrices are our goal. In this work, we have utilized Bernoulli polynomials a powerful tool to establish operational matrices of fractional-order integration and differentiation. The concerned matrices convert the considered FODE to an algebraic equation of Sylvester-type matrix equation. The obtained matrix equation is then solved by the use of the Gauss elimination procedure using computational software like MATLAB or Mathematica. In this way, we obtain the numerical scheme which needs no discretization as earlier used in research. Furthermore, the proposed polynomials have the ability to produce more accurate results than wavelet and other different methods. Here, it is interesting that in the last few years, spectral methods have been used very well for different problems of FODEs. For some more frequent works, we refer to [32–36] and [37]. Here, researchers have used optimal control and Bernstein and Legendre polynomials to study various problems of FODEs. Result devoted to stability has also been deduced by following the methodology of [38].

The rest of the work of this article is organized as follows. In Section 2, we provide some necessary definitions from the fundamental theory of calculus and FOBPs required in the subsequent development. The function approximation procedure and the OMs of fractional integration and differentiation are given in Section 3. Also, in the same Section 3, we give new OM corresponding to BCs. Section 4 is devoted to the establishment of numerical algorithms for the solutions of FOBVPS and coupled system FOBVPS. Part 5 is related to investigating the convergency of the proposed method. In Section 6, the proposed method is applied to various examples, and the concerned numerical results are presented through graphs. Also, the exact and numerical solutions are compared in the same Section 6. The last Section 7 is devoted to the conclusion of this article 7.

2. Preliminaries

Here, we give some important definitions from the basic fractional calculus theory and Bernoulli polynomials necessary for the subsequent development.

Definition 1. The Riemann–Liouville's arbitrary order integral operator is defined by [32].

$$\mathbf{I}^{\nu} y(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} y(s) \mathrm{d}s, \quad t > 0. \tag{5}$$

For $k_1, k_2 \in R$ and any arbitrary order ν, ν_1, ν_2 and $\alpha > -1$, the following properties are satisfied [32, 33]:

(i) $\mathbf{I}^{\nu}(k_1 y_1(t) + k_2 y_2(t)) = k_1 \mathbf{I}^{\nu} y_1(t) + k_2 \mathbf{I}^{\nu} y_2(t)$

(ii) $\mathbf{I}^{\nu_1}\mathbf{I}^{\nu_2} = \mathbf{I}^{\nu_1+\nu_2} y(t)$

(iii) $\mathbf{I}^{\nu_1}\mathbf{I}^{\nu_2} y(t) = \mathbf{I}^{\nu_2}\mathbf{I}^{\nu_1} y(t)$

(iv) $\mathbf{I}^{\nu} t^{\alpha} = \Gamma(\alpha+1)/\Gamma(\alpha+1+\nu) t^{\alpha+\nu}$

Definition 2. The Caputo's arbitrary order derivative is defined by [32]

$$\mathbf{D}^{\nu} y(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\nu-n+1}} \mathrm{d}s, \; n-1 < \nu \leq n, t > 0,$$

where $n = \lceil \nu \rceil + 1$.

(6)

The given properties are satisfied by Caputo's arbitrary order derivative (details can be seen in [32, 33]).

(i) $\mathbf{D}^{\nu}\mathbf{I}^{\nu} y(t) = y(t)$

(ii) $\mathbf{I}^{\nu}\mathbf{D}^{\nu} y(t) = y(t) - \sum_{i=0}^{m-1} y^{(i)}(0) t^i / i!$

(iii) $\mathbf{D}^{\nu} t^{\alpha} = \begin{cases} 0, & \alpha < \nu, \\ \Gamma(\alpha+1)/\Gamma(\alpha+1-\nu) t^{\alpha-\nu}, & \text{elsewhere} \end{cases}$

(iv) $\mathbf{D}^{\nu} c = 0$

(v) $\mathbf{D}^{\nu}(k_1 y_1(t) + k_2 y_2(t)) = k_1 \mathbf{D}^{\nu} y_1(t) + k_2 \mathbf{D}^{\nu} y_2(t)$

where c, k_1, k_2 are constants.

2.1. The Fractional-Order Bernoulli Polynomials. In this section, we define FOBPs and provide some of their properties.

The FOBPs $B_n^{\alpha}(x)$ are defined on [0, 1] in [35] as

$$B_n^{\alpha}(x) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j}^{\alpha} x^{j\alpha}, \quad x \in [0,1], \tag{7}$$

where $B_j^{\alpha} = B_j^{\alpha}(0) = B_j, \quad j = 0, 1, 2, \ldots, n$ are Bernoulli constants. Thus, the first four FOBPs are

$$\begin{aligned} B_0^{\alpha}(x) &= 1, \\ B_1^{\alpha}(x) &= x^{\alpha} - \frac{1}{2}, \\ B_2^{\alpha}(x) &= x^{2\alpha} - x^{\alpha} + \frac{1}{6}, \\ B_3^{\alpha}(x) &= x^{3\alpha} - \frac{3}{2} x^{2\alpha} + \frac{1}{2} x^{\alpha}. \end{aligned} \tag{8}$$

The fractional-order Bernoulli polynomials of order $n\alpha$ in the determinant form are defined by

$$B_0^\alpha(x) = 1,$$

$$B_n^\alpha(x) = \frac{(-1)^n}{(n-1)!}\begin{vmatrix} 1 & x^\alpha & x^{2\alpha} & x^{3\alpha} & \cdots & x^{(n-1)\alpha} & x^{n\alpha} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 2 & 3 & \cdots & n-1 & n \\ 0 & 0 & 0 & \binom{3}{2} & \cdots & \binom{n-1}{2} & \binom{n}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-1}{n-2} & \binom{n}{n-2} \end{vmatrix}, \quad (9)$$

for each $n = 1, 2, 3, \ldots$.

According to [30], FOBPs constitute a complete basis on $[0, 1]$. Also, FOBPs have the following property [35]:

$$\int_0^1 B_n^\alpha(x)B_m^\alpha(x)x^{\alpha-1}\mathrm{d}x = \frac{1}{\alpha}(-1)^{n-1}\frac{m!n!}{(m+n)!}B_{m+n}, \quad m, n \geq 1. \quad (10)$$

3. Function Approximation

In this section, we give the procedure for how to approximate a function in terms of FOBPs. Here, we define $I = \{y\colon 0 \leq y \leq 1\}$ and $\mathbf{L}^2(I) = \{g\colon I \longrightarrow Rg \text{ is measurable, and } \|g\|_2 < \infty\}$, where

$$\|g\|_2 = \left(\int_0^1 |g(x)|^2\mathrm{d}x\right)^{1/2}. \quad (11)$$

Now, suppose that

$$\{B_0^\alpha(x), B_1^\alpha(x), B_2^\alpha(x), \cdots, B_m^\alpha(x)\} \subset L^2(I), \quad m \in W, \quad (12)$$

is the set of FOBPs and

$$Z_m = \mathrm{Span}\{B_0^\alpha(x), B_1^\alpha(x), B_2^\alpha(x), \cdots, B_m^\alpha(x)\}. \quad (13)$$

Here, Z_m is finite-dimensional and closed vector space. So, every element g of $\mathbf{L}^2(I)$ has a unique best approximation g_0 out of Z_m, such that

$$\forall z \in Z_m, \|g - g_0\| \leq \|g - z\|, \quad (14)$$

which can be written as

$$<g - g_0, z> = 0, \quad \forall z \in Z_m, \quad (15)$$

where $\langle , \rangle$ stands for inner product. Now, as $g_0 \in Z_m$, the unique coefficients $c_0, c_1, c_2, \ldots, c_m$ are exist such that

$$f(x) \simeq f_0(x) = \sum_{i=0}^m c_i B_i^\alpha(x) = C_{M\times1}^T B_{M\times1}^\alpha(x), \quad (16)$$

where $M = m + 1$, $C_{M\times1}$ is the coefficients matrix and $B_{M\times1}^\alpha(x)$ is fractional-order Bernoulli functions vector given by

$$C_{M\times1} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix}, \quad B_{M\times1}^\alpha(x) = \begin{bmatrix} B_0^\alpha(x) \\ B_1^\alpha(x) \\ \vdots \\ B_m^\alpha(x) \end{bmatrix}. \quad (17)$$

For evaluating $C_{M\times1}$, consider

$$f_j = <f, B_j^\alpha> = \int_0^1 f(x)B_j^\alpha(x)x^{\alpha-1}\mathrm{d}x. \quad (18)$$

Using (11), we get

$$\begin{aligned} f_j &= \sum_{i=0}^{m} c_i \int_0^1 B_i^{\alpha}(x) B_j^{\alpha}(x) x^{\alpha-1} \mathrm{d}x \\ &= \sum_{i=0}^{m} c_i d_{ij}^{\alpha}, \quad j = 0, 1, 2, \ldots, m, \end{aligned} \tag{19}$$

where

$$d_{ij}^{\alpha} = \int_0^1 B_i^{\alpha}(x) B_j^{\alpha}(x) x^{\alpha-1} \mathrm{d}x, \quad i = j = 0, 1, 2, \ldots, m. \tag{20}$$

This implies that

$$f_j = C_{M\times 1}^{T} \left[d_{0j}^{\alpha}, d_{1j}^{\alpha}, d_{2j}^{\alpha}, \ldots, d_{mj}^{\alpha}\right]^{T}, \quad j = 0, 1, 2, \ldots, m, \tag{21}$$

which gives

$$F_{M\times 1}^{T} = C_{M\times 1}^{T} D_{M\times M}^{\alpha}, \tag{22}$$

where

$$\begin{aligned} F_{M\times 1}^{T} &= \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_m \end{bmatrix}, \\ D_{M\times M}^{\alpha} = \left[d_{ij}^{\alpha}\right] &= \begin{bmatrix} d_{00}^{\alpha} & d_{01}^{\alpha} & \cdots & d_{0m}^{\alpha} \\ d_{10}^{\alpha} & d_{11}^{\alpha} & \cdots & d_{1m}^{\alpha} \\ \vdots & \vdots & \cdots & \vdots \\ d_{m0}^{\alpha} & d_{m1}^{\alpha} & \cdots & d_{mm}^{\alpha} \end{bmatrix}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} d_{ij}^{\alpha} &= \int_0^1 B_i^{\alpha}(x) B_j^{\alpha}(x) x^{\alpha-1} \mathrm{d}x \\ &= \frac{1}{\alpha}(-1)^{i-1} \frac{j!i!}{(j+i)!} B_{j+i}, \quad i, j \geq 1. \end{aligned} \tag{24}$$

3.1. Derivation of OMs. Here, in this section, we derive Bernoulli-type OMs of fractional-order integration and differentiation. Moreover, we construct a new OM corresponding to some boundary value.

