Hindawi
Journal of Mathematics
Volume 2023, Article ID 5833837, 6 pages
https://doi.org/10.1155/2023/5833837

Research Article

Map-Pointed Fuzzy Hyperoperations

Mohammad Esmaeil Sharbati,[1] Hamid Rasouli ,[1] Abolghasem Karimi Feizabadi,[2] and Abolfazl Tehranian[1]

[1]*Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*
[2]*Department of Mathematics, Gorgan Branch, Islamic Azad University, Gorgan, Iran*

Correspondence should be addressed to Hamid Rasouli; hrasouli@srbiau.ac.ir

Received 4 April 2023; Revised 22 June 2023; Accepted 30 September 2023; Published 20 November 2023

Academic Editor: Francisco J. Garcia Pacheco

In this paper, we establish a correspondence between fuzzy hypergroupoids and certain types of hypergroupoids that possess map-pointed properties. Specifically, we introduce a new type of fuzzy hyperoperations, known as map-pointed fuzzy hyperoperations, and show that this correspondence yields an adjunction.

1. Introduction and Preliminaries

The concept of hyperstructures was introduced in 1934 by Marty [1], and since then, researchers have explored fuzzy algebraic structures, such as fuzzy subgroups of a group studied by Rosenfeld in 1971 [2]. Corsini and Tofan [3] proposed a new model for generalizing hyperstructures through fuzzy theory, where hyperoperations are generalized to fuzzy hyperoperations. This idea has been applied to semihypergroups in [4] and further generalized to fuzzy hyperrings and fuzzy hypermodules in [5, 6]. One interesting direction in the research is the construction of fuzzy hyperoperations from hyperoperations and vice versa. In [4], a possible way to obtain a hyperoperation from a fuzzy hyperoperation is presented. However, there is no fundamental approach to constructing fuzzy hyperoperations from hyperoperations, except for a trivial example using characteristic functions. In this paper, we take a category-theoretic approach to explore the correspondence between hyperoperations and fuzzy hyperoperations. We demonstrate that the constructions based on the notion of map-pointed fuzzy hyperoperation derived from a map-pointed hypergroupoid are appropriate, as the corresponding functors between the categories of map-pointed hypergroupoids and fuzzy hypergroupoids define an adjunction, and the restrictions of this correspond to equivalence on certain subcategories.

Let us recall from [3, 4] some basic notions in hypergroupoids and fuzzy hypergroupoids needed in the sequel.

Let S be a nonempty set. A *hyperoperation* on S is a map $\star : S \times S \longrightarrow P(S)$, where $P(S)$ is the set of all subsets of S. For $A, B \subseteq S$, $A \star B$ is defined by

$$A \star B = \bigcup_{a\in A, b\in B} a \star b. \tag{1}$$

The notations $x\star A$ and $A\star x$ are used for $\{x\}\star A$ and $A\star\{x\}$, respectively. Throughout, the symbol $\mathcal{U}$ stands for $S\star S$. A *hypergroupoid* is a structure $(S, \star)$ where $\star$ is a hyperoperation on S. A hypergroupoid $(S, \star)$ is called a *semihypergroup* if $\star$ is *associative*; that is, for all $a, b, c \in S$, $a\star(b\star c) = (a\star b)\star c$.

Let $(S_1, \star_1)$ and $(S_2, \star_2)$ be two hypergroupoids. We say that a map $f: S_1 \longrightarrow S_2$ is a *homomorphism* if

$$f(x\star_1 y) \subseteq f(x)\star_2 f(y) \text{ for all } x, y \in S_1, \tag{2}$$

and a *strong homomorphism* if

$$f(x\star_1 y) = f(x)\star_2 f(y) \text{ for all } x, y \in S_1. \tag{3}$$

Let S be a nonempty set. A *fuzzy subset* of S is a map $\alpha: S \longrightarrow [0, 1]$. A *fuzzy hyperoperation* on S is a map $\circ : S \times S \longrightarrow F(S)$, where $F(S)$ is the set of all fuzzy subsets of S.

Let α and β be two fuzzy subsets of a fuzzy hypergroupoid $(S, \circ)$. One can define a new fuzzy subset $\alpha \circ \beta$ of S by

$$(\alpha \circ \beta)(t) = \bigvee_{r,s \in S} ((r \circ s)(t) \wedge \alpha(r) \wedge \beta(s)), \tag{4}$$

for all $t \in S$, where $\vee$ and $\wedge$ are arbitrary supremum and finite infimum in the set of real numbers, respectively. Also, for any $x \in S$ and $\alpha \in F(S)$, $x \circ \alpha$ is defined by

$$(x \circ \alpha)(t) = \bigvee_{s \in S} ((x \circ s)(t) \wedge \alpha(s)), \tag{5}$$

for all $t \in S$. A *fuzzy hypergroupoid* is a structure $(S, \circ)$, where $\circ$ is a fuzzy hyperoperation on S. A fuzzy hypergroupoid $(S, \circ)$ is a *fuzzy hypersemigroup* if $\circ$ is *associative*; that is, for all $x, y, z \in S$, $x \circ (y \circ z) = (x \circ y) \circ z$.

