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Research Article

Decision-Making Based on Spherical Linear Diophantine Fuzzy Rough Aggregation Operators and EDAS Method

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In everyday life, decision-making is a difficult task fraught with ambiguity and uncertainty. Many researchers and scholars have suggested numerous fuzzy set theories to resolve these ambiguities and uncertainties. The EDAS method (evaluation based on distance from average answer) is extremely beneficial in decision-making situations. In multi-attribute group decision-making (MAGDM) situations, this is especially true when there are more competing criteria. In this paper, we introduce the concept of spherical linear diophantine fuzzy rough sets (SLDFRSs). We develop basic operational laws and a number of propose aggregation operators. Furthermore, the necessary and desired features of SLDFRS are explored. This study proposes a new technique known as the spherical linear diophantine fuzzy rough set EDAS (SLDFRS-EDAS) method to deal with these uncertainties in the MAGDM problem. A MAGDM technique is intended to evaluate the emergency system based on the newly introduced operators. Furthermore, a comparison study of the activity and applicability of the suggested approach with earlier procedures is used to validate its viability and applicability.

1. Introduction

1.1. Research Motivation. Decision-making (DM) is a common task in everyday life. The primary goal of DM is to choose the best and most desirable choice among a set of reasonable alternatives. The fundamental components of decision-making are the decision-makers' judgment and selected information. The combination of judgment information and the technique for presenting results are two significant difficulties in the DM process. A number of ideas and methodologies from various viewpoints are offered in the intended value selection process. Using traditional techniques, the DM professionals had to define their applicable value in precise numerical terms. If experts describe their data numerically, the calculation becomes simple and straightforward. Due to the increasing complexity and ambiguity in the field of DM, we must deal with many sorts of uncertain information that are erroneous, partial, and occasionally uncertain in nature. Many concepts and theories are defined for a good solution to the DM problem.

1.2. Literature Review. Zadeh's [1] fuzzy set (FS) notion is one of them. Atanassove [2] developed the intuitionistic fuzzy set (IFS) notion, which is an enlarged form of the fuzzy set theory. All IFS members are represented by an ordered pair, which is further categorized by a membership degree (MD) and a nonmembership degree (NMD). Furthermore, the sum of the MD and NMD is equal to or less than 1. The IFS concept has received a lot of attention since its discovery. The IFS theory, on the other hand, fails to characterize evaluation information; for example, if an expert assesses his assessment information to be $0.6 + 0.6 > 1$, the IFS technique will not solve the problem. Deabes and Amin [3] image reconstruction algorithm is based on the PSO-tuned fuzzy inference system for electrical capacitance tomography. Fallatah and Ikram [4] developed the selecting the right ERP system for SMEs: an intelligent ranking engine of cloud SAAS service providers based on fuzziness quality attributes.

As a result, Yager [5] for the first time developed the notion of the Pythagorean fuzzy set (PFS) to cope with undefined viewpoints in DM difficulties. PFS is also beneficial for presenting uncertainty in multi-criteria decision-making (MCDM) issues. Because it satisfies the criteria $\mu_Q^2 + \eta_Q^2 \leq 1$, the PFS idea can manage the unpredictably in the DM problem better than IFS. PFS is also classified using the MD and NMD, whose total squares are equal to or less than 1. As a result, the PFS is more general than the IFS. The PFS is capable of resolving difficulties that the IFS cannot. To summarize, the PFS is more universal and all IFSs are considered a part of the PFSs and so included in it. The Pythagorean fuzzy number (PFN) was proposed by Zhang and Xu [6]. They developed the PFS mathematical form and the Pythagorean fuzzy TOPSIS algorithm. This approach was used within PFNs to tackle the MCDM challenge. Peng and Yang [7] presented a Pythagorean fuzzy inferiority and superiority strategy using PFNs to tackle the MCGDM problem. They have also defined PFN subtraction and division operations. Gou et al. [8] studied the properties of continuous PF data. Ren et al. [9] solved the MCDM problem using the TODIM method under the PFNs. Sarkis and Bai [10] Green supplier development and analytical assessment were defined using rough set theory. Al-Shami et al. [11] defined new generalization of fuzzy soft sets: (a, b)-fuzzy soft sets. Jin et al. [12] proposed a reliable wireless communication mechanisms and decision support system for the IoT networks.

Despite this, Yager [13] devised q-rung orthopair fuzzy sets (q-ROFSs) to show more decision information and the sum of the qth powers of MD and NMD is less than or equal to 1, i.e., $\mu_Q^q + \eta_Q^q \leq 1$. Liu and Wang [14] developed some averaging and geometric AOs for the q-ROFS. Wei et al. [15] presented some q-ROF Heronian mean (HM) operators. Ali [16] defined two new algorithms to deal with the q-ROFSs. Yager et al. [17] investigated the concepts of possibility, certainty, plausibility, and belief in the q-ROFS data. Yang and Pang [18] devised a novel partitioned Bonferroni mean (BM) operator using q-ROF information. Xu et al. [19] present some q-rung dual hesitant orthopair fuzzy HM operators. Lei and Xu [20] suggested a strategy for the MCDM problem utilizing q-rung interval-valued orthopair fuzzy data for the green supplier q-ROFRH operator using q-rung interval-valued orthopair fuzzy data. Xing et al. [21] gave the point operators in the case of q-ROFSs. Garg and Chen [22] q-ROFS neutrality AOs for dealing with group decision-making (DM) difficulties were recently presented. Gao et al. [23] developed the concept of continuities and differentials of q-ROF functions. Ye et al. [24] studied differential calculus under q-ROFSs. Peng et al. [25] established the exponential and logarithmic operational rules for q-ROFSs. The q-ROFS is obviously more general than the IFS, allowing for more ambiguous information to be transmitted. Thus, the q-ROFS can handle MCGDM problems that IFS cannot, and since IFS is a component of the q-ROFS, the q-ROFS is more effective and powerful in dealing with fuzzy and uncertain DM problems. Al-Shami and Mhemdi [26] proposed a generalized frame for orthopair fuzzy sets: (m, n)-fuzzy sets and their applications to MCDM methods. Liu and Liu [27] investigated the DM problem in a q-rung fuzzy environment using the Bonferroni average operator. Liu and Wang [28] examined the application of Archimedean norm-based DM. Al-Shami [29] defined a (2, 1)-fuzzy sets: properties, weighted aggregated operators, and their applications to multi-criteria decision-making methods. Dalkılıç [30] developed an approach that takes into account interactions between parameters: pure (fuzzy) soft sets. Dalkılıç [31] proposed two novel approaches that reduce the effectiveness of the decision maker in decision making under uncertainty environments. Dalkılıç and Demirtaş [32] defined a novel perspective for Q-neutrosophic soft relations and their application in decision-making. Dalkılıç [33] determines the membership degrees in the range (0, 1) for hypersoft sets independently of the decision-maker. Dalkılıç [34] defined a novel approach to soft set theory in decision-making under uncertainty.