Theorem 3. *Let $B_{M\times 1}^{\alpha}(x)$ be the fractional-order Bernoulli functions vector, then*

$$\mathbf{I}^{\nu} B_{M\times 1}^{\alpha}(x) = P_{M\times M}^{(\nu,\alpha)} B_{M\times 1}^{\alpha}(x), \tag{25}$$

where $P_{M\times M}^{(\nu,\alpha)}$ is given by

$$P_{M\times M}^{(\nu,\alpha)} = \begin{bmatrix} \omega_{0,0,0}^{(\nu,\alpha)} & \omega_{0,1,0}^{(\nu,\alpha)} & \cdots & \omega_{0,m,0}^{(\nu,\alpha)} \\ \sum_{r=0}^{1} \omega_{1,0,r}^{(\nu,\alpha)} & \sum_{r=0}^{1} \omega_{1,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=0}^{1} \omega_{1,m,r}^{(\nu,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=0}^{m} \omega_{m,0,r}^{(\nu,\alpha)} & \sum_{r=0}^{m} \omega_{m,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=0}^{m} \omega_{m,m,r}^{(\nu,\alpha)} \end{bmatrix}. \tag{26}$$

Proof. Consider

$$\mathbf{I}^{\nu} B_i^{\alpha}(x) = I^{\nu}\left(\sum_{r=0}^{i} \binom{i}{r} B_{i-r}^{\alpha} x^{\alpha r}\right). \tag{27}$$

Using the properties of the Riemann–Liouville fractional integral operator, we have

$$\begin{aligned} \mathbf{I}^{\nu} B_i^{\alpha}(x) &= \sum_{r=0}^{i} \binom{i}{r} B_{i-r}^{\alpha} \mathbf{I}^{\nu} x^{\alpha r} \\ &= \sum_{r=0}^{i} \binom{i}{r} B_{i-r}^{\alpha} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 + \nu)} x^{\alpha r + \nu} \\ &= \sum_{r=0}^{i} \mu_{i,r}^{(\nu,\alpha)} x^{\alpha r + \nu}, \quad i = 0, 1, 2, \ldots, m, \end{aligned} \tag{28}$$

where

$$\mu_{i,r}^{(\nu,\alpha)} = \binom{i}{r} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 + \nu)} B_{i-r}^{\alpha}. \tag{29}$$

Now, we approximate $x^{\alpha r + \nu}$ as

$$x^{\alpha r + \nu} \simeq \sum_{j=0}^{m} \theta_{r,j}^{(\nu,\alpha)} B_j^{\alpha}(x). \tag{30}$$

Using (30) in (27), we get

$$\mathbf{I}^{\nu} B_i^{\alpha}(x) \simeq \sum_{r=0}^{i} \mu_{i,r}^{(\nu,\alpha)} \sum_{j=0}^{m} \theta_{r,j}^{(\nu,\alpha)} B_j^{\alpha}(x) = \sum_{j=0}^{m}\left(\sum_{r=0}^{i} \omega_{i,j,r}^{(\nu,\alpha)}\right) B_j^{\alpha}(x), \tag{31}$$

where

$$\omega_{i,j,r}^{(\nu,\alpha)} = \mu_{i,r}^{(\nu,\alpha)} \theta_{r,j}^{(\nu,\alpha)}. \tag{32}$$

(31) implies that

$$\mathbf{I}^{\nu} B_i^{\alpha}(x) \simeq \left[\sum_{r=0}^{i} \omega_{i,0,r}^{(\nu,\alpha)}, \sum_{r=0}^{i} \omega_{i,1,r}^{(\nu,\alpha)}, \ldots, \sum_{r=0}^{i} \omega_{i,m,r}^{(\nu,\alpha)}\right] B_{M\times 1}^{\alpha}(x), \quad i = 0, 1, 2, \ldots, m. \tag{33}$$

Hence, we have

$$P_{M\times M}^{(\nu,\alpha)} = \begin{bmatrix} \omega_{0,0,0}^{(\nu,\alpha)} & \omega_{0,1,0}^{(\nu,\alpha)} & \cdots & \omega_{0,m,0}^{(\nu,\alpha)} \\ \sum_{r=0}^{1}\omega_{1,0,r}^{(\nu,\alpha)} & \sum_{r=0}^{1}\omega_{1,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=0}^{1}\omega_{1,m,r}^{(\nu,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=0}^{m}\omega_{m,0,r}^{(\nu,\alpha)} & \sum_{r=0}^{m}\omega_{m,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=0}^{m}\omega_{m,m,r}^{(\nu,\alpha)} \end{bmatrix}. \tag{34}$$

□

Lemma 4. *Let $B_i^\alpha(x)$ be the FOBP, then*

$$\mathbf{D}^\omega B_i^\alpha(x) = 0, \quad i = 0, 1, \ldots, \left\lceil \frac{\omega}{\alpha} \right\rceil - 1, \omega > 0. \tag{35}$$

Proof. By applying the properties (iii – v) of Caputo arbitrary order differential operator in (11), one can easily prove the lemma. □

Theorem 5. *Let $B_{M\times 1}^\alpha(x)$ be the FOBPs vector, then*

$$\mathbf{D}^\nu B_{M\times 1}^\alpha(x) = D_{M\times M}^{(\nu,\alpha)} B_{M\times 1}^\alpha(x), \tag{36}$$

where $D_{M\times M}^{(\nu,\alpha)}$ is given by

$$D_{M\times M}^{(\nu,\alpha)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{r=\lceil \nu/\alpha \rceil}^{\lceil \nu/\alpha \rceil} \Omega_{\lceil \nu/\alpha \rceil,0,r}^{(\nu,\alpha)} & \sum_{r=\lceil \nu/\alpha \rceil}^{\lceil \nu/\alpha \rceil} \Omega_{\lceil \nu/\alpha \rceil,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=\lceil \nu/\alpha \rceil}^{\lceil \nu/\alpha \rceil} \Omega_{\lceil \nu/\alpha \rceil,m,r}^{(\nu,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,0,r}^{(\nu,\alpha)} & \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,m,r}^{(\nu,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=\lceil \nu/\alpha \rceil}^{m} \Omega_{m,0,r}^{(\nu,\alpha)} & \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{m,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{m,m,r}^{(\nu,\alpha)} \end{bmatrix}. \tag{37}$$

Proof. Consider

$$\mathbf{D}^\nu B_i^\alpha(x) = D^\nu\left(\sum_{r=0}^{i} \binom{i}{r} B_{i-r}^\alpha x^{\alpha r}\right). \tag{38}$$

By applying the properties of Caputo arbitrary order differential operator, we have

$$\begin{aligned} \mathbf{D}^\nu B_i^\alpha(x) &= \sum_{r=0}^{i} \binom{i}{r} B_{i-r}^\alpha \mathbf{D}^\nu x^{\alpha r} \\ &= \sum_{r=\lceil \nu/\alpha \rceil}^{i} \binom{i}{r} B_{i-r}^\alpha \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - \nu)} x^{\alpha r - \nu} \\ &= \sum_{r=0}^{i} \eta_{i,r}^{(\nu,\alpha)} x^{\alpha r - \nu}, \quad i = \left\lceil \frac{\nu}{\alpha} \right\rceil, \ldots, m, \end{aligned} \tag{39}$$

where

$$\eta_{i,r}^{(\nu,\alpha)} = \binom{i}{r} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - \nu)} B_{i-r}^\alpha. \tag{40}$$

Now, we approximate $x^{\alpha r - \nu}$ by

$$x^{\alpha r - \nu} \simeq \sum_{j=0}^{m} \phi_{r,j}^{(\nu,\alpha)} B_j^\alpha(x). \tag{41}$$

Using (41) in (38), we have

$$\mathbf{D}^\nu B_i^\alpha(x) \simeq \sum_{r=\lceil \nu/\alpha \rceil}^{i} \eta_{i,r}^{(\nu,\alpha)} \sum_{j=0}^{m} \phi_{r,j}^{(\nu,\alpha)} B_j^\alpha(x) = \sum_{j=0}^{m}\left(\sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,j,r}^{(\nu,\alpha)}\right) B_j^\alpha(x), \tag{42}$$

where

$$\Omega_{i,j,r}^{(\nu,\alpha)} = \eta_{i,r}^{(\nu,\alpha)} \phi_{r,j}^{(\nu,\alpha)}. \tag{43}$$

Equation (42) implies that

$$\mathbf{D}^{\nu} B_i^{\alpha}(x) \simeq \left[\sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,0,r}^{(\nu,\alpha)}, \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,1,r}^{(\nu,\alpha)}, \ldots, \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,m,r}^{(\nu,\alpha)} \right] B_{M\times 1}^{\alpha}(x), \quad i = \left\lceil \frac{\nu}{\alpha} \right\rceil, \ldots, m. \tag{44}$$

Also, using Lemma 4, we get from (44)

$$\mathbf{D}^{\nu} B_i^{\alpha}(x) = [0,0,0,\ldots,0] B_{M\times 1}^{\alpha}(x), \quad i = 0,1,\ldots, \left\lceil \frac{\nu}{\alpha} \right\rceil - 1. \tag{45}$$

Hence, from (44) and (45), we have

$$D_{M\times M}^{(\nu,\alpha)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{r=\lceil \nu/\alpha \rceil}^{\lceil \nu/\alpha \rceil} \Omega_{\lceil \nu/\alpha \rceil,0,r}^{(\nu,\alpha)} & \sum_{r=\lceil \nu/\alpha \rceil}^{\lceil \nu/\alpha \rceil} \Omega_{\lceil \nu/\alpha \rceil,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=\lceil \nu/\alpha \rceil}^{\lceil \nu/\alpha \rceil} \Omega_{\lceil \nu/\alpha \rceil,m,r}^{(\nu,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,0,r}^{(\nu,\alpha)} & \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{i,m,r}^{(\nu,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=\lceil \nu/\alpha \rceil}^{m} \Omega_{m,0,r}^{(\nu,\alpha)} & \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{m,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=\lceil \nu/\alpha \rceil}^{i} \Omega_{m,m,r}^{(\nu,\alpha)} \end{bmatrix}. \tag{46}$$

□

Theorem 6. *Let $B_{M\times 1}^{\alpha}(x)$ be a FOBPs matrix and let $\phi(x)$ be any function given $\phi(x) = lx^n, l = 0,1,2,\ldots, l \in R$ and $U(x) = C_{M\times 1}^T B_{M\times 1}^{\alpha}(x)$, then*

$$\phi(x)\, {}_0\mathbf{I}_1^{\alpha} U(x) = C_{M\times 1}^T Q_{M\times M}^{(\alpha,\phi)} B_{M\times 1}^{\alpha}(x), \tag{47}$$

where $Q_{M\times M}^{(\alpha,\phi)}$ is given by

$$Q_{M\times M}^{(\alpha,\phi)} = \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,j} & \cdots & \Phi_{0,m} \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,j} & \cdots & \Phi_{1,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{i,0} & \Phi_{i,1} & \cdots & \Phi_{i,j} & \cdots & \Phi_{i,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{m,0} & \Phi_{m,1} & \cdots & \Phi_{m,j} & \cdots & \Phi_{m,m} \end{bmatrix}, \tag{48}$$

where Φ can be calculated by using equation (38).