We refer to [7] for the unexplained terminology from category theory used in this paper.

2. Map-Pointed Hypergroupoids and Fuzzy Hyperoperations

In this section, we introduce the notion of a map-pointed fuzzy hyperoperation and give some equivalent conditions to this basic concept which plays a key role in the next section.

Let $(S, \star)$ be a hypergroupoid and α be a fuzzy subset of S. We call the triple $(S, \star, \alpha)$ a *map-pointed hypergroupoid*.

Definition 1. Let $(S, \star, \alpha)$ be a map-pointed hypergroupoid. Then,

(i) $(S, \star, \alpha)$ is said to be *coclosed* if it satisfies the following condition:

$$z \in x \star y \Rightarrow \alpha(z) \le \alpha(x), \alpha(z) \le \alpha(y). \tag{6}$$

(ii) The fuzzy *hyperoperation* $\overset{\alpha}{\star}$ on S is given by

$$(x \overset{\alpha}{\star} y)(s) = \alpha(s) \chi_{x \star y}(s), \tag{7}$$

which is called a *map-pointed fuzzy hyperoperation*.

Theorem 2. *Let $(S, \star, \alpha)$ be a map-pointed hypergroupoid for which $(S, \star)$ is a semihypergroup. If $(S, \star, \alpha)$ is coclosed, then $(S, \overset{\alpha}{\star})$ is a fuzzy hypersemigroup.*

Proof. Let $x, y, z \in S$. For any $s \in S$, we have

$$\begin{aligned}(x \overset{\alpha}{\star} (y \overset{\alpha}{\star} z))(s) &= \bigvee_{t \in S} ((x \overset{\alpha}{\star} t)(s) \wedge (y \overset{\alpha}{\star} z)(t)) \\ &= \bigvee_{t \in y \star z} \left(\alpha(s) \chi_{x \star t}(s) \wedge \alpha(t) \chi_{y \star z}(t)\right) \\ &= \bigvee_{\substack{t \in y \star z \\ s \in x \star t}} (\alpha(s) \wedge \alpha(t)).\end{aligned} \tag{8}$$

Since $(S, \star, \alpha)$ is coclosed, $\alpha(s) \le \alpha(t)$ for any $t \in S$ and $s \in x \star t$. Hence,

$$\begin{aligned}(x \overset{\alpha}{\star} (y \overset{\alpha}{\star} z))(s) &= \bigvee_{\substack{t \in y \star z \\ s \in x \star t}} \alpha(s) \\ &= \alpha(s) \chi_{x \star (y \star z)}(s).\end{aligned} \tag{9}$$

Similarly, one can show that for any $s \in S$,

$$((x \overset{\alpha}{\star} y) \overset{\alpha}{\star} (z))(s) = \alpha(s) \chi_{(x \star y) \star z}(s). \tag{10}$$

Since $\star$ is associative by the assumption, $x \overset{\alpha}{\star} (y \overset{\alpha}{\star} z) = (x \overset{\alpha}{\star} y) \overset{\alpha}{\star} z$.

In the following, we present an equivalent condition for a fuzzy hyperoperation to be map-pointed. □

Definition 3. Let S be a nonempty set. A fuzzy hyperoperation $\circ$ on S is called *smooth* if for every $a, b, c, d, s \in S$, $(a \circ b)(s) > 0$ and $(c \circ d)(s) > 0$ imply $(a \circ b)(s) = (c \circ d)(s)$.

Theorem 4. *A fuzzy hyperoperation is map-pointed if and only if it is smooth.*

Proof. Consider a map-pointed fuzzy hyperoperation $\overset{\alpha}{\star}$ on S for a map-pointed hypergroupoid $(S, \star, \alpha)$. We show that $\overset{\alpha}{\star}$ is smooth. Let $(a \overset{\alpha}{\star} b)(s) > 0$ and $(c \overset{\alpha}{\star} d)(s) > 0$ for some $a, b, c, d, s \in S$. We have $0 < (a \overset{\alpha}{\star} b)(s) = \alpha(s) \chi_{a \star b}(s)$ and so, $\chi_{a \star b}(s) = 1$. Similarly, $\chi_{c \star d}(s) = 1$. Therefore,

$$\begin{aligned}(a \overset{\alpha}{\star} b)(s) &= \alpha(s) \chi_{a \star b}(s) \\ &= \alpha(s) \\ &= \alpha(s) \chi_{c \star d}(s) \\ &= (c \overset{\alpha}{\star} d)(s).\end{aligned} \tag{11}$$