The idea of spherical fuzzy set (SFS) was first proposed by Ashraf et al. [35] to address this issue that picture fuzzy set (PFS) cannot handle. They discovered a plethora of spherical fuzzy information-based aggregation approaches. In contrast to PFSs, where all membership degrees must fulfil the criterion $\mu_Q + \upsilon_Q + \eta_Q \leq 1$, SFSs require $\mu_Q^2 + \upsilon_Q^2 + \eta_Q^2 \leq 1$. Ashraf et al. [36] defined Spherical fuzzy aggregation operators using t-norm and t-conorms. Kutlu Gundogdu and Kahraman [37] proposed spherical fuzzy TOPSIS method. Rafiq et al. [38] defined cosine similarity measures of SFSs and their applications in DM. Deli and Çagman presented the concept of spherical fuzzy numbers and an MCDM method using spherical fuzzy set data in [39]. Qiyas et al. [40] defined spherical fuzzy AOs with sine trigonometry and its application in DM problem. Qiyas et al. [41] Hamacher AOs for spherical uncertain linguistics were defined, and their application in group DM to achieve consistent opinion fusion was investigated. Abdullah et al. [42] analyzed the decision support system based on 2-tuple spherical fuzzy linguistic aggregation information. Jin et al. [43] defined linguistic spherical fuzzy AOs and their applications in MADM problems.

In 2019, Riaz and Hashmi [44] critically analyzed the restrictions associated with membership and nonmembership functions in FS, IFS, PyFS, and q-ROFS structures, and these limitations were statistically highlighted. To overcome these constraints, they developed the linear diophantine fuzzy set (LDFS) by adding reference parameters to the structure of IFS. They claimed that the LDFS idea will eliminate limits in present methodologies for other sets and allow for the free selection of data in practice. In the creation of LDFS, the grades of membership, nonmembership, and reference parameters are actually valued. Almagrabi et al. [45] defined various linear diophantine fuzzy similarity metrics and used them to solve a DM problem. Qiyas et al. [46] presented the concept of q-rung linear diophantine fuzzy set-based similarity metrics and their use in logistics and supply chain management. Hanif et al. [47] created linear diophantine fuzzy graphs using a novel decision-making method. Riaz et al. [48] developed spherical linear diophantine fuzzy sets with modelling uncertainty. Hashmi et al. [49] introduced spherical linear diophantine fuzzy soft rough sets with multi-criteria decision-making.

Keshavarz Ghorabaee et al. [50] were the first to investigate the EDAS method for dealing with DM issues. This technique works well in DM scenarios, particularly when the conflicting requirements are more rigorous than those in MCDM problems. The EDAS approach's goal is to select the best option among a large number of alternatives by employing PDAS (positive distance from average solution) and NDAS (negative distance from average solution) based on the average solution (AS). These two phases highlight the differences between each solution and the AS. As a result, the superior candidate should have a lower NDAS and a higher PDAS. Feng et al. [51] conducted research on EDAS techniques based on the hesitant fuzzy idea. Li et al. [52] created the hybrid operator and examined its implementation in DM utilising the EDAS approach. The IF region was subjected to an EDAS system study, and an energy-saving scheme was conducted by Liang [53]. For the site selection problem based on IF information, Kahraman [54] used the EDAS technique. Ilieva [55] developed the EDAS approach for group diabetes using interval fuzzy data. Karasan and Kahraman [56] developed an EDAS technique based on interval-valued neutrosophic data. Stanujkic et al. [57] built the EDAS technique using the grey number specification. Mehdi [58] developed a characterization for the rank reversal phenomenon and used the EDAS methods to study its combined analysis. Hedar et al. [59] defined simulation-based EDAs for stochastic programming problems.

1.3. Necessity of the Research. Spherical fuzzy set and spherical fuzzy rough set notions have several applications in diverse domains of real life, but both theories have their own limits linked to membership and nonmembership grades. To overcome these constraints, we offer the innovative idea of linear diophantine fuzzy set (LDFS) with reference parameters. Because of the usage of reference parameters, the suggested LDFS model is more efficient and adaptable than existing techniques. By modifying the physical sense of reference parameters, LDFS can also categorise data in MADM difficulties. With the use of reference parameters, this set covers the spaces of current structures and expands the space for membership and neutral and nonmembership grades.

To the best of our knowledge, the linear diophantine fuzzy set is not defined in terms of the spherical fuzzy rough set. For addressing some kind of issues, we explore the idea of spherical linear diophantine fuzzy rough sets (SLDFRSs). The membership degree (MD), neutral membership degree (NuMD), and nonmembership degree (NMD) are constrained by the SFS concepts, which have wide applications in a variety of spheres of daily life. To address these difficulties, we created a brand-new extended concept of SLDFRSs. In the SLDFRS framework that has been proposed, there are three membership degrees with the linear parameter.

1.4. Main Contributions. The primary objectives of the work are listed as follows:

(1) To construct a new notion of spherical linear diophantine fuzzy rough sets and analyze their basic operational laws.

(2) The concept of SLDFRS has been utilized to express the uncertainties in the data.

(3) To develop several weighted aggregation operators to aggregate the collective information.

(4) Some special cases of the proposed operators are deduced under the existing environment.

(5) To establish an MAGDM method based on the proposed operators to solve the problems.

(6) To show the significance and superiority of proposed aggregation operators over existing aggregation operators numerically.

1.5. Structure of the Paper. The study's structure is as follows: Section 2 delves into the fundamental definitions and ideas, such as FS, IFS, and PFS. Section 3 discusses the concept of SLDFRS, methods for comparison, basic operational rules, and several key theorems. Section 4 presented new operators, such as SLDFRWA, SLDFROWA, and SLDFRHA, and explored their desirable properties. In Section 5, we defined and examined new operators such as SLDFRWG, SLDFROWG, and SLDFRHG operators. In Section 6, we introduced the SLDFR-EDAS method for MAGDM and showed their technique. Section 7 provides a numerical illustration of emergency programs selection. A comprehensive comparison of the analyzed models with numerous recognized methodologies is also offered, proving that the studied model is more efficient and advantageous than current procedures. Section 8 is where we write the article's conclusion.

2. Preliminaries

The aim of this part is to present the preexisting basic definitions for SLDFS, rough set, and several related notions in a concise manner.

Definition 1 [1]. A fuzzy set Q on fixed set N is given by

$$Q = \{(\nu, \mu_Q(\nu)) \mid \nu \in N\}, \tag{1}$$

where $\mu_Q(\nu)$: $N \longrightarrow [0,1]$ is a membership function of Q.