Proof. Consider

$$\begin{aligned} {}_0\mathbf{I}_1^{\alpha} B_i^{\nu}(x) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} B_i^{\nu}(t)\mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} \sum_{i=0}^{m} \binom{m}{i} B_{m-i}^{\nu} t^{i\nu}\mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{m} \binom{m}{i} B_{m-i}^{\nu} \int_0^1 (1-t)^{\alpha-1} t^{i\nu}\mathrm{d}t. \end{aligned} \tag{49}$$

Using the well-known property of the Beta function given by

$$\beta(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}\mathrm{d}y = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{50}$$

in equation (47), we have

$$ {}_0\mathbf{I}_1^{\alpha} B_i^{\nu}(x) = \sum_{i=0}^{m} \binom{m}{i} B_{m-i}^{\nu} \frac{\Gamma(i\nu+1)}{\Gamma(i\nu+1+\alpha)} = \Delta_{\mathbf{i}}, \tag{51} $$

which implies

$$ \phi(x)_0 I_1^{\alpha} B_i^{\alpha}(x) = \Delta_{\mathbf{i}}\phi(x), \tag{52} $$

which can be approximated in FOBPs as

$$ \Delta_{\mathbf{i}}\phi(x) = \sum_{j=0}^{m} \Phi_{i,j} B_i^{\alpha}(x). \tag{53} $$

Hence, we have the required result. □

4. Numerical Algorithms

Here, we show the fundamental importance of OMs developed in the previous section by applying them to various FOBVPs. For this purpose, we consider the following two steps.

4.1. Numerical Scheme for Scaler Problem (1). Here, we develop a numerical scheme for a single problem given in (1). To obtain the solution in terms of FOBPs, we assume that

$$ \mathbf{D}^{\nu} U(x) = C_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x). \tag{54} $$

Applying $\mathbf{I}^{\nu}$ and property (ii) for the Caputo differential operator, we get from (54)

$$ U(x) = C_{M\times 1}^{T} P_{M\times M}^{(\nu,\alpha)} B_{M\times 1}^{\alpha}(x) + c_0 + c_1 x. \tag{55} $$

Using the given boundary conditions, we have $c_0 = u_0$ and

$$ c_1 = u_1 - u_0 - C_{M\times 1}^{T} P_{M\times M}^{(\nu,\alpha)} B_{M\times 1}^{\alpha}(1). \tag{56} $$

Putting the values of c_0 and c_1 in (55), one can get

$$ U(x) = C_{M\times 1}^{T} P_{M\times M}^{(\nu,\alpha)} B_{M\times 1}^{\alpha}(1) + u_0 + (u_1 - u_0)x - x C_{M\times 1}^{T} P_{M\times M}^{(\nu,\alpha)} B_{M\times 1}^{\alpha}(1). \tag{57} $$

Using the afore-established OMs, and after simplification, we get from (57)

$$ U(x) = C_{M\times 1}^{T}\left(P_{M\times M}^{(\nu,\alpha)} - Q_{M\times M}^{(\nu,\phi)}\right) B_{M\times 1}^{\alpha}(x) + F_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x), \tag{58} $$

where $F_{M\times 1}^{T} B_{M\times 1}^{\alpha}(1) = u_0 + (u_1 - u_0)x$ and $\phi = x$. Now by applying $\mathbf{D}^{\nu_1}$ to (58), we have

$$ \mathbf{D}^{\nu_1} U(x) = C_{M\times 1}^{T}\left(P_{M\times M}^{(\nu,\alpha)} - Q_{M\times M}^{(\nu,\phi)}\right) D_{M\times M}^{(\nu_1,\alpha)} B_{M\times 1}^{\alpha}(x) + F_{M\times 1}^{T} D_{M\times M}^{(\nu_1,\alpha)} B_{M\times 1}^{\alpha}(x). \tag{59} $$

Putting (54), (58), and (59) in (1) yields

$$ \begin{aligned} C_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x) = {} & k_1 C_{M\times 1}^{T}\left(P_{M\times M}^{(\nu,\alpha)} - Q_{M\times M}^{(\nu,\phi)}\right) D_{M\times M}^{(\nu_1,\alpha)} B_{M\times 1}^{\alpha}(x) + k_1 F_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x) \\ & + k_2 C_{M\times 1}^{T}\left(P_{M\times M}^{(\nu,\alpha)} - Q_{M\times M}^{(\nu,\phi)}\right) B_{M\times 1}^{\alpha}(x) + k_2 F_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x) + G_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x), \end{aligned} \tag{60} $$

where $G_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x) = f(x)$. After simplification, we obtain

$$ \left(C_{M\times 1}^{T} - C_{M\times 1}^{T}\left(P_{M\times M}^{(\nu,\alpha)} - Q_{M\times M}^{(\nu,\phi)}\right)\left(k_1 D_{M\times M}^{(\nu_1,\alpha)} + k_2 I_{M\times M}\right) - L_{M\times M}\right) B_{M\times 1}^{\alpha}(x) = 0, \tag{61} $$

where $k_1 F_{M\times 1}^{T} D_{M\times M}^{(\nu_1,\alpha)} + k_2 F_{M\times 1}^{T} + G_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x) = L_{M\times M}$. Equation (61) implies that

$$C_{M\times 1}^{T}-C_{M\times 1}^{T}\left(P_{M\times M}^{(\nu,\alpha)}-Q_{M\times M}^{(\nu,\phi)}\right)\left(k_1 D_{M\times M}^{(\nu_1,\alpha)}+k_2 I_{M\times M}\right)-L_{M\times M}=0. \tag{62}$$

This is a Sylvester-type equation. Calculating $C_{M\times 1}$ from (23) and substituting back in (20), one can have the desired approximate solution of the given FOBVP.

4.2. Numerical Scheme for Coupled System (2) with BCs (3). Here, we develop a numerical scheme for the computation of an approximate solution for a coupled system (2) with BCs (3). In this regard, we assume that

$$\begin{aligned}\mathbf{D}^{\nu}U(x)&=C_{M\times 1}^{T}B_{M\times 1}^{\alpha}(x),\\ \mathbf{D}^{\omega}V(x)&=E_{M\times 1}^{T}B_{M\times 1}^{\alpha}(x).\end{aligned} \tag{63}$$

Applying $\mathbf{I}^{\nu},\mathbf{I}^{\omega}$ and property (ii) for the Caputo differential operator, we get

$$\begin{aligned}U(x)&=C_{M\times 1}^{T}P_{M\times M}^{(\nu,\alpha)}B_{M\times 1}^{\alpha}(x)+c_0+c_1x,\\ V(x)&=E_{M\times 1}^{T}P_{M\times M}^{(\omega,\alpha)}B_{M\times 1}^{\alpha}(x)+d_0+d_1x.\end{aligned} \tag{64}$$

Using the given BCs, we have

$$\begin{aligned}c_0&=u_0, c_1=u_1-u_0-C_{M\times 1}^{T}P_{M\times M}^{(\nu,\alpha)}B_{M\times 1}^{\alpha}(1),\\ d_0&=v_0, d_1=v_1-v_0-E_{M\times 1}^{T}P_{M\times M}^{(\omega,\alpha)}B_{M\times 1}^{\alpha}(x).\end{aligned} \tag{65}$$

Inserting (65) in (64), we obtain

$$\begin{aligned}U(x)&=C_{M\times 1}^{T}P_{M\times M}^{(\nu,\alpha)}B_{M\times 1}^{\alpha}(x)+u_0+(u_1-u_0)x-xC_{M\times 1}^{T}P_{M\times M}^{(\nu,\alpha)}B_{M\times 1}^{\alpha}(1),\\ V(x)&=E_{M\times 1}^{T}P_{M\times M}^{(\omega,\alpha)}B_{M\times 1}^{\alpha}(x)+v_0+(v_1-v_0)x-xE_{M\times 1}^{T}P_{M\times M}^{(\omega,\alpha)}B_{M\times 1}^{\alpha}(1).\end{aligned} \tag{66}$$

After simplification of (66), we get

$$\begin{aligned}U(x)&=C_{M\times 1}^{T}\left(P_{M\times M}^{(\nu,\alpha)}-Q_{M\times M}^{(\nu,\phi)}\right)B_{M\times 1}^{\alpha}(x)+F_{M\times 1}^{T}B_{M\times 1}^{\alpha}(x),\\ V(x)&=E_{M\times 1}^{T}\left(P_{M\times M}^{(\omega,\alpha)}-Q_{M\times M}^{(\omega,\phi)}\right)B_{M\times 1}^{\alpha}(x)+J_{M\times 1}^{T}B_{M\times 1}^{\alpha}(x),\end{aligned} \tag{67}$$

where $F_{M\times 1}^{T}B_{M\times 1}^{\alpha}(x)=u_0+(u_1-u_0)x, J_{M\times 1}^{T}B_{M\times 1}^{\alpha}(x)=v_0+(v_1-v_0)x$ and $\phi=x$. Now, by applying $\mathbf{D}^{\nu_1},\mathbf{D}^{\nu_2},\mathbf{D}^{\omega_1}$, and $\mathbf{D}^{\omega_2}$ to (67), we have

$$\begin{aligned}\mathbf{D}^{\nu_1}U(x)&=C_{M\times 1}^{T}\left(P_{M\times M}^{(\nu,\alpha)}-Q_{M\times M}^{(\nu,\phi)}\right)D_{M\times M}^{(\nu_1,\alpha)}B_{M\times 1}^{\alpha}(x)+F_{M\times 1}^{T}D_{M\times M}^{(\nu_1,\alpha)}B_{M\times 1}^{\alpha}(x),\\ \mathbf{D}^{\nu_2}V(x)&=E_{M\times 1}^{T}\left(P_{M\times M}^{(\omega,\alpha)}-Q_{M\times M}^{(\omega,\phi)}\right)D_{M\times M}^{(\nu_2,\alpha)}B_{M\times 1}^{\alpha}(x)+J_{M\times 1}^{T}D_{M\times M}^{(\nu_2,\alpha)}B_{M\times 1}^{\alpha}(x),\\ \mathbf{D}^{\omega_1}U(x)&=C_{M\times 1}^{T}\left(P_{M\times M}^{(\nu,\alpha)}-Q_{M\times M}^{(\nu,\phi)}\right)D_{M\times M}^{(\omega_1,\alpha)}B_{M\times 1}^{\alpha}(x)+F_{M\times 1}^{T}D_{M\times M}^{(\omega_1,\alpha)}B_{M\times 1}^{\alpha}(x),\\ \mathbf{D}^{\omega_2}V(x)&=E_{M\times 1}^{T}\left(P_{M\times M}^{(\omega,\alpha)}-Q_{M\times M}^{(\omega,\phi)}\right)D_{M\times M}^{(\omega_2,\alpha)}B_{M\times 1}^{\alpha}(x)+J_{M\times 1}^{T}D_{M\times M}^{(\omega_2,\alpha)}B_{M\times 1}^{\alpha}(x).\end{aligned} \tag{68}$$