For the converse, let $\circ$ be a smooth fuzzy hyperoperation on S. We define $\star$ on S by

$$x \star y = \{s \in S \mid (x \circ y)(s) > 0\}, \tag{12}$$

and α: $S \longrightarrow [0, 1]$ by

$$\alpha(s) = \bigvee_{x,y \in S} (x \circ y)(s), \tag{13}$$

for every $s \in S$. We claim that $\circ = \overset{\alpha}{\star}$. Let us first assume that $s \in a \star b$. Then, $\chi_{a \star b}(s) = 1$ and hence

$$\begin{aligned}(a \overset{\alpha}{\star} b)(s) &= \alpha(s) \chi_{a \star b}(s) \\ &= \alpha(s) \\ &= \bigvee_{x,y \in S} (x \circ y)(s) \\ &= \bigvee_{s \in x \star y} (x \circ y)(s) \\ &= (a \circ b)(s).\end{aligned} \tag{14}$$

The last equality follows from the smoothness of $\circ$. Indeed, for every $x, y \in S$ with $s \in x \star y$, we have $(x \circ y)(s) > 0$; also, $s \in a \star b$ gives that $(a \circ b)(s) > 0$ and so $(x \circ y)(s) = (a \circ b)(s)$. Now, let $s \notin a \star b$. Then,

$$\begin{aligned}(a \overset{\alpha}{\star} b)(s) &= \alpha(s) \chi_{a \star b}(s) \\ &= 0 \\ &= (a \circ b)(s).\end{aligned} \tag{15}$$

Therefore, $a \overset{\alpha}{\star} b = a \circ b$, which means that $\circ$ is map-pointed. □

Notation 5. Let $\circ$: $S \times S \to F(S)$ be a fuzzy hyperoperation. We define the fuzzy subset $\widehat{\circ}$: $S \to [0,1]$ of S by

$$\widehat{\circ}(s) = \bigvee_{x,y\in S} (x \circ y)(s), \tag{16}$$

for every $s \in S$. Moreover, we use the symbol $\otimes$ for the hyperoperation with respect to $\circ$, that is,

$$x \otimes y = \{s \in S \mid (x \circ y)(s) > 0\}, \tag{17}$$

for all $x, y \in S$.

In view of the proof of Theorem 4, we get the following:

Corollary 6. *Let $\circ: S \times S \to F(S)$ be a fuzzy hyperoperation. Then, $\circ$ is smooth if and only if $\circ = \overset{\widehat{\circ}}{\otimes}$.*

Proposition 7. *Let $(S, \star, \alpha)$ be a map-pointed hypergroupoid. Then,*

(i) $\widehat{\overset{\alpha}{\star}} = \alpha\chi_{\mathscr{U}}$.

(ii) $\widehat{\overset{\alpha}{\star}} = \alpha$ if and only if $\alpha(s) = 0$ for all $s \notin \mathscr{U}$. In particular, if $\mathscr{U} = S$, then $\widehat{\overset{\alpha}{\star}} = \alpha$.

(iii) If $(S, \star, \beta)$ is a map-pointed hypergroupoid, then $\overset{\alpha}{\star} = \overset{\beta}{\star}$ if and only if $\alpha|_{\mathscr{U}} = \beta|_{\mathscr{U}}$.

Proof

(i) For every $s \in S$, we have

$$\begin{aligned} \widehat{\overset{\alpha}{\star}}(s) &= \bigvee_{x,y\in S} (x \overset{\alpha}{\star} y)(s) \\ &= \bigvee_{x,y\in S} \left(\alpha(s)\chi_{x\star y}(s)\right) \\ &= \begin{cases} \alpha(s), & s \in x\star y \text{ for some } x, y \in S, \\ 0, & \text{otherwise,} \end{cases} \\ &= \alpha(s)\chi_{\mathscr{U}}(s). \end{aligned} \tag{18}$$

(ii) ($\Rightarrow$) Suppose that $\widehat{\overset{\alpha}{\star}}(s) = \alpha$. Then, $\alpha\chi_{\mathscr{U}} = \alpha$. Let $s \notin \mathscr{U}$. So, $\alpha(s) = \alpha(s)\chi_{\mathscr{U}}(s) = \alpha(s)0 = 0$.

($\Leftarrow$) Let $s \in S$. If $s \in \mathscr{U}$, then

$$\begin{aligned} \widehat{\overset{\alpha}{\star}}(s) &= \alpha(s)\chi_{\mathscr{U}}(s) \\ &= \alpha(s)1 \\ &= \alpha(s). \end{aligned} \tag{19}$$

If $s \notin \mathscr{U}$, then

$$\widehat{\overset{\alpha}{\star}}(s) = \alpha(s)\chi_{\mathscr{U}}(s) = 0 = \alpha(s). \tag{20}$$

Hence, $\widehat{\overset{\alpha}{\star}} = \alpha$.