Definition 2 [2]. An intuitionistic fuzzy set Q on fixed set N is given by

$$Q = \{\langle \nu, \mu_Q(\nu), \upsilon_Q(\nu)\rangle \mid \nu \in N\}. \tag{2}$$

Here, $\mu_Q(v): N \longrightarrow [0,1]$ and $v_Q(v): N \longrightarrow [0,1]$ show the MD and NMD of an alternative $v \in N$ with the condition $0 \leq \mu_Q + v_Q(v) \leq 1$. Furthermore, hesitant degree is given as

$$\pi_Q(v) = 1 - \mu_Q(v) - v_Q(v). \quad (3)$$

Definition 3 [5]. A Pythagorean fuzzy set Q on fixed set N is given by

$$Q = \left\{\left\langle\left(v, \mu_Q(v), v_Q(v)\right\rangle \mid v \in N\right\}. \quad (4)$$

Here, $\mu_Q(v): N \longrightarrow [0,1]$ and $v_Q(v): N \longrightarrow [0,1]$ show the MD and NMD of an alternative $v \in N$ with the condition $0 \leq \mu_Q^2 + v_Q^2(v) \leq 1$. Furthermore, hesitant degree is given as

$$\pi_Q(v) = \sqrt{1 - \mu_Q^2(v) - v_Q^2(v)}. \quad (5)$$

Definition 4 [13]. A q-ROFS Q on fixed set N is given by

$$Q = \left\{\left\langle\left(v, \mu_Q(v), v_Q(v)\right\rangle \mid v \in N\right\}. \quad (6)$$

Here, $\mu_Q(v): N \longrightarrow [0,1]$, and $v_Q(v): N \longrightarrow [0,1]$ show the MD and NMD of an alternative $v \in N$ with the condition $0 \leq \mu_Q^q + v_Q^q(v) \leq 1$. Furthermore, hesitant degree is given as

$$\pi_Q(v) = \sqrt{1 - \mu_Q^q(v) - v_Q^q(v)}. \quad (7)$$

Definition 5 [44]. A A LDFS G on fixed set N is given by

$$G = \{(v, \langle u_G(v), v_G(v)\rangle, \langle \alpha, \beta\rangle) \mid v \in N\}, \quad (8)$$

where $u_G, v_G, \alpha, \beta \in [0,1]$ are MD, NMD, and RPs, respectively, and satisfy the condition $0 \leq (\alpha)u_G + (\beta)v_G \leq 1$, $\forall v \in M$ with $0 \leq \alpha + \beta \leq 1$. The hesitant grade is given as

$$\pi_D = 1 - (\alpha)u_G(v) - (\beta)v_G(v). \quad (9)$$

Definition 6 [60]. Let N be a fixed set and $Q \in N \times N$ be an arbitrary relation on N. A set valued function $Q^*: N \longrightarrow \mathfrak{I}(N)$ is defined as

$$Q^*(v) = \{v \in N \mid (v, c) \in Q\} \text{ for } v \in N, \quad (10)$$

where $Q^*(v)$ is the successor neighborhood of the object v with respect to Q and the pair (N, Q) is called crisp approximation space. Now, lower and upper approximation of the $\Bbbk\,(\Bbbk \subseteq N)$ based on approximation space (N, Q) is given as

$$\begin{aligned} \underline{Q}(\Bbbk) &= \left\{v \in N \,\middle|\, Q^*(v) \subseteq \Bbbk\right\}, \\ \overline{Q}(\Bbbk) &= \left\{v \in N \,\middle|\, Q^*(v) \cap \Bbbk \neq \phi\right\}. \end{aligned} \quad (11)$$

Thus, $(\underline{Q}(\Bbbk), \overline{Q}(\Bbbk))$ is called the rough set and $\underline{Q}(\Bbbk), \overline{Q}(\Bbbk): \mathfrak{I}(N) \longrightarrow \mathfrak{I}(N)$ are the lower approximation and upper approximation operators.

Definition 7 [61]. Let $Q \in \mathrm{IFS}(N \times N)$ be a subset of intuitionistic fuzzy relation. The pair (N, Q) is known as intuitionistic fuzzy approximation space. For any upper approximation and lower approximation $\Bbbk\,(\Bbbk \subseteq \mathrm{IFS}(N))$, with respect to intuitionistic fuzzy approximation space, (N, Q) have two IFSs, $(\overline{Q}(\Bbbk)$ and $\underline{Q}(\Bbbk))$ defined as

$$\begin{aligned} \overline{Q}(\Bbbk) &= \left\{\left(v, \mu_{\overline{Q}(\Bbbk)}(v), v_{\overline{Q}(\Bbbk)}(v) \mid v \in N\right\}, \\ \underline{Q}(\Bbbk) &= \left\{\left(v, \mu_{\underline{Q}(\Bbbk)}(v), v_{\underline{Q}(\Bbbk)}(v) \mid v \in N\right\}, \end{aligned} \quad (12)$$

where

$$\begin{aligned} \mu_{\overline{Q}(\Bbbk)}(v) &= \bigvee_{c \in v}\left\{\mu_Q(v, c) \vee \mu_\Bbbk(c)\right\}, \\ v_{\overline{Q}(\Bbbk)}(v) &= \bigwedge_{c \in v}\left\{v_Q(v, c) \wedge v_\Bbbk(c)\right\}, \\ \mu_{\underline{Q}(\Bbbk)}(v) &= \bigwedge_{c \in v}\left\{\mu_Q(v, c) \wedge \mu_\Bbbk(c)\right\}, \\ v_{\underline{Q}(\Bbbk)}(v) &= \bigvee_{c \in v}\left\{v_Q(v, c) \vee v_\Bbbk(c)\right\}. \end{aligned} \quad (13)$$

Such that $0 \leq \mu_{\overline{Q}(\Bbbk)}(v) + v_{\overline{Q}(\Bbbk)}(v) \leq 1$ and $0 \leq \mu_{\underline{Q}(\Bbbk)}(v) + v_{\underline{Q}(\Bbbk)}(v) \leq 1$, $\overline{Q}(\Bbbk)$ and $\underline{Q}(\Bbbk)$ are IFSs and $\underline{Q}(\Bbbk), \overline{Q}(\Bbbk): \mathfrak{I}(N) \longrightarrow \mathfrak{I}(N)$ are lower and upper approximation operators. Then, the pair is called IFRS.

$$\begin{aligned} Q(\Bbbk) &= \left(\underline{Q}(\Bbbk), \overline{Q}(\Bbbk)\right) \\ &= \left\{\left(v, \left(\mu_{\underline{Q}(\Bbbk)}(v), v_{\underline{Q}(\Bbbk)}(v)\right), \left(\mu_{\overline{Q}(\Bbbk)}(v), v_{\overline{Q}(\Bbbk)}(v)\right) \mid v \in N\right\}. \end{aligned} \quad (14)$$

Definition 8 [45]. A q-RLDFS T on fixed set N is given by

$$T = \{(v, \langle \mu_T(v), v_T(v)\rangle, \langle \alpha, \beta\rangle) \mid v \in N\}, \quad (15)$$

where $\mu_T(v), v_T(v), \alpha, \beta \in [0,1]$ are MD, NMD, and reference parameters (RPs), respectively. They also satisfy restrictions $0 \leq (\alpha)^q \mu_T + (\beta)^q v_T \leq 1$, $\forall v \in N, q \geq 1$, with