Putting (63), (66), and (68) in coupled system (2) yields

$$\begin{aligned}\begin{bmatrix}C_{M\times 1}^{T}B_{M\times 1}^{\alpha}(x)\\ E_{M\times 1}^{T}B_{M\times 1}^{\alpha}(x)\end{bmatrix}&=\begin{bmatrix}k_1C_{M\times 1}^{T}H_{M\times M}^{(\nu_1,\alpha)}B_{M\times 1}^{\alpha}(x)\\ m_2E_{M\times 1}^{T}H_{M\times M}^{(\omega_2,\alpha)}B_{M\times 1}^{\alpha}(x)\end{bmatrix}+\begin{bmatrix}k_2E_{M\times 1}^{T}H_{M\times M}^{(\nu_2,\alpha)}B_{M\times 1}^{\alpha}(x)\\ m_1C_{M\times 1}^{T}H_{M\times M}^{(\omega_1,\alpha)}B_{M\times 1}^{\alpha}(x)\end{bmatrix}+\begin{bmatrix}k_3C_{M\times 1}^{T}M_{M\times M}^{(\nu,\alpha)}B_{M\times 1}^{\alpha}(x)\\ m_4E_{M\times 1}^{T}M_{M\times M}^{(\omega,\alpha)}B_{M\times 1}^{\alpha}(x)\end{bmatrix}\\ &\quad+\begin{bmatrix}k_4E_{M\times 1}^{T}M_{M\times M}^{(\omega,\alpha)}B_{M\times 1}^{\alpha}(x)\\ m_3C_{M\times 1}^{T}M_{M\times M}^{(\nu,\alpha)}B_{M\times 1}^{\alpha}(x)\end{bmatrix}+\begin{bmatrix}\overbrace{F_{M\times 1}^{1}}B_{M\times 1}^{\alpha}(x)\\ \overbrace{F_{M\times 1}^{2}}B_{M\times 1}^{\alpha}(x)\end{bmatrix},\end{aligned} \tag{69}$$

where

$$
\begin{aligned}
H_{M\times M}^{(\nu_i,\alpha)} &= \left(P_{M\times M}^{(\nu,\alpha)} - Q_{M\times M}^{(\nu,\phi)}\right) D_{M\times M}^{(\nu_i,\alpha)}; \quad i = 1, 2,\\
H_{M\times M}^{(\omega_i,\alpha)} &= \left(P_{M\times M}^{(\omega,\alpha)} - Q_{M\times M}^{(\omega,\phi)}\right) D_{M\times M}^{(\omega_i,\alpha)}; \quad i = 1, 2,\\
M_{M\times M}^{(\nu,\alpha)} &= P_{M\times M}^{(\nu,\alpha)} - Q_{M\times M}^{(\nu,\phi)},\\
M_{M\times M}^{(\omega,\alpha)} &= P_{M\times M}^{(\omega,\alpha)} - Q_{M\times M}^{(\omega,\phi)},\\
\overbrace{F_{M\times 1}^{1}} &= k_1 F_{M\times 1}^{T} D_{M\times M}^{(\nu_1,\alpha)} + k_2 J_{M\times 1}^{T} D_{M\times M}^{(\nu_2,\alpha)} + k_3 F_{M\times 1}^{T} + k_4 J_{M\times 1}^{T} + W_{M\times 1}^{T},\\
\overbrace{F_{M\times 1}^{2}} &= m_1 F_{M\times 1}^{T} P_{M\times M}^{(\omega_1,\alpha)} + m_2 J_{M\times 1}^{T} D_{M\times M}^{(\omega_2,\alpha)} + m_3 F_{M\times 1}^{T} + m_4 J_{M\times 1}^{T} + N_{M\times 1}^{T},\\
W_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x) &= f(x),\\
N_{M\times 1}^{T} B_{M\times 1}^{\alpha}(x) &= g(x).
\end{aligned}
\tag{70}
$$

By taking transpose and rearranging the terms of (69), we have

$$
\begin{aligned}
&\left[\, C_{M\times 1}^{T} \;\; E_{M\times 1}^{T} \,\right] \begin{bmatrix} B_{M\times 1}^{\alpha}(x) & O_M \\ O_M & B_{M\times 1}^{\alpha}(x) \end{bmatrix}\\
&= \left[\, C_{M\times 1}^{T} \;\; E_{M\times 1}^{T} \,\right] \begin{bmatrix} k_1 H_{M\times M}^{(\nu_1,\alpha)} & O_{M\times M} \\ O_{M\times M} & m_2 H_{M\times M}^{(\omega_2,\alpha)} \end{bmatrix} \begin{bmatrix} B_{M\times 1}^{\alpha}(x) & O_M \\ O_M & B_{M\times 1}^{\alpha}(x) \end{bmatrix}\\
&\quad + \left[\, C_{M\times 1}^{T} \;\; E_{M\times 1}^{T} \,\right] \begin{bmatrix} O_{M\times M} & m_1 H_{M\times M}^{(\omega_1,\alpha)} \\ k_2 H_{M\times M}^{(\nu_2,\alpha)} & O_{M\times M} \end{bmatrix} \begin{bmatrix} B_{M\times 1}^{\alpha}(x) & O_M \\ O_M & B_{M\times 1}^{\alpha}(x) \end{bmatrix}\\
&\quad + \left[\, C_{M\times 1}^{T} \;\; E_{M\times 1}^{T} \,\right] \begin{bmatrix} k_3 M_{M\times M}^{(\nu,\alpha)} & O_{M\times M} \\ O_{M\times M} & m_4 M_{M\times M}^{(\omega,\alpha)} \end{bmatrix} \begin{bmatrix} B_{M\times 1}^{\alpha}(x) & O_M \\ O_M & B_{M\times 1}^{\alpha}(x) \end{bmatrix}\\
&\quad + \left[\, C_{M\times 1}^{T} \;\; E_{M\times 1}^{T} \,\right] \begin{bmatrix} O_{M\times M} & m_3 M_{M\times M}^{(\nu,\alpha)} \\ k_4 M_{M\times M}^{(\omega,\alpha)} & O_{M\times M} \end{bmatrix} \begin{bmatrix} B_{M\times 1}^{\alpha}(x) & O_M \\ O_M & B_{M\times 1}^{\alpha}(x) \end{bmatrix}\\
&\quad + \left[\, \overbrace{F_{M\times 1}^{1}} \;\; \overbrace{F_{M\times 1}^{2}} \,\right] \begin{bmatrix} B_{M\times 1}^{\alpha}(x) & O_M \\ O_M & B_{M\times 1}^{\alpha}(x) \end{bmatrix},
\end{aligned}
\tag{71}
$$

where $O_{M\times M}$ and O_M are zero matrix and zero vector, respectively. By simplification, (71) can be written as

$$
\left[\, C_{M\times 1}^{T} \;\; E_{M\times 1}^{T} \,\right] - \left[\, C_{M\times 1}^{T} \;\; E_{M\times 1}^{T} \,\right] \begin{bmatrix} k_1 H_{M\times M}^{(\nu_1,\alpha)} + k_3 M_{M\times M}^{(\nu,\alpha)} & m_1 H_{M\times M}^{(\omega_1,\alpha)} + m_3 M_{M\times M}^{(\nu_1,\alpha)} \\ k_2 H_{M\times M}^{(\nu_2,\alpha)} + k_4 M_{M\times M}^{(\omega,\alpha)} & m_2 H_{M\times M}^{(\omega_2,\alpha)} + m_4 M_{M\times M}^{(\nu,\alpha)} \end{bmatrix} - \left[\, \overbrace{F_{M\times 1}^{1}} \;\; \overbrace{F_{M\times 1}^{2}} \,\right] = 0. \tag{72}
$$

This is a system of Sylvester-type equations. Calculating the matrices $C_{M\times 1}^{T}$ and $E_{M\times 1}^{T}$ from (72) and substituting back in (67), one can obtain the desired solution.

5. Convergence Analysis

Here, we discuss the convergence of the proposed method developed above. For this purpose, we provide the following theorems to show that the error matrices $\mathbf{E}_{\mathbf{I}}^{(\nu,\alpha)}$ and $\mathbf{E}_{\mathbf{D}}^{(\nu,\alpha)}$ getting smaller and smaller as we increase the number of FOBPs.

Theorem 7. *Let $V \subseteq H$ be a closed subspace of H.space H with finite dimension and $S = \{v_1, v_2, v_3, \ldots, v_n\}$ form basis for V. Let $h \in H$ and h have the unique best approximation v^* in V.*

Then,

$$\left\|h - v^*\right\|_2^2 = \frac{G(h, v_1, v_2, \ldots, v_n)}{G(v_1, v_2, v_3, \ldots, v_n)}, \tag{73}$$

where

$$G(h, v_1, v_2, \ldots, v_n) = \begin{vmatrix} <h, h> & <h, v_1> & \cdots & <h, v_n> \\ <v_1, h> & <v_1, v_1> & \cdots & <v_1, v_n> \\ \vdots & \vdots & \cdots & \vdots \\ <v_n, h> & <v_n, v_1> & \cdots & <v_n, v_n> \end{vmatrix}. \tag{74}$$

Proof. The proof is given in [30]. □

Theorem 8. *Let $\mathbf{D}^{l\alpha} g$ be a continuous real-valued function defined on $[0,1]$, where $l = 0, 1, 2, \ldots, n$ and $Z_n = \text{Span}\left\{B_0^\alpha(x), B_1^\alpha(x), \ldots, B_n^\alpha(x)\right\}$. Let g is approximated by $g_n \in Z_n$ as*

$$\begin{aligned} g(x) \simeq g_n(x) &= \sum_{i=0}^{n} c_i B_i^\alpha(x) = C_{M\times 1}^T B_{M\times 1}^\alpha(x), \\ K_n(g) &= \int_0^1 \left[g(x) - g_n(x)\right]^{2dx}. \end{aligned} \tag{75}$$

Then, we have

$$\lim_{n\longrightarrow\infty} K_n(g) = 0. \tag{76}$$

Proof. The proof is given in [35]. □

Theorem 9. *The error matrix $\mathbf{E}_{\mathbf{I}}^{(\nu,\alpha)}$ of OM $P^{(\nu,\alpha)}$ defined by*

$$\mathbf{E}_{\mathbf{I}}^{(\nu,\alpha)} = P^{(\nu,\alpha)} B_{M\times 1}^\alpha(x) - \mathbf{I}^\nu B_{M\times 1}^\alpha(x), \mathbf{E}_{\mathbf{I}}^{(\nu,\alpha)} = \begin{bmatrix} \mathbf{e}_{\mathbf{I}0}^{(\nu,\alpha)} \\ \mathbf{e}_{\mathbf{I}1}^{(\nu,\alpha)} \\ \vdots \\ \mathbf{e}_{\mathbf{I}n}^{(\nu,\alpha)} \end{bmatrix}, \tag{77}$$

is bounded by the following inequality:

$$\left\|\mathbf{e}_{\mathbf{I}i}^{(\nu,\alpha)}\right\|_2 \leq \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 + \nu)} B_{i-r}^\alpha \left(\frac{G\left(x^{\alpha r+\nu}, B_0^\alpha(x), B_1^\alpha(x), \ldots, B_n^\alpha(x)\right)}{G\left(B_0^\alpha(x), B_1^\alpha(x), \ldots, B_n^\alpha(x)\right)}\right)^{1/2}, \quad 0 \leq i \leq n, \tag{78}$$

where

$$x^{\alpha r+\nu} \simeq \sum_{j=0}^{m} \theta_{r,j}^{(\nu,\alpha)} B_j^\alpha(x). \tag{79}$$