(iii) ($\Rightarrow$) Let $\widehat{\overset{\alpha}{\star}} = \widehat{\overset{\beta}{\star}}$. Then, $\alpha\chi_{\mathscr{U}} = \beta\chi_{\mathscr{U}}$ by (i). Hence, $\alpha|_{\mathscr{U}} = \beta|_{\mathscr{U}}$.

($\Leftarrow$) Suppose that $\alpha|_{\mathscr{U}} = \beta|_{\mathscr{U}}$. Then, let $x, y, s \in S$ and we have

$$\begin{aligned} (x \overset{\alpha}{\star} y)(s) &= \alpha(s)\chi_{x\star y}(s) \\ &= \beta(s)\chi_{x\star y}(s) \\ &= (x \overset{\beta}{\star} y)(s). \end{aligned} \tag{21}$$

Hence, $\overset{\alpha}{\star} = \overset{\beta}{\star}$. □

Corollary 8. *Let $\circ: S \times S \to F(S)$ be a fuzzy hyperoperation. Then, the following are equivalent:*

(i) $\circ$ is smooth.

(ii) $\circ$ is map-pointed.

(iii) $\circ = \overset{\widehat{\circ}}{\otimes}$.

(iv) $\circ = \overset{\beta}{\star}$ for a hyperoperation $\star$ and a unique map β with $\beta(s) = 0$ for all $s \notin \mathscr{U}$.

Proof. The equivalences of (i), (ii), and (iii) follow from Theorem 4 and Corollary 6. For (ii) $\Rightarrow$ (iv) define β: $S \to [0,1]$ by

$$\beta(s) = \begin{cases} \alpha(s), & s \in \mathscr{U}, \\ 0, & s \notin \mathscr{U}. \end{cases} \tag{22}$$

We have $\alpha|_{\mathscr{U}} = \beta|_{\mathscr{U}}$ and so $\overset{\alpha}{\star} = \overset{\beta}{\star}$ by Proposition 7. Thus, $\circ = \overset{\beta}{\star}$ with $\beta(s) = 0$ for all $s \notin \mathscr{U}$. Also, β is unique by Proposition 7 and (iv) $\Rightarrow$ (ii) is trivial. □

Remark 9. $\widehat{\circ}$ is the unique map mentioned in Corollary 8 because $\widehat{\circ}(s) = 0$ for all $s \notin \mathscr{U}$. Let $s \notin \mathscr{U}$, then $s \notin x \otimes y$ for all $x, y \in S$. Therefore, $(x \circ y)(s) = 0$ for all $x, y \in S$ and hence

$$\begin{aligned} \widehat{\circ}(s) &= \bigvee_{x,y\in S} (x \circ y)(s) \\ &= 0. \end{aligned} \tag{23}$$

Proposition 10. *Let $\star$ be a hyperoperation on S and α: $S \to [0,1]$ be a map. If $\overset{\alpha}{\star}$ is associative and $\alpha|_{\mathscr{U}} > 0$, then $\star$ is associative.*

Proof. It suffices to show that for all $x, y \in S$,

$$x \star y = \left\{s \in S \mid (x \overset{\alpha}{\star} y)(s) > 0\right\}. \tag{24}$$

To this end, let $x, y, s \in S$. Then, we have

$$\begin{aligned} (x \overset{\alpha}{\star} y)(s) > 0 &\Leftrightarrow \alpha(s)\chi_{x\star y}(s) > 0 \\ &\Leftrightarrow \alpha(s) > 0, s \in x\star y \\ &\Leftrightarrow s \in x\star y \quad (\alpha|_{\mathscr{U}} > 0). \end{aligned} \tag{25}$$

□

3. Adjoint Functors

In this section, two functors between the categories of map-pointed hypergroupoids and fuzzy hypergroupoids are defined. These functors give an adjunction whose restrictions on certain subcategories form an equivalence.

Definition 11. Let $(S, \star, \alpha)$ and $(P, \square, \beta)$ be two map-pointed hypergroupoids. A map f: $S \to P$ is said to be a *homomorphism* if

(1) $f(x\star y) \subseteq f(x)\square f(y)$ for all $x, y \in S$.

(2) $\alpha(s) \le \beta(f(s))$ for all $s \in S$.

Also, f is called a strong homomorphism if

(1) $f(x\star y) = f(x)\Box f(y)$ for all $x, y \in S$.

(2) $\alpha(s) = \beta(f(s))$ for all $s \in S$.

Note that the classes of all map-pointed hypergroupoids with homomorphisms and strong homomorphisms between them form categories denoted by **mHgr** and **smHgr**, respectively.

Definition 12. Let $(S,\circ)$ and $(P,\Box)$ be two fuzzy hypergroupoids. A map $f\colon S\rightarrow P$ is said to be a *homomorphism* if for every $x, y \in S$,

$$f(x\circ y)\leq f(x)\Box f(y), \tag{26}$$

in which $f(x\circ y)$ is a fuzzy subset of P defined by

$$f(x\circ y)(p) = \bigvee_{f(s)=p}(x\circ y)(s), \tag{27}$$

for any $p\in P$ (see [4]). Also, f is called a *strong homomorphism* if

$$f(x\circ y) = f(x)\Box f(y). \tag{28}$$

The categories of all fuzzy hypergroupoids with homomorphisms and strong homomorphisms between them are denoted by **FHgr** and **FsHgr**, respectively.