$0 \le \alpha^q + \beta^q \le 1, \quad q \ge 1$. Such reference parameters can aid in the description or identification of a particular model. The hesitant grade is given as

$$\pi_D = \sqrt{[q]}1 - \left((\alpha)^q \mu_T + (\beta)^q \upsilon_T\right). \tag{16}$$

Definition 9 [48]. A SLDFS T on fixed set N is given by

$$T = \{(v, \langle \mu_T(v), \upsilon_T(v), \eta_T(v)\rangle, \langle \alpha(v), \beta(v), \gamma(v)\rangle) \mid v \in N\}, \tag{17}$$

where $\mu_T(v), \upsilon_T(v), \eta_T(v), \alpha(v), \beta(v), \gamma(v) \in [0,1]$ are MD, NuMD, NMD, and reference parameters (RPs), respectively. They also satisfy restriction $0 \le (\alpha)^2 \mu_T + (\beta)^2 \upsilon_T + (\gamma)^2 \eta_T \le 1, \quad \forall v \in N$, with $0 \le \alpha + \beta + \gamma \le 1$. The hesitant degree is given as

$$\pi_D = \sqrt{1 - \left((\alpha)^2 \mu_T + (\beta)^2 \upsilon_T + (\gamma)^2 \eta_T\right)}. \tag{18}$$

Definition 10 [48]. A spherical linear diophantine fuzzy number (SLDFN) is defined as

$$T = \{\langle \mu_T, \upsilon_T, \eta_T\rangle, \langle \alpha, \beta, \gamma\rangle\}, \tag{19}$$

where T represent the SLDFN with the following conditions;

(1) $0 \le (\alpha)^2 + (\beta)^2 + (\gamma)^2 \le 1$;
(2) $0 \le (\alpha)^2 \mu_T(v) + (\beta)^2 \upsilon_T(v) + (\gamma)^2 \eta_T(v) \le 1$;
(3) $0 \le \alpha + \beta + \gamma \le 1$.

Definition 11 [61]. Let N be a universal set and $Q \in (N \times N)$ be spherical fuzzy relation. Then, we have

(i) Q is reflexive if $\mu_Q(v,v) = 1$ and $\upsilon_Q(v,v) = 0, \quad \forall Q \in N$
(ii) Q is symmetric if $\forall (v,c) \in (N \times N), \mu_Q(v,c) = \mu_Q(c,v), \quad \upsilon_Q(v,c) = \upsilon_Q(c,v)$, and $\eta_Q(v,c) = \eta_Q(c,v)$
(iii) Q is transitive if $\forall (v,d) \in (v \times v), \mu_Q(v,d) \ge \bigvee_{c \in N} [\mu_Q(v,c) \wedge \mu_Q(c,d)], \quad \upsilon_Q(v,d) \ge \bigvee_{c \in N} [\upsilon_Q(v,c) \wedge \upsilon_Q(c,d)]$, and $\eta_Q(v,d) \ge \bigvee_{c \in N} [\eta_Q(v,c) \wedge \eta_Q(c,d)]$

3. Spherical Linear Diophantine Fuzzy Rough Set

This part illustrates a hybrid notion of SLDFS and rough sets. Moreover, the basic technique, score function, accuracy function, and essential operating laws of SLDFRS are addressed.

Definition 12. Let $Q \in \text{SLDFS}(N \times N)$ is the subset of SLFDF relation. Then, the pair (N, Q) is known as SLDF approximation space. For any upper and lower approximations $\Bbbk (\Bbbk \subseteq \text{SLDFS}(N))$, with respect to SLDF approximation space, (N, Q) are two SLDFSs, $\overline{Q}(\Bbbk)$ and $\underline{Q}(\Bbbk)$ defined as

$$\begin{aligned}
\overline{Q}(\Bbbk) &= \left\{\left(v, \left\langle \mu_{\overline{Q}(\Bbbk)}(v), \upsilon_{\overline{Q}(\Bbbk)}(v), \eta_{\overline{Q}(\Bbbk)}(v)\right\rangle, \left\langle \alpha_{\overline{Q}(\Bbbk)}(v), \beta_{\overline{Q}(\Bbbk)}(v), \gamma_{\overline{Q}(\Bbbk)}(v)\right\rangle \mid v \in N\right\},\\
\underline{Q}(\Bbbk) &= \left\{\left(v, \left\langle \mu_{\underline{Q}(\Bbbk)}(v), \upsilon_{\underline{Q}(\Bbbk)}(v), \eta_{\underline{Q}(\Bbbk)}(v)\right\rangle, \left\langle \alpha_{\underline{Q}(\Bbbk)}(v), \beta_{\underline{Q}(\Bbbk)}(v), \eta_{\underline{Q}(\Bbbk)}(v)\right\rangle \mid v \in N\right\},
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\mu_{\overline{Q}(\Bbbk)}(v) &= \bigvee_{c \in v} \left\{\mu_Q(v,c) \vee \mu_{\Bbbk}(c)\right\}; \alpha_{\overline{Q}(\Bbbk)}(v) = \bigvee_{c \in v} \left\{\alpha_Q(v,c) \vee \alpha_{\Bbbk}(c)\right\},\\
\upsilon_{\overline{Q}(\Bbbk)}(v) &= \bigwedge_{c \in v} \left\{\upsilon_Q(v,c) \wedge \upsilon_{\Bbbk}(c)\right\}; \beta_{\overline{Q}(\Bbbk)}(v) = \bigwedge_{c \in v} \left\{\beta_Q(v,c) \wedge \beta_{\Bbbk}(c)\right\},\\
\eta_{\overline{Q}(\Bbbk)}(v) &= \bigwedge_{c \in v} \left\{\eta_Q(v,c) \wedge \eta_{\Bbbk}(c)\right\}; \gamma_{\overline{Q}(\Bbbk)}(v) = \bigwedge_{c \in v} \left\{\gamma_Q(v,c) \wedge \gamma_{\Bbbk}(c)\right\},\\
\mu_{\underline{Q}(\Bbbk)}(v) &= \bigwedge_{c \in v} \left\{\mu_Q(v,c) \wedge \mu_{\Bbbk}(c)\right\}; \alpha_{\underline{Q}(\Bbbk)}(v) = \bigwedge_{c \in v} \left\{\alpha_Q(v,c) \wedge \alpha_{\Bbbk}(c)\right\},\\
\upsilon_{\underline{Q}(\Bbbk)}(v) &= \bigvee_{c \in v} \left\{\upsilon_Q(v,c) \vee \upsilon_{\Bbbk}(c)\right\}; \beta_{\underline{Q}(\Bbbk)}(v) = \bigvee_{c \in v} \left\{\beta_Q(v,c) \vee \beta_{\Bbbk}(c)\right\},\\
\eta_{\underline{Q}(\Bbbk)}(v) &= \bigvee_{c \in v} \left\{\eta_Q(v,c) \vee \eta_{\Bbbk}(c)\right\}; \gamma_{\underline{Q}(\Bbbk)}(v) = \bigvee_{c \in v} \left\{\gamma_Q(v,c) \vee \gamma_{\Bbbk}(c)\right\}.
\end{aligned} \tag{21}$$