Proof. It has been proved in [35]. □

Theorem 10. *The error matrix $\mathbf{E}_{\mathbf{D}}^{(\nu,\alpha)}$ of the OM $D^{(\nu,\alpha)}$ given by*

$$\mathbf{E}_{\mathbf{D}}^{(\nu,\alpha)} = D^{(\nu,\alpha)} B_{M\times 1}^\alpha(x) - \mathbf{D}^\nu B_{M\times 1}^\alpha(x), \mathbf{E}_{\mathbf{D}}^{(\nu,\alpha)} = \begin{bmatrix} \mathbf{e}_{\mathbf{D}0}^{(\nu,\alpha)} \\ \mathbf{e}_{\mathbf{D}1}^{(\nu,\alpha)} \\ \vdots \\ \mathbf{e}_{\mathbf{D}n}^{(\nu,\alpha)} \end{bmatrix}, \tag{80}$$

is bounded by the following inequality:

$$\left\|\mathbf{e}_{\mathbf{D}i}^{(\nu,\alpha)}\right\|_2 \leq \sum_{r=\lceil \nu/\alpha \rceil}^{i} \binom{i}{r} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - \nu)} B_{i-r}^\alpha \left(\frac{G\left(x^{\alpha r-\nu}, B_0^\alpha(x), B_1^\alpha(x), \ldots, B_n^\alpha(x)\right)}{G\left(B_0^\alpha(x), B_1^\alpha(x), \ldots, B_n^\alpha(x)\right)}\right)^{1/2}, \left\lceil \frac{\nu}{\alpha} \right\rceil \leq i \leq n, \tag{81}$$

where

$$x^{\alpha r-\nu}\simeq\sum_{j=0}^{m}\theta_{r,j}^{(\nu,\alpha)}B_j^{\alpha}(x),$$
$$\left\|\mathbf{e}_{\mathbf{D}i}^{(\nu,\alpha)}\right\|_2=0,\quad 0\le i\le\left\lceil\frac{\nu}{\alpha}\right\rceil-1. \tag{82}$$

Proof. It has been proved in [35].

From the above theorems, we come to the conclusion that as we increase the number of FOBPs, the error matrices $\mathbf{E}_{\mathbf{I}}^{(\nu,\alpha)}$ and $\mathbf{E}_{\mathbf{D}}^{(\nu,\alpha)}$ approach to zero.

Furthermore, here, we deduce some results regarding stability following [38]. Let $U(x)\in C[0,1]$ be exact solution, and $U_M(x)$ be the approximate solution of the proposed problem, then the error bounded is computed as

$$\mathscr{E}(U)=\left\|U(x)-U_M(x)\right\|\le\frac{B_M^2U_M}{(\Gamma(M+1))^2}=\varepsilon, \tag{83}$$

where B_M, U_M are the maximum values of $B_M^{\alpha}(x), U_M(x)$ in $[0,1]$, respectively. Also, if $M\longrightarrow\infty$, then $\varepsilon\longrightarrow 0$. Furthermore, the given remark holds. □

Remark 11. For $\mathbf{K}>0$, the relation

$$\widehat{U}\left(B_M^{\alpha}(x)\right)^T\le\mathbf{K}U(x),\text{for all }x\in[0,1], \tag{84}$$

holds.

Theorem 12. *Let $U(x)$ be the exact solution to the problem under our consideration, and $U_M(x)=\widehat{U}(B_M^{\alpha}(x))^T$ be the approximate solution of the proposed problem, where $\widehat{U}$ is the Bernoulli coefficient matrix that is determined by solving the algebraic equation (62), then using Remark 11, the method is stable if $\mathbf{K}<1$.*

Proof. Consider the exact solution as $U(x)$ and $\widetilde{U}(x)$ be any approximate solution to the proposed problem, we have using the above remark

$$\begin{aligned}|U(x)-\widetilde{U}(x)|&=\left|U(x)-U_M(x)+U_M(x)-\widetilde{U}(x)\right|\\&\le\left|U(x)-U_M(x)\right|+\left|U_M(x)-\widetilde{U}(x)\right|\\&\le\varepsilon+\left|\widehat{U}\left(B_M^{\alpha}(x)\right)^T-\widetilde{U}(x)\right|\\&\le\varepsilon+\mathbf{K}|U(x)-\widetilde{U}(x)|.\end{aligned} \tag{85}$$

Taking maxima over $[0,1]$ of both sides and rearranging the terms, (85) implies that

$$\|U(x)-\widetilde{U}(x)\|\le\frac{\varepsilon}{1-\mathbf{K}}. \tag{86}$$

Hence, the solution is stable. Furthermore, if there exists a nondecreasing function say φ: $(0,1)\longrightarrow(0,\infty)$, such that $\varphi(\varepsilon)=\varepsilon$, where $\varphi(0)=0$, then from relation (87), we conclude that

$$\|U(x)-\widetilde{U}(x)\|\le\frac{\varphi(\varepsilon)}{1-\mathbf{K}}, \tag{87}$$

which further yields that the solution of the proposed linear problem of fractional order is generalized stable. □

6. Numerical Examples

Here, we solve some examples through the scheme we developed in Section 4 and present the concerned numerical results through graphs. Also, we compare the exact and numerical solutions of the said examples.

Example 13. Consider the general FOBVP [36].

$$\begin{aligned}\mathbf{D}^{\nu}U(x)&=4\mathbf{D}_1^{\nu}U(x)+8U(x)+f(x),\\U(0)&=1,U(1)=-8,\quad k_1,k_2\in R,\end{aligned} \tag{88}$$

where $1<\nu\le 2$, $0<\nu_1\le 1$.

Let us assume that the given problem has an exact solution at $\alpha=2$ and $\beta=1$ which is given by

$$U(x)=\frac{x}{(x-3)^{-3}}+\frac{1}{(1-x)^{-4}}. \tag{89}$$

Additionally, $f(x)$ can be approximated as

$$\begin{aligned}f(x)&=2(x-3)\left[(x-3)^2(1-10x)+3x\right]\\&\quad+4(1-x)^2\left(5-2x^2\right).\end{aligned} \tag{90}$$

In Figure 1, we compare the actual and approximate solutions at different fractional orders, and the corresponding absolute errors are also presented graphically. We see that as the fractional order approaches the corresponding integer value 2, the concerned curve tends to the corresponding integer order curve. The concerned error also reduces as the order increases. In Figure 2, we provide a graphical presentation of numerical solutions at various scale levels. Here, we see that as the value of M is increasing, the corresponding absolute error also decreases. Hence, the mentioned method is also scale-oriented.

Example 14. Here, we take another FOBVP given by

$$\begin{aligned}\mathbf{D}^{\nu}U(x)&=5\mathbf{D}^{\nu_1}U(x)+6U(x)+f(x),\\U(x)|_{x=0}&=1,U(x)|_{x=1}=4,\end{aligned} \tag{91}$$

where $1<\nu\le 2$, $0<\nu_1\le 1$. We solve this problem for various fractional orders. Let us assume that the given problem has an exact solution at $\nu=2$ and $\nu_1=1$ which is given by

$$U(x)=x^6-x^5+x^4+x^3+x+1, \tag{92}$$

with the source term given by

$$f(x)=-6x^6-24x^5+49x^4-46x^3-3x^2-11. \tag{93}$$

In Figure 3, we compare the exact and approximate solutions at different fractional orders, and the corresponding absolute errors are also presented graphically. We see that as the fractional order approaches the corresponding integer value 2, the concerned curve tends to the corresponding integer order curve. The concerned error also reduces as the order increases. In Figure 4, we provide

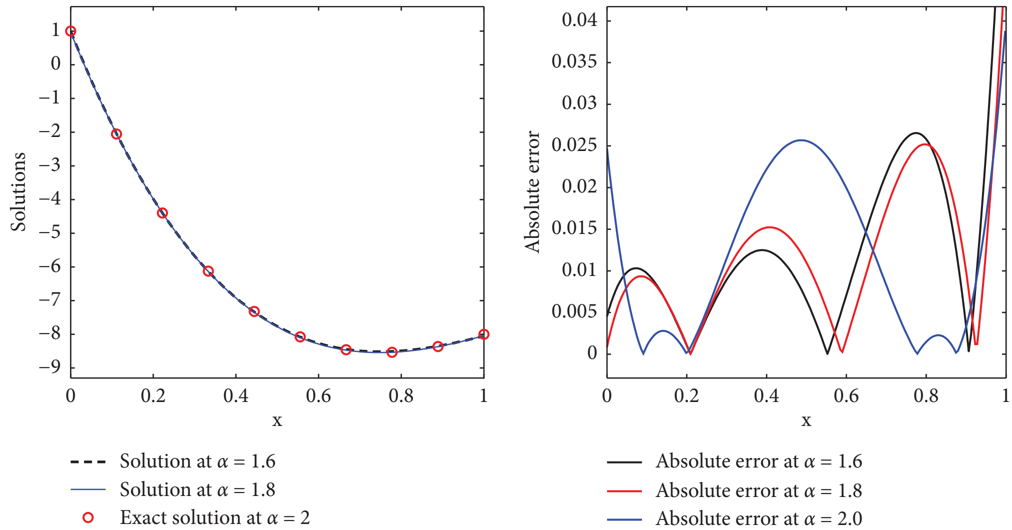

FIGURE 1: Actual and numerical solutions comparison at $M = 4$ and $\nu_1 = 0.8$ and various values of α.

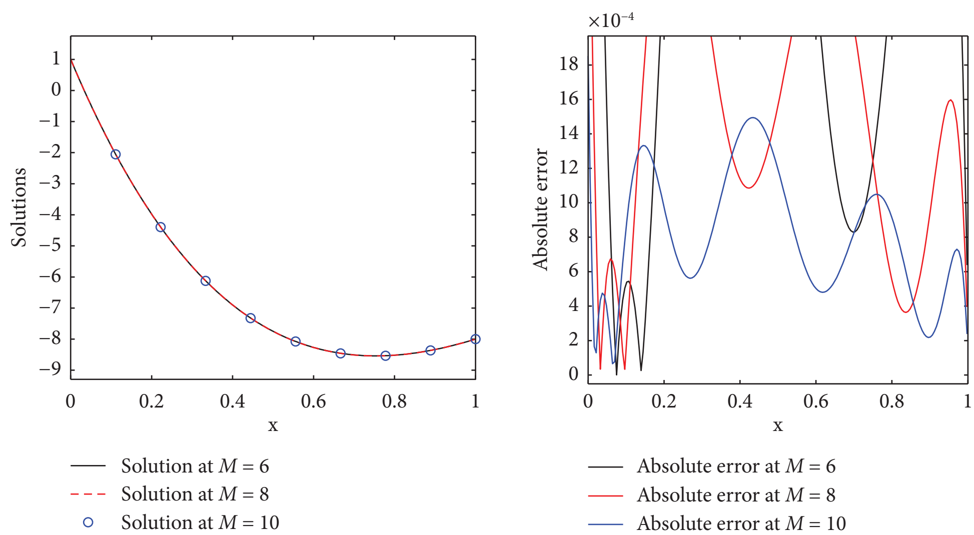

FIGURE 2: Graphical presentation of approximate solutions at $\alpha = 1.8$ and $\nu_1 = 0.8$ and different values of M.

a graphical presentation of numerical solutions at various scale levels. We provide a graphical presentation of numerical solutions at various scale levels. Here, we see that as the value of M is increasing, the corresponding absolute error also decreases. Hence, the mentioned method is also scale-oriented.