Lemma 13. *Let $(S,\star,\alpha)$ and $(P,\Box,\beta)$ be two map-pointed hypergroupoids. If $f\colon S\rightarrow P$ is a (strong) homomorphism of map-pointed hypergroupoids, then it is a (strong) homomorphism of fuzzy hypergroupoids with respect to $(S,\overset{\alpha}{\star})$ and $(P,\overset{\beta}{\Box})$. The converse also holds provided that $\alpha(s)\neq 0$ for all $s\in x\star y, x, y\in S$.*

Proof. We suppose that f is a homomorphism of map-pointed hypergroupoids. Let $x, y\in S$. Then, for any $p\in P$, we have

$$\begin{aligned} f(x\overset{\alpha}{\star}y)(p) &= \bigvee_{f(s)=p}(x\overset{\alpha}{\star}y)(s)\\ &= \bigvee_{f(s)=p}\alpha(s)\chi_{x\star y}(s). \end{aligned} \tag{29}$$

Since f is a homomorphism of map-pointed hypergroupoids, we have

$$\alpha(s)\leq\beta(f(s) \text{ and } \chi_{x\star y}(s)\leq\chi_{f(x)\Box f(y)}(f(s)). \tag{30}$$

Then,

$$\begin{aligned} f(x\overset{\alpha}{\star}y)(p) &= \bigvee_{f(s)=p}\alpha(s)\chi_{x\star y}(s)\\ &\leq\beta(p)\chi_{f(x)\Box f(y)}(p)\\ &= (f(x)\overset{\beta}{\Box}f(y))(p). \end{aligned} \tag{31}$$

A similar argument is applied for the case of strong homomorphism by replacing each $\leq$ with equality. Conversely, we suppose that $f\colon(S,\overset{\alpha}{\star})\rightarrow(P,\overset{\beta}{\Box})$ is a homomorphism of fuzzy hypergroupoids where $\alpha(s)\neq 0$ for all $s\in x\star y, x, y\in S$. Let $s\in x\star y$ for $x, y\in S$ and let $p = f(s)$. Since f is a homomorphism, $f(x\overset{\alpha}{\star}y)(p)\leq(f(x)\overset{\beta}{\Box}f(y))(p)$, hence

$$\bigvee_{f(t)=p}(x\overset{\alpha}{\star}y)(t)\leq\beta(p)\chi_{f(x)\Box f(y)}(p), \tag{32}$$

and so since $f(s) = p$, $(x\overset{\alpha}{\star}y)(s)\leq\beta(p)\chi_{f(x)\Box f(y)}(p)$. Therefore, $\alpha(s)\chi_{x\star y}(s)\leq\beta(p)\chi_{f(x)\Box f(y)}(p)$. Since $\chi_{x\star y}(s) = 1$ and $\alpha(s)\neq 0$, $\chi_{f(x)\Box f(y)}(p)\neq 0$ and $\beta(p)\geq\alpha(s)$. Hence, $f(s)\in f(x)\Box f(y)$ and $\alpha(s)\leq\beta(f(s))$. □

Theorem 14. *The assignment defined by $(S,\star,\alpha)\rightsquigarrow(S,\overset{\alpha}{\star})$ on objects and $f\rightsquigarrow f$ on morphisms gives us functors denoted by F: **mHgr** $\rightarrow$ **FHgr** and F_s: **smHgr** $\rightarrow$ **FsHgr**.*

Proof. This is proved by Lemma 13. □

Lemma 15. *Let $(S,\circ)$ and $(P,\Box)$ be two fuzzy hypergroupoids. Let $\otimes$ and $\boxtimes$ be the hyperoperations with respect to $\circ$ and $\Box$. If $f\colon(S,\circ)\rightarrow(P,\Box)$ is a homomorphism of fuzzy hypergroupoids, then $f\colon(S,\otimes,\widehat{\circ})\rightarrow(P,\boxtimes,\widehat{\Box})$ is a homomorphism of map-pointed hypergroupoids.*

Proof. Let $x, y\in S$. First, we prove that $f(x\otimes y)\subseteq f(x)\boxtimes f(y)$. Let $t\in x\otimes y$ so that $(x\circ y)(t)>0$. Let $p = f(t)$, then we have

$$\begin{aligned} (f(x)\Box f(y))(p) &\geq (f(x\circ y))(p)\\ &= \bigvee_{f(s)=p}(x\circ y)(s)\\ &\geq (x\circ y)(t)>0. \end{aligned} \tag{33}$$