Such that $0 \le \alpha_{\overline{Q}(\Bbbk)} \mu_{\overline{Q}(\Bbbk)} + \beta_{\overline{Q}(\Bbbk)} \upsilon_{\overline{Q}(\Bbbk)} + \gamma_{\overline{Q}(\Bbbk)} \eta_{\overline{Q}(\Bbbk)} \le 1, 0 \le \alpha_{\underline{Q}(\Bbbk)} \mu_{\underline{Q}(\Bbbk)} + \beta_{\underline{Q}(\Bbbk)} \upsilon_{\underline{Q}(\Bbbk)} + \gamma_{\underline{Q}(\Bbbk)} \eta_{\underline{Q}(\Bbbk)} \le 1, \quad 0 \le \alpha_{\overline{Q}(\Bbbk)} + \beta_{\overline{Q}(\Bbbk)} + \gamma_{\overline{Q}(\Bbbk)} \le 1$, and $0 \le \alpha_{\underline{Q}(\Bbbk)} + \beta_{\underline{Q}(\Bbbk)} + \gamma_{\underline{Q}(\Bbbk)} \le 1$. $\overline{Q}(\Bbbk)$ and $\underline{Q}(\Bbbk)$ are SLDFSs and $\underline{Q}(\Bbbk), \overline{Q}(\Bbbk)$: $\mathfrak{I}(N) \longrightarrow \mathfrak{I}(N)$ are lower and upper approximation operators. Then, the pair is called SLDFRS.

$$Q(\Bbbk) = \left(\underline{Q}(\Bbbk), \overline{Q}(\Bbbk)\right) = \left\{ \left(v, \begin{pmatrix} \left\langle \mu_{\underline{Q}(\Bbbk)}(v), v_{\underline{Q}(\Bbbk)}(v), \eta_{\underline{Q}(\Bbbk)}(v) \right\rangle, \\ \left\langle \alpha_{\underline{Q}(\Bbbk)}(v), \beta_{\underline{Q}(\Bbbk)}(v), \gamma_{\underline{Q}(\Bbbk)}(v) \right\rangle \end{pmatrix}, \begin{pmatrix} \left\langle \left(\mu_{\overline{Q}(\Bbbk)}(v), v_{\overline{Q}(\Bbbk)}(v), \eta_{\overline{Q}(\Bbbk)}(v) \right\rangle, \\ \left\langle \alpha_{\overline{Q}(\Bbbk)}(v), \beta_{\overline{Q}(\Bbbk)}(v), \gamma_{\overline{Q}(\Bbbk)}(v) \right\rangle \end{pmatrix} \right) \mid v \in N \right\}. \tag{22}$$

Definition 13. Let $Q(\Bbbk_1) = (\underline{Q}(\Bbbk_1), \overline{Q}(\Bbbk_1))$ and $Q(\Bbbk_2) = (\underline{Q}(\Bbbk_2), \overline{Q}(\Bbbk_2))$ be two SLDFRSs. Then, the following operation is defined.

(1) $Q(\Bbbk_1) \cup Q(\Bbbk_2) = \left\{(\underline{Q}(\Bbbk_1) \cup \underline{Q}(\Bbbk_2)), \overline{Q}(\Bbbk_1) \cup \overline{Q}(\Bbbk_2)\right\}$;

(2) $Q(\Bbbk_1) \cap Q(\Bbbk_2) = \left\{(\underline{Q}(\Bbbk_1) \cap \underline{Q}(\Bbbk_2)), \overline{Q}(\Bbbk_1) \cap \overline{Q}(\Bbbk_2)\right\}$;

(3) $Q(\Bbbk_1) \oplus Q(\Bbbk_2) = \left\{(\underline{Q}(\Bbbk_1) \oplus \underline{Q}(\Bbbk_2)), \overline{Q}(\Bbbk_1) \oplus \overline{Q}(\Bbbk_2)\right\}$;

(4) $Q(\Bbbk_1) \otimes Q(\Bbbk_2) = \left\{(\underline{Q}(\Bbbk_1) \otimes \underline{Q}(\Bbbk_2)), \overline{Q}(\Bbbk_1) \otimes \overline{Q}(\Bbbk_2)\right\}$;

(5) $Q(\Bbbk_1) \subseteq Q(\Bbbk_2) = \left\{(\underline{Q}(\Bbbk_1) \subseteq \underline{Q}(\Bbbk_2)), \overline{Q}(\Bbbk_1) \subseteq \overline{Q}(\Bbbk_2)\right\}$;

(6) $\lambda Q(\Bbbk_1) = (\lambda \underline{Q}(\Bbbk_1), \lambda \overline{Q}(\Bbbk_1))$, for $\lambda \geq 1$;

(7) $(Q(\Bbbk_1))^{\lambda} = (\underline{Q}(\Bbbk_1))^{\lambda}, (\overline{Q}(\Bbbk_1))^{\lambda}$, for $\lambda \geq 1$;

(8) $(Q(\Bbbk_1))^{c} = (\underline{Q}(\Bbbk_1))^{c}, (\overline{Q}(\Bbbk_1))^{c}$, where $(\underline{Q}(\Bbbk_1))^{c} = (\underline{\mu}_1, \underline{v}_1, \underline{\eta}_1, \underline{\alpha}_1, \underline{\beta}_1, \underline{\gamma}_1)$

(9) $Q(\Bbbk_1) = Q(\Bbbk_2) \Leftrightarrow \underline{Q}(\Bbbk_1) = \overline{Q}(\Bbbk_1)$ and $\underline{Q}(\Bbbk_2) = \overline{Q}(\Bbbk_2)$

Definition 14. Let $Q(\Bbbk) = (\underline{Q}(\Bbbk), \overline{Q}(\Bbbk)) = (\langle \underline{\mu}, \underline{v}, \underline{\eta}, \underline{\alpha}, \underline{\beta}, \underline{\gamma} \rangle, \langle \overline{\mu}, \overline{v}, \overline{\eta}, \overline{\alpha}, \overline{\beta}, \overline{\gamma} \rangle)$ is an SLDFRN. Then, score function $\mathrm{Sco}^*(Q(\Bbbk))$ is given as

$$\mathrm{Sco}^*(Q(\Bbbk)) = \frac{\left(8 + \underline{\mu} - \underline{v} - \underline{\eta} + \underline{\alpha} - \underline{\beta} - \underline{\gamma} + \overline{\mu} - \overline{v} - \overline{\eta} + \overline{\alpha} - \overline{\beta} - \overline{\gamma}\right)}{12}. \tag{23}$$

Such that $\mathrm{Sco}^*(Q(\Bbbk)) \in [-1, 1]$. The function $\mathrm{Sco}^*(Q(\Bbbk))$ calculates the score of the SLFRN $Q(\Bbbk)$.