Example 15. Consider the system of FOBVPs [36]

$$\begin{aligned} \mathbf{D}^{\nu}U(x) &= 2\mathbf{D}^{\nu_1}U(x) + 2\mathbf{D}^{\nu_2}V(x) + 3U(x) + 5V(x) + f(x), \\ \mathbf{D}^{\omega}V(x) &= 2\mathbf{D}^{\omega_1}U(x) + 2\mathbf{D}^{\omega_2}V(x) + 4U(x) + 5V(x) + g(x), \end{aligned} \tag{94}$$

with BCs given by

$$\begin{aligned} U(x)|_{x=0} &= 100, U(x)|_{x=1} = 100, \\ V(x)|_{x=0} &= 50, V(x)|_{x=1} = 50. \end{aligned} \tag{95}$$

We solve this problem under the set of parameters

$$S_3 = \{\nu = \omega = 2, \nu_1 = \omega_1 = 1, \nu_2 = \omega_2 = 1\}. \tag{96}$$

Let us assume that this problem has the exact solution given below under the given parameters

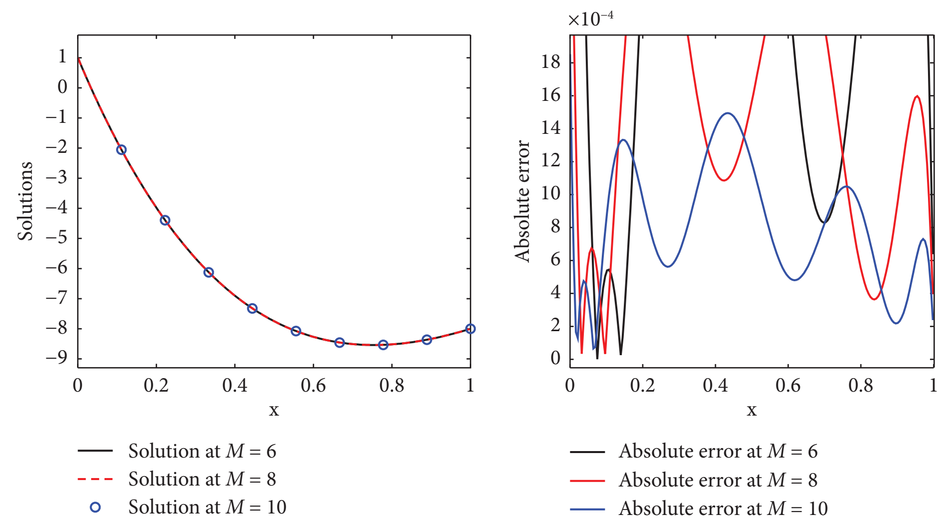

Figure 3: Actual and numerical solutions comparison at $M = 4$ and $\nu_1 = 0.8$ and various values of α.

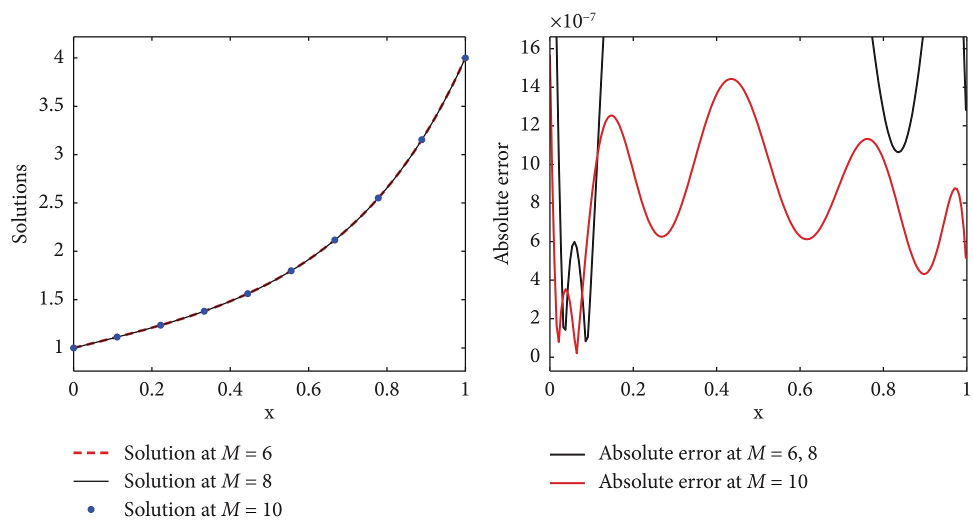

Figure 4: Graphical presentation of approximate solutions at $\alpha = 1.8$ and $\nu_1 = 0.8$ and different values of M.

$$\begin{aligned} U(x) &= \frac{x^2}{(1-x)^2} + 100, \\ V(x) &= \frac{x}{(1-x)^{-3}} + 50. \end{aligned} \tag{97}$$

Then, $f(x)$ and $g(x)$ are given by

$$\begin{aligned} f(x) &= \frac{4}{(2x-2)^{-1}}x + \frac{2}{(x-1)^{-2}}x + \frac{5}{(x-1)^{-3}}x + \frac{2}{(x-1)^{-2}} + 2(x-1)^3 - 2x^2(2x-2) - \frac{3}{(x-1)^{-2}}x^2 + 2x^2 - 550, \\ g(x) &= \frac{2}{(x-1)^{-2}}x - 3x(2x-2) + \frac{5}{(x-1)^{-3}}x - \frac{6}{(x-1)^{-2}} + \frac{2}{(x-1)^{-3}} - \frac{2}{(2x-2)^{-1}}x^2 - \frac{4}{(x-1)^{-2}}x^2 - 650. \end{aligned} \tag{98}$$

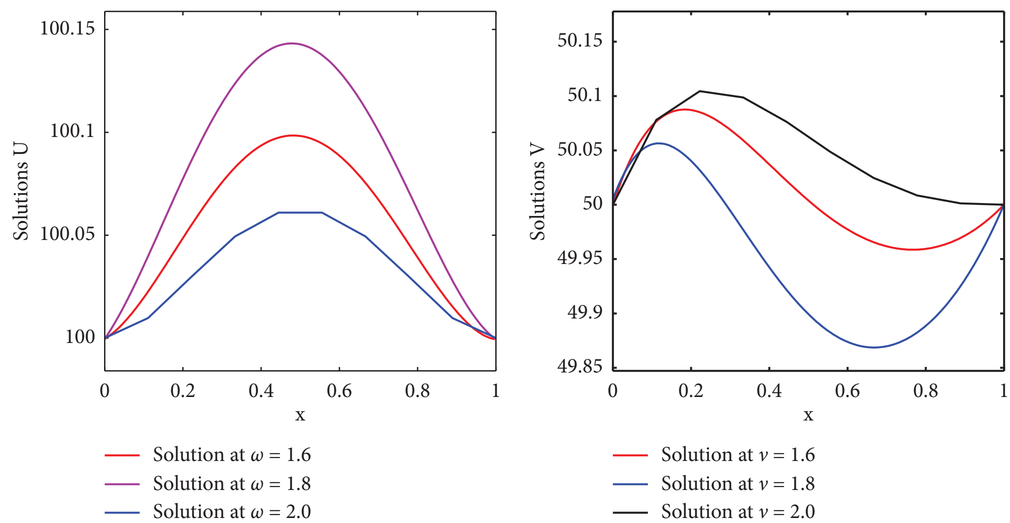

FIGURE 5: Graphical presentation of approximate solutions at different values of ν, ω and $\nu_i = 0.8, \omega_i = 0.8, i = 1, 2$ for $M = 6$.

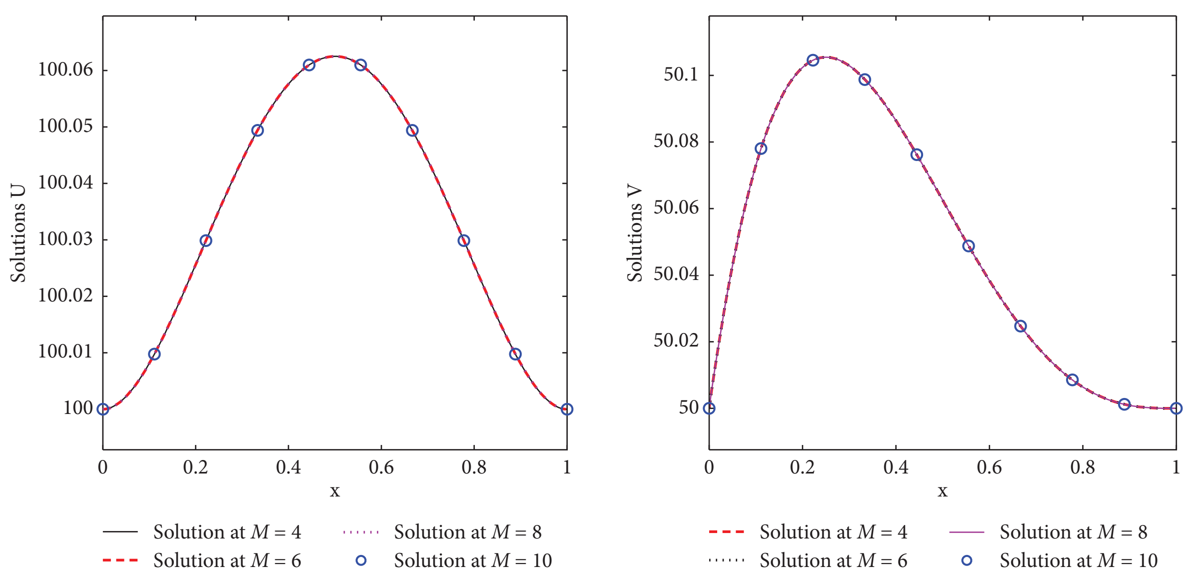

FIGURE 6: Graphical presentation of approximate solutions at $\nu = \omega = 2$ and $\nu_i = 1$ and different values of M.

Here, we give a plot at $M = 6$ and different values of fractional order ω, ν. In Figure 5, we compare the actual and approximate solutions at different fractional orders, and the corresponding absolute errors are also presented graphically. We see that as the fractional order approaches the corresponding integer value 2, the concerned curves tend to the corresponding integer order curves. The concerned error also reduces as the order increases. In Figures 6 and 7, we provide plots of approximate solutions at various scale levels. Also, we provide a graphical presentation of numerical solutions at various scale levels. Here, we see that as the value of M is increasing, the corresponding absolute error also decreases. Hence, the mentioned method is also scale-oriented. In Figure 7, we provide plots of absolute errors at different scale levels. Here, we compare the numerical results for Example 15 with those given in [36] in Table 1. We see that the existing method produces slightly good results than the numerical results obtained in [36] for the given problem by using the shifted Legendre polynomials spectral method. In addition, here in Table 2, we compare the CPU time of our proposed method with that of [36].