Hence, $f(t) = p\in f(x)\boxtimes f(y)$. This means that $f(x\otimes y) = f(x)\boxtimes f(y)$. Now, we prove that $\widehat{\circ}(s)\leq\widehat{\Box}(f(s))$. Let $p = f(s)$, then we have

$$\begin{aligned} \widehat{\Box}(f(s)) = \widehat{\Box}(p) &= \bigvee_{w,z\in P}(w\Box z)(p)\\ &\geq\bigvee_{x,y\in S}(f(x)\Box f(y))(p)\\ &\geq\bigvee_{x,y\in S}(f(x\circ y))(p)\\ &= \bigvee_{x,y\in S}\bigvee_{f(s)=p}(x\circ y)(s)\\ &\geq\bigvee_{x,y\in S}(x\circ y)(s)\\ &= \widehat{\circ}(s). \end{aligned} \tag{34}$$

□

Theorem 16. *The assignment defined by $(S,\circ)\rightsquigarrow(S,\otimes,\widehat{\circ})$ on objects and $f\rightsquigarrow f$ on morphisms gives us a functor denoted by G: **FHgr** $\rightarrow$ **mHgr**.*

Proof. This follows from Lemma 15.

In the following, ${}^{\alpha}\otimes$ denotes the hyperoperation with respect to the fuzzy hyperoperation $\overset{\alpha}{\star}$, that is,

$$x^{\alpha}\otimes y = \left\{s\in S\mid(x\overset{\alpha}{\star}y)(s)>0\right\}. \tag{35}$$

□

Proposition 17. *Let* $(S, \star, \alpha) \in$ **mHgr** *and* $(S, \circ) \in$ **FHgr**. *Then,*

(i) id_S: $GF(S, \star, \alpha) \to (S, \star, \alpha)$ *is a morphism in mHgr*.

(ii) id_S: $(S, \circ) \to FG(S, \circ)$ *is a morphism in* **FHgr**.

Proof

(i) We have $GF(S, \star, \alpha) = (S, {}^{\alpha}\otimes, \widehat{\overset{\alpha}{\star}})$, and for any $x, y \in S$,

$$\begin{aligned} x \overset{\alpha}{\otimes} y &= \left\{s \in S \mid (x \overset{\alpha}{\star} y)(s) > 0\right\} \\ &= \left\{s \in S \mid \alpha(s)\chi_{x\star y}(s) > 0\right\} \\ &= \left\{s \in x\star y \mid \alpha(s) > 0\right\} \\ &\subseteq x\star y. \end{aligned} \tag{36}$$

Also, for any $s \in S$,

$$\begin{aligned} \widehat{\overset{\alpha}{\star}}(s) &= \bigvee_{x,y\in S} (x \overset{\alpha}{\star} y)(s) \\ &= \bigvee_{x,y\in S} \alpha(s)\chi_{x\star y}(s) \\ &\le \alpha(s) \\ &= \alpha(\mathrm{id}_S(s)). \end{aligned} \tag{37}$$

(ii) Note that $FG(S, \circ) = (S, \widehat{\overset{\circ}{\otimes}})$. Let $x, y, s \in S$. Then, we have

$$\begin{aligned} (x \overset{\circ}{\otimes}{}^{\widehat{\circ}} y)(s) &= \widehat{\circ}(s)\chi_{x\otimes y}(s) \\ &= \bigvee_{w,z\in S} (w \circ z)(s)\chi_{x\otimes y}(s) \\ &\ge (x \circ y)(s)\chi_{x\otimes y}(s) \\ &= (x \circ y)(s). \end{aligned} \tag{38}$$

□

Definition 18. A map-pointed hypergroupoid $(S, \star, \alpha)$ is called *super* if

$$\alpha(s) > 0 \Leftrightarrow s \in \mathscr{U}. \tag{39}$$

Theorem 19. *The following statements are equivalent:*

(i) id_S: $GF(S, \star, \alpha) \to (S, \star, \alpha)$ *is an isomorphism in* **mHgr**.

(ii) id_S: $(S, \star, \alpha) \to GF(S, \star, \alpha)$ *is a morphism in* **mHgr**.

(iii) id_S: $GF(S, \star, \alpha) \to (S, \star, \alpha)$ *is a morphism in* **smHgr**.