Definition 15. Let $Q(\Bbbk) = (\underline{Q}(\Bbbk), \overline{Q}(\Bbbk)) = (\langle \underline{\mu}, \underline{v}, \underline{\eta}, \underline{\alpha}, \underline{\beta}, \underline{\gamma} \rangle, \langle \overline{\mu}, \overline{v}, \overline{\eta}, \overline{\alpha}, \overline{\beta}, \overline{\gamma} \rangle)$ is an SLDFRN. Then, accuracy function $\mathrm{Hco}^*(Q(\Bbbk))$ is defined as

$$\mathrm{Hco}^* Q(\Bbbk) = \frac{\left(\underline{\mu} + \underline{v} + \underline{\eta} + \underline{\alpha} + \underline{\beta} + \underline{\gamma} + \overline{\mu} + \overline{v} + \overline{\eta} + \overline{\alpha} + \overline{\beta} + \overline{\gamma}\right)}{12}. \tag{24}$$

Such that $\mathrm{Hco}^*(Q(\Bbbk)) \in [0, 1]$. The function $\mathrm{Hco}^*(Q(\Bbbk))$ calculates the accuracy of the SLDFRN $Q(\Bbbk)$.

4. Spherical Linear Diophantine Fuzzy Rough Averaging Operator

We defined the notion of SLDFR aggregation operators in this section of the paper by combining both notions (rough sets and SLDFSs) to generate some aggregation operators.

4.1. Spherical Linear Diophantine Fuzzy Rough Weighted Averaging (SLDFRWA) Aggregation Operator. This section focuses on SLDFRWA aggregation operators and their desirable properties.

Definition 16. Let $Q(\Bbbk_\imath) = (\underline{Q}(\Bbbk_\imath), \overline{Q}(\Bbbk_\imath))\,(\imath = 1, \ldots, n)$ is a family of the SLDFRNs with the weight vector $\omega = (\omega_1, \ldots, \omega_n)^T$, such as $\sum_{\imath=1}^{n} \omega_\imath = 1$ and $0 \leq \omega_\imath \leq 1$. Then, the SLDFRWA operator is defined as

$$\mathrm{SLDFRWA}(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)) = \left\{ \mathop{\oplus}_{\imath=1}^{n} \omega_\imath \underline{Q}(\Bbbk_\imath), \mathop{\oplus}_{\imath=1}^{n} \omega_\imath \overline{Q}(\Bbbk_\imath) \right\}. \tag{25}$$

Based on Definition 16, the aggregated value for SLDFRWA operator is given in Theorem 17.

Theorem 17. *Let* $Q(\Bbbk_\imath) = (\underline{Q}(\Bbbk_\imath), \overline{Q}(\Bbbk_\imath))$ $(\imath = 1, \ldots, n)$ *is a family of SLDFRNs with weight vector* $\omega = (\omega_1, \ldots, \omega_n)^T$. *Then, SLDFRWA operator is obtained as*

$$\mathrm{SLDFRWA}(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)) = \left\{\overset{n}{\underset{\imath=1}{\oplus}} \omega_\imath \underline{Q}(\Bbbk_\imath), \overset{n}{\underset{\imath=1}{\oplus}} \omega_\imath \overline{Q}(\Bbbk_\imath)\right\}$$
$$= \left\{\begin{array}{l} \left(\begin{array}{c} \left\langle \sqrt{1 - \prod_{\imath=1}^{n}\left(1 - (\underline{\mu}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n}\left(1 - (\underline{\alpha}_\imath)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \prod_{\imath=1}^{n} (\underline{\upsilon}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{n} (\underline{\beta}_\imath)^{\omega_\imath} \right\rangle, \left\langle \prod_{\imath=1}^{n} (\underline{\eta}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{n} (\underline{\gamma}_\imath)^{\omega_\imath} \right\rangle \end{array}\right), \\ \left(\begin{array}{c} \left\langle \sqrt{1 - \prod_{\imath=1}^{n}\left(1 - (\overline{\mu}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n}\left(1 - (\overline{\alpha}_\imath)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \prod_{\imath=1}^{n} (\overline{\upsilon}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{n} (\overline{\beta}_\imath)^{\omega_\imath} \right\rangle, \left\langle \prod_{\imath=1}^{n} (\overline{\eta}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{n} (\overline{\gamma}_\imath)^{\omega_\imath} \right\rangle \end{array}\right) \end{array}\right\}. \tag{26}$$

Proof. Using mathematical induction principle as

$$Q(\Bbbk_1) \oplus Q(\Bbbk_2) = \left\{\left(\underline{Q}(\Bbbk_1) \oplus \underline{Q}(\Bbbk_2)\right), \left(\overline{Q}(\Bbbk_1) \oplus \overline{Q}(\Bbbk_2)\right)\right\},$$
$$\omega_1 Q(\Bbbk_1) = \left(\omega_1 \underline{Q}(\Bbbk_1), \omega_1 \overline{Q}(\Bbbk_1)\right)$$
$$= \left\{\begin{array}{l} \left(\begin{array}{c} \left\langle \sqrt{1 - \left(1 - (\underline{\mu}_1)^2\right)^{\omega_1}}, \sqrt{1 - \left(1 - (\underline{\alpha}_1)^2\right)^{\omega_1}} \right\rangle, \\ \left\langle (\underline{\upsilon}_1)^{\omega_1}, (\underline{\beta}_1)^{\omega_1} \right\rangle, \left\langle (\underline{\eta}_1)^{\omega_1}, (\underline{\gamma}_1)^{\omega_1} \right\rangle \end{array}\right), \\ \left(\begin{array}{c} \left\langle \sqrt{1 - \left(1 - (\overline{\mu}_1)^2\right)^{\omega_1}}, \sqrt{1 - \left(1 - (\overline{\alpha}_1)^2\right)^{\omega_1}} \right\rangle, \\ \left\langle (\overline{\upsilon}_1)^{\omega_1}, (\overline{\beta}_1)^{\omega_1} \right\rangle, \left\langle (\overline{\eta}_1)^{\omega_1}, (\overline{\gamma}_1)^{\omega_1} \right\rangle \end{array}\right) \end{array}\right\}. \tag{27}$$