Example 16. Consider the system of FOBVPs

$$\begin{aligned}
\mathbf{D}^{\gamma} Y(x) &= \mathbf{D}^{\gamma_1} Y(x) + \mathbf{D}^{\gamma_2} Z(x) + 3Y(x) - 2Z(x) + h(x), \\
\mathbf{D}^{\eta} Z(x) &= 4\mathbf{D}^{\eta_1} Y(x) + 3\mathbf{D}^{\eta_2} Z(x) + 2Y(x) + Z(x) + k(x),
\end{aligned} \tag{99}$$

with BCs given by

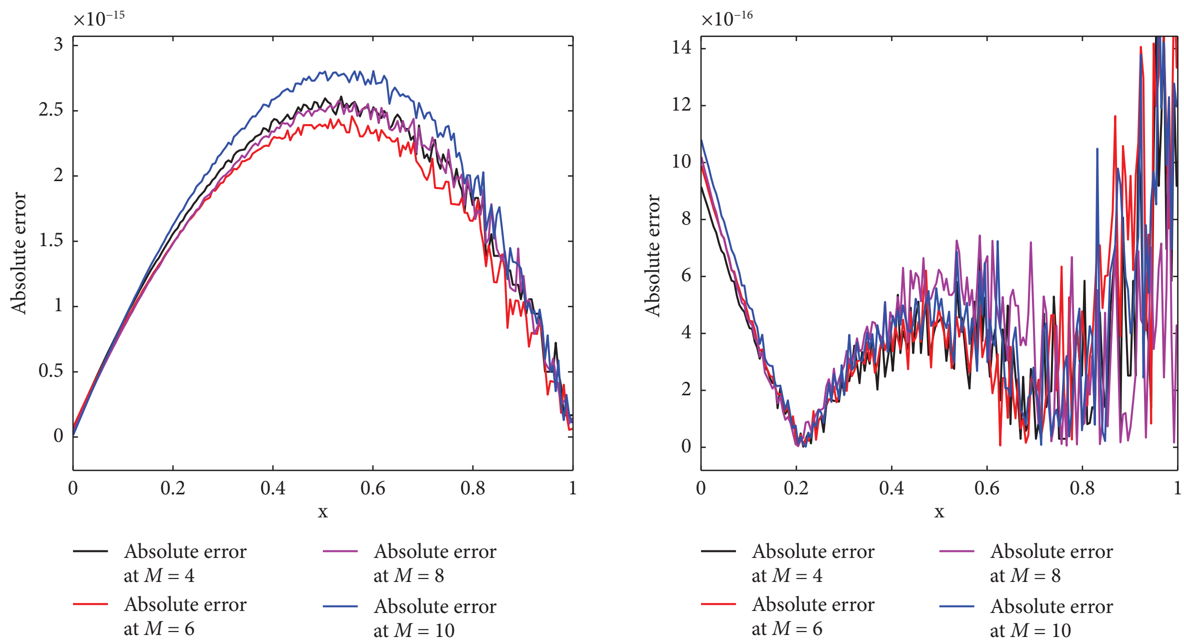

Figure 7: Graphical presentation of absolute errors at $\nu = \omega = 2$, $\nu_i = 1$ and various values of M.

Table 1: Comparison between numerical results of Example 15 by our proposed method and that used in [36].

Scale M	Fractional order $\nu = \omega = 1.9, \nu_1 = \nu_2 = 0.9$	Proposed method results $\|U\|_e$	$\|V\|_e$	Results of [36] $\|U\|_e$	$\|V\|_e$
4		9.9×10^{-3}	1.9×10^{-3}	1.3×10^{-2}	2×10^{-3}
5		1.1×10^{-3}	1.0×10^{-4}	6.7×10^{-3}	1.53×10^{-4}
6		6.5×10^{-4}	2.5×10^{-4}	6.7×10^{-4}	7×10^{-4}
7		1.2×10^{-5}	2.8×10^{-5}	7.3×10^{-5}	9.27×10^{-5}
8		2.05×10^{-5}	1.1×10^{-5}	6.17×10^{-5}	6.23×10^{-5}
9		6.5×10^{-6}	1.7×10^{-6}	3.67×10^{-6}	8.28×10^{-6}
10		4.91×10^{-7}	1.5×10^{-8}	4.7×10^{-7}	6.3×10^{-7}

Table 2: Comparison of CPU time to compute solution in Example 15 by our proposed method and that used in [36] using Cori7 generation HP machine of 1024 GHz.

Scale M	Fractional order $\nu = \omega = 1.9, \nu_1 = \nu_2 = 0.9$	CPU time using the proposed method Y (sec)	Z (sec)	CPU time using the method of [36] Y (sec)	Z (sec)
4		80	81	84	85
5		90	90.5	91	92
6		92	93	94	95
7		95	96	97	97.4
8		97	98	98	99.8
9		100	101	105	106
10		110	110	115	117

$$Y(x)|_{x=0} = 2, Y(x)|_{x=1} = 2,$$
$$Z(x)|_{x=0} = 1, Z(x)|_{x=1} = 1. \quad (100)$$

Let us assume that this problem has the following exact solution under the set of parameters.

$$S_4 = \{\gamma = 1.8, \gamma_1 = 0.6, \gamma_2 = 0.5, \eta = 1.6, \eta_1 = 0.6, \eta_2 = 0.5\},$$
$$Y(x) = x^6 - x^5 + 2,$$
$$Z(x) = x^5 - x^4 + 1. \quad (101)$$

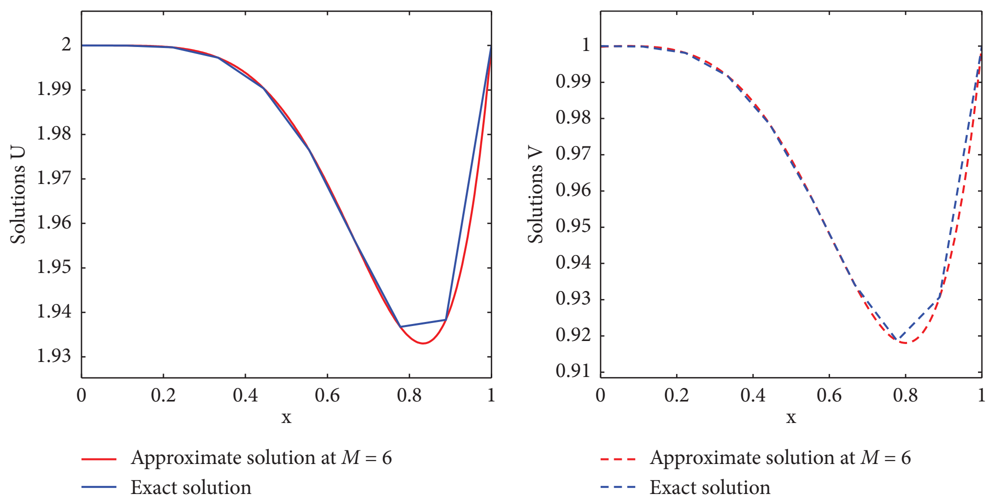

Figure 8: Actual and numerical solutions comparison at $M= 6$.

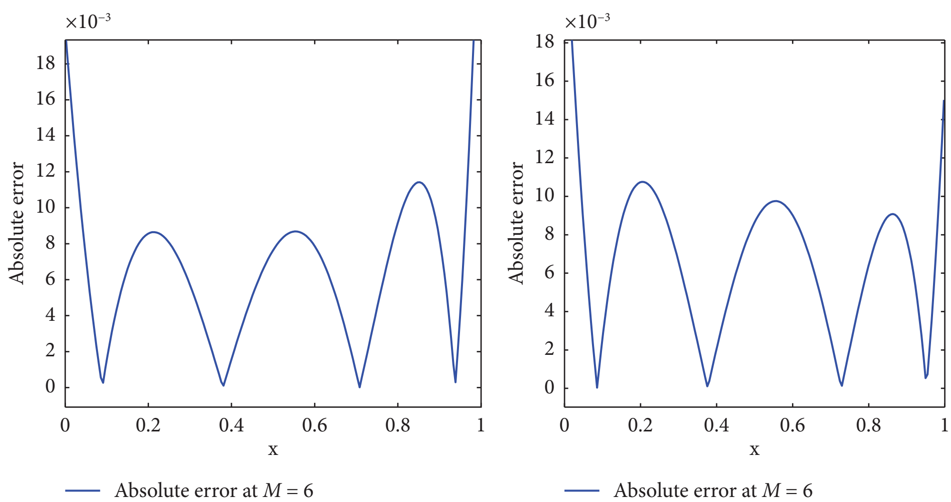

Figure 9: Absolute error at both solutions of the given system at $M= 6$.

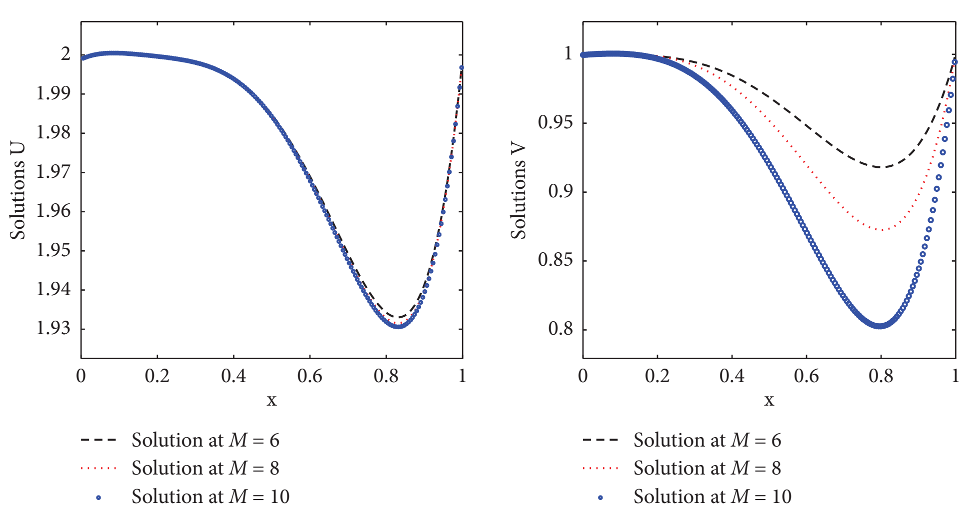

Figure 10: Graphical presentation of numerical solutions at various scale levels of M.

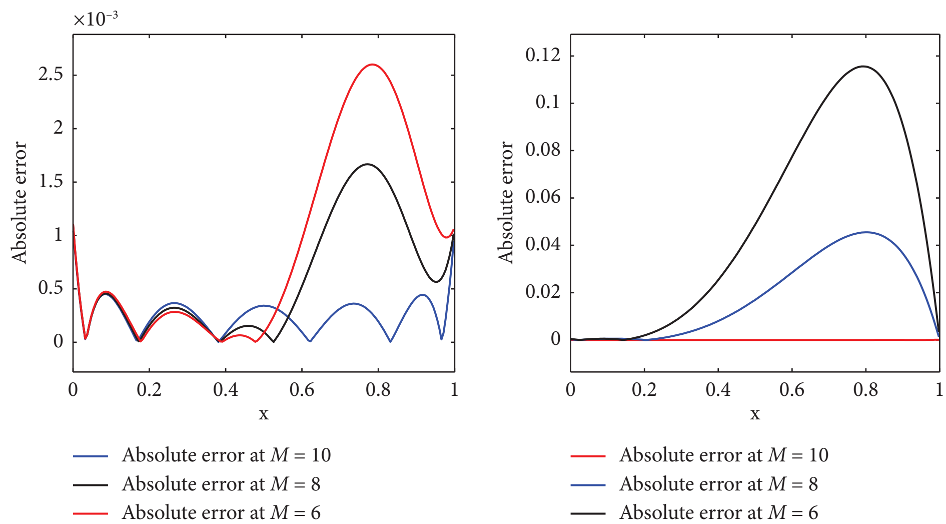

FIGURE 11: Absolute errors at different scale levels of M.