(iv) $(S, \star, \alpha)$ *is a super map-pointed hypergroupoid.*

Proof. The implications (iii) ⇒ (i) ⇒ (ii) are trivial, and (ii) ⇒ (iii) follows from Proposition 17(i). For (iii) ⇒ (iv), we have $GF(S, \star, \alpha) = (S, \overset{\alpha}{\otimes}, \widehat{\overset{\alpha}{\star}})$. So, id_S: $(S, {}^{\alpha}\otimes, \widehat{\overset{\alpha}{\star}}) \to (S, \star, \alpha)$ is a strong homomorphism by the assumption. Then, ${}^{\alpha}\otimes = \star$ and $\widehat{\overset{\alpha}{\star}} = \alpha$. Suppose that $\alpha(s) > 0$ for $s \in S$. Then, $\widehat{\overset{\alpha}{\star}}(s) > 0$. So, $\alpha(s)\chi_{\mathscr{U}}(s) > 0$ and $s \in \mathscr{U}$. Now, let $s \in \mathscr{U}$. Then, there exist $x, y \in S$ with $s \in x \star y = x \overset{\alpha}{\otimes} y$. Hence, $0 < (x \overset{\alpha}{\star} y)(s) = \alpha(s)\chi_{x\star y}(s)$ whence $\alpha(s) > 0$. For (iv) ⇒ (iii), let $(S, \star, \alpha)$ be super. It must be shown that ${}^{\alpha}\otimes = \star$ and $\widehat{\overset{\alpha}{\star}} = \alpha$. Let $x, y \in S$. Then,

$$\begin{aligned} x \overset{\alpha}{\otimes} y &= \left\{s \in S \mid (x \overset{\alpha}{\star} y)(s) > 0\right\} \\ &= \left\{s \in S \mid \alpha(s)\chi_{x\star y}(s) > 0\right\} \\ &= \left\{s \in S \mid \alpha(s) > 0, s \in x \star y\right\} \\ &= \left\{s \in S \mid s \in \mathscr{U}, s \in x \star y\right\} \\ &= x \star y. \end{aligned} \tag{40}$$

Also, by Proposition 7 (ii), $\widehat{\overset{\alpha}{\star}} = \alpha$. □

Theorem 20. *The following statements are equivalent:*

(i) id_S: $(S, \circ) \to FG(S, \circ)$ *is an isomorphism in* **FHgr**.

(ii) id_S: $FG(S, \circ) \to (S, \circ)$ *is a morphism in* **FHgr**.

(iii) id_S: $(S, \circ) \to FG(S, \circ)$ *is a morphism in* **FsHgr**.

(iv) $(S, \circ)$ *is a smooth fuzzy hypergroupoid.*

Proof. The implications (iii) ⇒ (i) ⇒ (ii) are trivial, and (ii) ⇒ (iii) follows from Proposition 17 (ii). For (iii) ⇔ (iv), we have $FG(S, \circ) = (S, \widehat{\overset{\circ}{\otimes}})$. So, id_S: $(S, \circ) \to (S, \widehat{\overset{\circ}{\otimes}})$ is strong if and only if $\circ = \widehat{\overset{\circ}{\otimes}}$, which is equivalent to the smoothness of $\circ$ by Corollary 8. □

Notation 21. The subcategory of all super map-pointed hypergroupoids is denoted by $\mathscr{C}$, and the subcategory of all smooth fuzzy hypergroupoids is denoted by $\mathscr{D}$.

Theorem 22. *The restriction of F on* $\mathscr{C}$ *and G on* $\mathscr{D}$ *make an equivalence between* $\mathscr{C}$ *and* $\mathscr{D}$. *In other words,* $\mathscr{C} \approx \mathscr{D}$.

Proof. By Theorem 19, $GF \overset{\mathrm{id}_S}{\to} 1$ is a natural isomorphism on $\mathscr{C}$. Also, by Theorem 20, $1 \overset{\mathrm{id}_S}{\to} FG$ is a natural isomorphism on $\mathscr{D}$.

Theorem 22 is in fact a part of a more general fact, as follows: □

Theorem 23. *G is a left adjoint to F.*

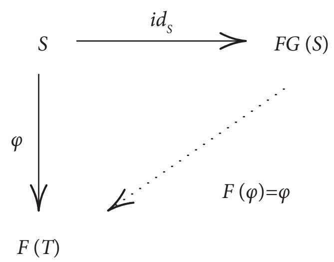
Proof. Let φ: $(S, \circ) \to F(T, \star, \alpha) = (T, \overset{\alpha}{\star})$ be a morphism in **FHgr**. We prove that φ: $G(S, \circ) = (S, \otimes, \widehat{\circ}) \to (T, \star, \alpha)$ is a morphism in **mHgr**. Let $x, y \in S$ and $s \in x \otimes y$. Also, let $t = \varphi(s)$, then we have

$$\begin{aligned} s \in x \otimes y &\Rightarrow (x \circ y)(s) > 0 \\ &\Rightarrow \varphi(x \circ y)(t) > 0 \\ &\Rightarrow (\varphi(x) \overset{\alpha}{\star} \varphi(y))(t) > 0 \\ &\Rightarrow \alpha(t)\chi_{\varphi(x)\star\varphi(y)}(t) > 0 \\ &\Rightarrow t \in \varphi(x)\star\varphi(y). \end{aligned} \tag{41}$$