Let $n = 2$, then

$$\mathrm{SLDFRWA}(Q(\Bbbk_1), Q(\Bbbk_2)) = \left\{\overset{2}{\underset{\imath=1}{\oplus}} \omega_\imath \underline{Q}(\Bbbk_\imath), \overset{2}{\underset{\imath=1}{\oplus}} \omega_\imath \overline{Q}(\Bbbk_\imath)\right\}$$
$$= \left\{\begin{array}{l} \left(\begin{array}{c} \left\langle \sqrt{1 - \prod_{\imath=1}^{2}\left(1 - (\underline{\mu}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{2}\left(1 - (\underline{\alpha}_\imath)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \prod_{\imath=1}^{2} (\underline{\upsilon}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{2} (\underline{\beta}_\imath)^{\omega_\imath} \right\rangle, \left\langle \prod_{\imath=1}^{2} (\underline{\eta}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{2} (\underline{\gamma}_\imath)^{\omega_\imath} \right\rangle \end{array}\right), \\ \left(\begin{array}{c} \left\langle \sqrt{1 - \prod_{\imath=1}^{2}\left(1 - (\overline{\mu}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{2}\left(1 - (\overline{\alpha}_\imath)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \prod_{\imath=1}^{2} (\overline{\upsilon}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{2} (\overline{\beta}_\imath)^{\omega_\imath} \right\rangle, \left\langle \prod_{\imath=1}^{2} (\overline{\eta}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{2} (\overline{\gamma}_\imath)^{\omega_\imath} \right\rangle \end{array}\right) \end{array}\right\}. \tag{28}$$

The results hold for $n = 2$.

Now, equation (26) holds for $n = k$. Then,

$$\begin{aligned}
&\mathrm{SLDFRWA}\left(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_k)\right)=\left\{\overset{k}{\underset{\imath=1}{\oplus}}\omega_\imath\underline{Q}(\mathbb{k}_\imath),\overset{k}{\underset{\imath=1}{\oplus}}\omega_\imath\overline{Q}(\mathbb{k}_\imath)\right\}\\
&=\left\{\begin{array}{l}
\left(\begin{array}{c}\left\langle\sqrt{1-\prod_{\imath=1}^{k}\left(1-\left(\underline{\mu}_\imath\right)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k}\left(1-\left(\underline{\alpha}_\imath\right)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\prod_{\imath=1}^{k}\left(\underline{\upsilon}_\imath\right)^{\omega_\imath},\prod_{\imath=1}^{k}\left(\underline{\beta}_\imath\right)^{\omega_\imath}\right\rangle,\left\langle\prod_{\imath=1}^{k}\left(\underline{\eta}_\imath\right)^{\omega_\imath},\prod_{\imath=1}^{k}\left(\underline{\gamma}_\imath\right)^{\omega_\imath}\right\rangle\end{array}\right),\\
\left(\begin{array}{c}\left\langle\sqrt{1-\prod_{\imath=1}^{k}\left(1-\left(\overline{\mu}_\imath\right)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k}\left(1-\left(\overline{\alpha}_\imath\right)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\prod_{\imath=1}^{k}\left(\overline{\upsilon}_\imath\right)^{\omega_\imath},\prod_{\imath=1}^{k}\left(\overline{\beta}_\imath\right)^{\omega_\imath}\right\rangle,\left\langle\prod_{\imath=1}^{k}\left(\overline{\eta}_\imath\right)^{\omega_\imath},\prod_{\imath=1}^{k}\left(\overline{\gamma}_\imath\right)^{\omega_\imath}\right\rangle\end{array}\right)
\end{array}\right\}.
\end{aligned}\tag{29}$$

Next, equation (26) holds for $n = k + 1$

,

$$\begin{aligned}
&\mathrm{SLDFRWA}\left(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_k)\oplus Q(\mathbb{k}_{k+1})\right)\\
&=\left\{\left(\overset{k}{\underset{\imath=1}{\oplus}}\omega_\imath\underline{Q}(\mathbb{k}_\imath)\right)\oplus\left(\omega_{k+1}\underline{Q}(\mathbb{k}_{k+1})\right),\left(\overset{k}{\underset{\imath=1}{\oplus}}\omega_\imath\overline{Q}(\mathbb{k}_\imath)\right)\oplus\left(\omega_{k+1}\overline{Q}(\mathbb{k}_{k+1})\right)\right\}\\
&=\left\{\begin{array}{l}
\left(\begin{array}{c}\left\langle\sqrt{1-\prod_{\imath=1}^{k}\left(1-\left(\underline{\mu}_\imath\right)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k}\left(1-\left(\underline{\alpha}_\imath\right)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\prod_{\imath=1}^{k}\left(\underline{\upsilon}_\imath\right)^{\omega_\imath},\prod_{\imath=1}^{k}\left(\underline{\beta}_\imath\right)^{\omega_\imath}\right\rangle,\left\langle\prod_{\imath=1}^{k}\left(\underline{\eta}_\imath\right)^{\omega_\imath},\prod_{\imath=1}^{k}\left(\underline{\gamma}_\imath\right)^{\omega_\imath}\right\rangle\end{array}\right),\\
\left(\begin{array}{c}\left\langle\sqrt{1-\prod_{\imath=1}^{k}\left(1-\left(\overline{\mu}_\imath\right)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k}\left(1-\left(\overline{\alpha}_\imath\right)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\prod_{\imath=1}^{k}\left(\overline{\upsilon}_\imath\right)^{\omega_\imath},\prod_{\imath=1}^{k}\left(\overline{\beta}_\imath\right)^{\omega_\imath}\right\rangle,\left\langle\prod_{\imath=1}^{k}\left(\overline{\eta}_\imath\right)^{\omega_\imath},\prod_{\imath=1}^{k}\left(\overline{\gamma}_\imath\right)^{\omega_\imath}\right\rangle\end{array}\right)
\end{array}\right\}\\
&\oplus\left\{\begin{array}{l}
\left(\begin{array}{c}\left\langle\sqrt{1-\left(1-\left(\underline{\mu}_{k+1}\right)^2\right)^{\omega_{k+1}}},\sqrt{1-\left(1-\left(\underline{\alpha}_{k+1}\right)^2\right)^{\omega_{k+1}}}\right\rangle,\\ \left\langle\left(\underline{\upsilon}_{k+1}\right)^{\omega_{k+1}},\left(\underline{\beta}_{k+1}\right)^{\omega_{k+1}}\right\rangle,\left\langle\left(\underline{\eta}_{k+1}\right)^{\omega_{k+1}},\left(\underline{\gamma}_{k+1}\right)^{\omega_{k+1}}\right\rangle\end{array}\right),\\
\left(\begin{array}{c}\left\langle\sqrt{1-\left(1-\left(\overline{\mu}_{k+1}\right)^2\right)^{\omega_{k+1}}},\sqrt{1-\left(1-\left(\overline{\alpha}_{k+1}\right)^2\right)^{\omega_{k+1}}}\right\rangle,\\ \left\langle\left(\overline{\upsilon}_{k+1}\right)^{\omega_{k+1}},\left(\overline{\beta}_{k+1}\right)^{\omega_{k+1}}\right\rangle,\left\langle\left(\overline{\eta}_{k+1}\right)^{\omega_{k+1}},\left(\overline{\gamma}_{k+1}\right)^{\omega_{k+1}}\right\rangle\end{array}\right)
\end{array}\right\}
\end{aligned}$$

$$= \left\{ \begin{array}{l} \left(\begin{array}{l} \left\langle \sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\underline{\mu}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\underline{\alpha}_\imath)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \prod_{\imath=1}^{k+1}(\underline{\upsilon}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{k+1}(\underline{\beta}_\imath)^{\omega_\imath} \right\rangle, \left\langle \prod_{\imath=1}^{k+1}(\underline{\eta}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{k+1}(\underline{\gamma}_\imath)^{\omega_\imath} \right\rangle \end{array} \right), \\ \left(\begin{array}{l} \left\langle \sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\overline{\mu}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\overline{\alpha}_\imath)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \prod_{\imath=1}^{k+1}(\overline{\upsilon}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{k+1}(\overline{\beta}_\imath)^{\omega_\imath} \right\rangle, \left\langle \prod_{\imath=1}^{k+1}(\overline{\eta}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{k+1}(\overline{\gamma}_\imath)^{\omega_\imath} \right\rangle \end{array} \right). \end{array} \right\}. \tag{30}$$