Then, $h(x)$ and $k(x)$ are given by

$$\begin{aligned} h(x) &= \frac{\left(2229536516744740625x^{(16/5)}(10x-7)\right)}{1008806316530991104} - \frac{\left(5081767996463981x^{(7/2)}(10x-9)\right)}{22166154415964160} \\ &\quad - \frac{\left(1982745796388234375x^{(22/5)}(10x-9)\right)}{6632113401256476672} - 2x^4 + 5x^5 - 3x^6 - 4, \\ k(x) &= \frac{\left(89181460669789625x^{(11/5)}(25x-16)\right)}{144115188075855872} - \frac{\left(1982745796388234375x^{(22/5)}(10x-9)\right)}{1658028350314119168} \\ &\quad - \frac{\left(5081767996463981x^{(7/2)}(10x-9)\right)}{7388718138654720} + x^4 + x^5 - 2x^6 - 5. \end{aligned} \tag{102}$$

Here, we compare the actual and numerical solutions and the corresponding absolute errors in Figures 8 and 9, respectively. We compare the actual and approximate solutions at different fractional orders, and the corresponding absolute errors are also presented graphically. We see that as the fractional orders approach the corresponding integer value 2, the concerned curves tend to the corresponding integer order curves. The concerned error also reduces as the order increases. Furthermore, we provide a graphical presentation of approximate solutions at various scale levels and the corresponding absolute errors in Figures 10 and 11, respectively. In addition, we provide a graphical presentation of numerical solutions at various scale levels. Here, we see that as the value of M is increasing, the corresponding absolute error also decreases. Hence, the mentioned method is also scale-oriented.

7. Conclusion and Discussion

In this article, we have investigated different classes of FODEs under boundary conditions for numerical solutions. We have used FOBPs along with some important properties to construct some operational matrices corresponding to fractional-order derivative and integration. Based on these matrices, we have converted the proposed problems to a system of Sylvester-type operational matrices. Upon using MATLAB, we solved several examples to demonstrate the procedure. From the numerical investigation, we see that the method is powerful and can be used to investigate various FODEs for numerical solutions. The efficiency of the method can be further improved by enlarging the scale level. The greater the scale level, the greater the accuracy, and vice versa. Also, we have computed the maximum absolute error by using different scale levels. In some examples, we have also investigated the problem by using various fractional orders to simulate the results. From the numerical results, we see that the operational method based on FOBPs can also be used as a powerful spectral method to handle FODEs for numerical purposes. We have compared our numerical results with those given in [36] by using the shifted Legendre polynomials spectral method. Our results are good than those given in the mentioned reference. The CPU time of the proposed method has been compared with the CPU time of the shifted Legendre spectral method. The CPU time of the proposed method is better than the spectral numerical method based on Legendre polynomials. In the future, we will extend the mentioned OMs method to variable order problems. Also, we will study FODEs involving nonsingular type derivatives by using our proposed method.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] B. Rose, *Fractional Calculas and its Applications*, Wiley, New York, NY, USA, 1975.
[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, Amesterdam, Netherlands, 2006.
[3] R. Hilfer, "Threefold Introduction to Fractional Derivatives," *Anomalous Transport: Foundations and Applications*, Springer, Berlin, Germany, 2008.
[4] S. Murtaza, Z. Ahmad, I. E. Ali et al., "Analysis and numerical simulation of fractal-fractional order non-linear couple stress nanofluid with cadmium telluride nanoparticles," *Journal of King Saud University Science*, vol. 35, no. 4, Article ID 102618, 2023.
[5] I. Podlubny, *Fractional Differential Equations*, Acadmic Press, New York, NY, USA, 1998.
[6] A. Ali, K. J. Ansari, H. Alrabaiah, A. Aloqaily, and N. Mlaiki, "Coupled system of fractional impulsive problem involving power-law kernel with piecewise order," *Fractal and Fractional*, vol. 7, no. 6, p. 436, 2023.
[7] A. A. Kilbas and J. J. Trujillo, "Differential equations of fractional order: methods, results and problems. II," *Applicable Analysis*, vol. 81, no. 2, pp. 435–493, 2002.
[8] J. Hristov, "Non-local kinetics: revisiting and updates emphasizing fractional calculus applications," *Symmetry*, vol. 15, no. 3, p. 632, 2023.
[9] K. Shah, T. Abdeljawad, and H. Alrabaiah, "On coupled system of drug therapy via piecewise equations," *Fractals*, vol. 30, no. 8, Article ID 2240206, 2022.
[10] S. W. Yao, Y. Sughra, M. I. Asma, M. Inc, and K. J. Ansari, "Qualitative analysis of implicit delay mittag-leffler-type fractional differential equations," *Fractals*, vol. 30, no. 8, Article ID 2240208, 2022.
[11] F. Ali, Z. Ahmad, M. Arif, I. Khan, and K. S. Nisar, "A time fractional model of generalized Couette flow of couple stress nanofluid with heat and mass transfer: applications in engine oil," *IEEE Access*, vol. 8, pp. 146944–146966, 2020.
[12] C. Xu, S. Tahir, K. J. Ansari, M. Ur Rahman, and F. S. Al-Duais, "ON systems of fractional-order differential equations for order $1<\vartheta\leq2$," *Fractals*, 2023.
[13] O. Abdulaziz, I. Hashim, and S. Momani, "Solving systems of fractional differential equations by homotopy-perturbation method," *Physics Letters A*, vol. 372, no. 4, pp. 451–459, 2008.
[14] S. Momani, "Application of variational iteration method to nonlinear differential equations of fractional order," *International Journal of Nonlinear Sciences and Numerical Stimulation*, vol. 7, pp. 15–27, 2006.
[15] V. S. Erturk and S. Momani, "Solving systems of fractional differential equations using differential transform method," *Journal of Computational and Applied Mathematics*, vol. 215, no. 1, pp. 142–151, 2008.
[16] B. Li, Z. Eskandari, and Z. Avazzadeh, "Strong resonance bifurcations for a discrete-time prey-predator model," *Journal of Applied Mathematics and Computing*, vol. 69, no. 3, pp. 2421–2438, 2023.
[17] B. Li, Z. Eskandari, and Z. Avazzadeh, "Dynamical behaviors of an SIR epidemic model with discrete time," *Fractal and Fractional*, vol. 6, no. 11, p. 659, 2022.
[18] K. Shah, T. Abdeljawad, F. Jarad, and Q. Al-Mdallal, "On nonlinear conformable fractional order dynamical system via differential transform method," *Computer Modeling in Engineering and Sciences*, vol. 136, no. 2, pp. 1457–1472, 2023.
[19] V. Daftardar-Gejji and H. Jafari, "Solving a multi-order fractional differential equation using adomian decomposition," *Applied Mathematics and Computation*, vol. 189, no. 1, pp. 541–548, 2007.
[20] E. Sousa and C. Li, "A weighted finite difference method for the fractional diffusion equation based on the Riemann-Liouville derivative," *Applied Numerical Mathematics*, vol. 90, pp. 22–37, 2015.
[21] A. Saadatmandi and M. Dehghan, "A tau approach for solution of the space fractional diffusion equation," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1135–1142, 2011.
[22] A. Saadatmandi and M. Mohabbati, "Numerical solution of fractional telegraph equation via the tau method," *Mathematical Reports*, vol. 17, pp. 155–166, 2015.
[23] A. Saadatmandi, "Bernstein operational matrix of fractional derivatives and its applications," *Applied Mathematical Modelling*, vol. 38, no. 4, pp. 1365–1372, 2014.
[24] P. Rahimkhani, Y. Ordokhani, and E. Babolian, "A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations," *Numerical Algorithms*, vol. 74, no. 1, pp. 223–245, 2017.
[25] D. Fathima, R. A. Alahmadi, A. Khan, A. Akhter, and A. H. Ganie, "An efficient analytical approach to investigate fractional Caudrey-Dodd-Gibbon equations with non-singular kernel derivatives," *Symmetry*, vol. 15, no. 4, p. 850, 2023.
[26] N. K. Mishra, M. M. AlBaidani, A. Khan, and A. H. Ganie, "Numerical investigation of time-fractional phi-four equation via novel transform," *Symmetry*, vol. 15, no. 3, p. 687, 2023.
[27] M. M. Al-Sawalha, R. Shah, K. Nonlaopon, and O. Y. Ababneh, "Numerical investigation of fractional-order wave-like equation," *AIMS Mathematics*, vol. 8, no. 3, pp. 5281–5302, 2022.
[28] M. K. Alaoui, R. Fayyaz, A. Khan, R. Shah, and M. S. Abdo, "Analytical investigation of Noyes-Field model for time-

fractional Belousov-Zhabotinsky reaction," *Complexity*, vol. 2021, Article ID 3248376, 21 pages, 2021.
[29] K. Nonlaopon, A. M. Alsharif, A. M. Zidan, A. Khan, Y. S. Hamed, and R. Shah, "Numerical investigation of fractional-order Swift-Hohenberg equations via a Novel transform," *Symmetry*, vol. 13, no. 7, p. 1263, 2021.
[30] E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley, New York, NY, USA, 1978.
[31] A. Saadatmandi and M. Dehghan, "A new operational matrix for solving fractional-order differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1326–1336, 2010.
[32] P. Rahimkhani, Y. Ordokhani, and E. Babolian, "An efficient approximate method for solving delay fractional optimal control problems," *Nonlinear Dynamics*, vol. 86, no. 3, pp. 1649–1661, 2016.
[33] Ş. Yüzbaşı, "Numerical solutions of fractional Riccati type differential equations by means of the Bernstein polynomials," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 6328–6343, 2013.
[34] S. Kazem, S. Abbasbandy, and S. Kumar, "Fractional-order Legendre functions for solving fractional-order differential equations," *Applied Mathematical Modelling*, vol. 37, no. 7, pp. 5498–5510, 2013.
[35] P. Rahimkhani, Y. Ordokhani, and E. Babolian, "Fractional-order Bernoulli functions and their applications in solving fractional Fredholem–Volterra integro-differential equations," *Applied Numerical Mathematics*, vol. 122, pp. 66–81, 2017.
[36] H. Khalil, K. Shah, and R. A. Khan, "Approximate solution of boundary value problems using shifted Legendre polynomials," *Applied and Computational Mathematics*, vol. 16, no. 3, pp. 269–285, 2017.
[37] K. Shah, F. Jarad, and T. Abdeljawad, "Stable numerical results to a class of time-space fractional partial differential equations via spectral method," *Journal of Advanced Research*, vol. 25, pp. 39–48, 2020.
[38] S. Bazm, "Bernoulli polynomials for the numerical solution of some classes of linear and nonlinear integral equations," *Journal of Computational and Applied Mathematics*, vol. 275, pp. 44–60, 2015.