Also, it must be shown that $\widehat{\circ}(s) \le \alpha(\varphi(s))$, that is,

$$\begin{aligned}\widehat{\circ}\,(s) &= \bigvee_{x,y\in S}(x\circ y)(s)\\ &\le \bigvee_{x,y\in S}\varphi(x\circ y)(t)\\ &\le \bigvee_{x,y\in S}(\varphi(x)\overset{\alpha}{\star}\varphi(y))(t)\\ &= \bigvee_{x,y\in S}\alpha(t)\chi_{\varphi(x)\star\varphi(y)}(t)\\ &\le \alpha(t)\\ &= \alpha(\varphi(s)).\end{aligned}\tag{42}$$

Moreover, it is clear that $\varphi\colon G(S)\to T$ is the unique homomorphism which completes the following diagram:



(43)

To prove the other side of adjunction, let $\psi\colon G(T,\circ)\to(S,\star,\alpha)$ be a morphism in **mHgr**. We prove that $\psi\colon (T,\circ)\to F(S,\star,\alpha)=(S,\overset{\alpha}{\star})$ is a morphism in **FHgr**. Let $x,y\in T$. So, for every $s\in S$, we have

$$\begin{aligned}\psi(x\circ y)(s) &= \bigvee_{\psi(t)=s}(x\circ y)(t)\\ &\le \widehat{\circ}\,(s)\\ &\le \alpha(s).\end{aligned}\tag{44}$$

If $\psi(x\circ y)(s)=0$, then we have

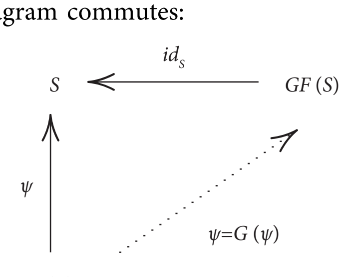
$$\psi(x\circ y)(s)\le(\psi(x)\overset{\alpha}{\star}\psi(y))(s).\tag{45}$$

Suppose that $\psi(x\circ y)(s)>0$. So, there exists $t\in T$ such that $(x\circ y)(t)>0$ and $\psi(t)=s$.

We have $t\in x\otimes y$. Hence, $s=\psi(t)\in\psi(x\otimes y)\subseteq\psi(x)\star\psi(y)$. Therefore, $\chi_{\psi(x)\star\psi(y)}(s)=1$ and

$$\begin{aligned}\psi(x\circ y)(s) &\le \alpha(s)\\ &= \alpha(s)\chi_{\psi(x)\star\psi(y)}(s)\\ &= (\psi(x)\overset{\alpha}{\star}\psi(y))(s).\end{aligned}\tag{46}$$

Hence, ψ is a fuzzy homomorphism. This means that $\psi\colon T\to F(S)$ is the unique homomorphism such that the following diagram commutes:



(47)

□

4. Conclusion

In this paper, we establish a correspondence between map-pointed hypergroupoids and fuzzy hypergroupoids. Specifically, for every map-pointed hypergroupoid $(S,\star,\alpha)$, where $\star$ is a hyperoperation on S and $\alpha\colon S\to[0,1]$ is a fuzzy subset, a fuzzy hyperoperation denoted by $\overset{\alpha}{\star}\colon S\times S\to F(S)$ is defined. This new operation is called a map-pointed fuzzy hyperoperation. We demonstrate that any fuzzy hyperoperation represented as a map-pointed fuzzy hyperoperation of the form $\overset{\alpha}{\star}$ must possess a property known as smoothness and vice versa. Conversely, for every fuzzy hypergroupoid $(S,\circ)$, a hyperoperation $\otimes$ on S and a fuzzy subset on S, denoted by $\widehat{\circ}\colon S\to[0,1]$, are defined. These two assignments give us two functors that are inverse to each other in the sense of adjunction in category theory.

Overall, this correspondence sheds light on the relationship between hypergroupoids and fuzzy hypergroupoids, providing insight into how these structures can be related through the lens of fuzzy set theory and category theory.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] F. Marty, "Sur une généralisation de La notion de group," in *Proceedings of the 4th Congress Math. Scandinaves*, pp. 45–49, Stockholm, Sweden, December 1934.

[2] A. Rosenfeld, "Fuzzy groups," *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512–517, 1971.

[3] R. Corsini and I. Tofan, "On fuzzy hypergroups," *Pure Mathematics and Applications*, vol. 8, pp. 29–37, 1997.

[4] M. K. Sen, R. Ameri, and G. Chowdhury, "Fuzzy hypersemigroups," *Soft Computing*, vol. 12, no. 9, pp. 891–900, 2008.

[5] V. Leoreanu-Fotea, "Fuzzy hypermodules," *Computers and Mathematics with Applications*, vol. 57, no. 3, pp. 466–475, 2009.

[6] V. Leoreanu-Fotea and B. Davvaz, "Fuzzy hyperrings," *Fuzzy Sets And Systems*, vol. 160, no. 16, pp. 2366–2378, 2009.

[7] J. Adamek, H. Herrlich, and G. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, New York, NY, USA, 1990.