This demonstrates that equation (26) is correct for $= k + 1$. As a result, the following result holds for $n \geq 1$.

From the above result, $\underline{Q}(\Bbbk_\imath)$ and $\overline{Q}(\Bbbk_\imath)$ are SLDFRNs. So, by Def. (3.2), $\oplus_{\imath=1}^{n}\omega_\imath \underline{Q}(\Bbbk_\imath)$ and $\oplus_{\imath=1}^{n}\omega_\imath \overline{Q}(\Bbbk_\imath)$ are also SLDFRNs. Therefore, $\text{SLDFRWA}(Q(\Bbbk_1), \ldots, Q(\Bbbk_k))$ is still an SLDFRN with SLDF approximation space (N, Q).

The SLDFRWA operator satisfied the following properties. □

Theorem 18. *Let* $Q(\Bbbk_\imath) = (\underline{Q}(\Bbbk_\imath), \overline{Q}(\Bbbk_\imath))\,(\imath = 1, \ldots, n)$ *is a family of SLDFRNs with the weight vector* $\omega = (\omega_1, \ldots, \omega_n)^T$, *such as* $\sum_{\imath=1}^{n}\omega_\imath = 1$ *and* $0 \leq \omega_\imath \leq 1$. *Then, the following properties hold.*

(1) *(Idempotency). If* $Q(\Bbbk_\imath) = Q(\Bbbk)$ *for all* $(\imath = 1, \ldots, n)$. *Here,*

$$\begin{aligned} Q(\Bbbk) &= \left(\underline{Q}(\Bbbk), \overline{Q}(\Bbbk)\right) \\ &= \left(\left(\underline{\mu}, \underline{\upsilon}, \underline{\eta}, \underline{\alpha}, \underline{\beta}, \underline{\gamma}\right), \left(\overline{\mu}, \overline{\upsilon}, \overline{\eta}, \overline{\alpha}, \overline{\beta}, \overline{\gamma}\right)\right). \end{aligned} \tag{31}$$

Then,

$$\text{SLDFRWA}\left(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)\right) = Q(\Bbbk). \tag{32}$$

(2) *(Boundedness). If* $Q(\Bbbk_\imath)^- = (\min_\imath \underline{Q}(\Bbbk_\imath), \max_\imath \overline{Q}(\Bbbk_\imath))$ *and* $Q(\Bbbk_\imath)^+ = (\max_\imath \underline{Q}(\Bbbk_\imath), \min_\imath \overline{Q}(\Bbbk_\imath))$. *Then,*

$$Q(\Bbbk_\imath)^- \leq \text{SLDFRWA}\left(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)\right) \leq Q(\Bbbk_\imath)^+. \tag{33}$$

(3) *(Monotonicity). Let* $Q(\Bbbk_\imath) = (\underline{Q}(\Bbbk_\imath), \overline{Q}(\Bbbk_\imath))$ *and* $\mathfrak{I}(\mathscr{L}_\imath) = (\underline{\mathfrak{I}}(\mathscr{L}_\imath), \overline{\mathfrak{I}}(\mathscr{L}_\imath))\,(\imath = 1, \ldots, n)$ *any two family of SLDFRNs, such that* $\underline{\mathfrak{I}}(\mathscr{L}_\imath) \leq \underline{Q}(\Bbbk_\imath)$ *and* $\overline{\mathfrak{I}}(\mathscr{L}_\imath) \leq \overline{Q}(\Bbbk_\imath)$. *Then,*

$$\text{SLDFRWA}\left(\mathfrak{I}(\mathscr{L}_1), \ldots, \mathfrak{I}(\mathscr{L}_n)\right) \leq \text{SLDFRWA}\left(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)\right). \tag{34}$$

(4) *(Commutativity). Let* $Q'(\Bbbk_\imath) = (\underline{Q}'(\Bbbk_\imath), \overline{Q}'(\Bbbk_\imath))$ $(\imath = 1, \ldots, n)$. *Then,*

$$\text{SLDFRWA}\left(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)\right) = \text{SLDFRWA}\left(Q'(\Bbbk_1), \ldots, Q'(\Bbbk_n)\right). \tag{35}$$

4.2. Spherical Linear Diophantine Fuzzy Rough Ordered Weighted Averaging (SLDFROWA) Aggregation Operator. We presented the SLDFROWA operator and reviewed its basic features in this section.

Definition 19. Let $Q(\Bbbk_\imath) = (\underline{Q}(\Bbbk_\imath), \overline{Q}(\Bbbk_\imath))\,(\imath = 1, \ldots, n)$ is a family of SLDFRNs with the weight vector $\omega = (\omega_1, \ldots, \omega_n)^T$, such as $\sum_{\imath=1}^{n}\omega_\imath = 1$ and $0 \leq \omega_\imath \leq 1$. The SLDFROWA operator is defined as

$$\text{SLDFROWA}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_n)) = \left\{\overset{n}{\underset{\imath=1}{\oplus}}\omega_\imath \underline{Q}\left(\mathbb{k}_{\sigma(\imath)}\right), \overset{n}{\underset{\imath=1}{\oplus}}\omega_\imath \overline{Q}\left(\mathbb{k}_{\sigma(\imath)}\right)\right\}. \tag{36}$$

Based on Definition 19, the aggregated value for SLDFROWA operator is given in Theorem 20.

Theorem 20. *Let* $Q(\mathbb{k}_\imath) = (\underline{Q}(\mathbb{k}_\imath), \overline{Q}(\mathbb{k}_\imath))\,(\imath = 1,\ldots,n)$ *is a family of SLDFRNs with weight vector* $\omega = (\omega_1,\ldots,\omega_n)^T$. *Then, SLDFROWA operator is described as*

$$\text{SLDFROWA}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_n)) = \left\{\overset{n}{\underset{\imath=1}{\oplus}}\omega_\imath \underline{Q}\left(\mathbb{k}_{\sigma(\imath)}\right), \overset{n}{\underset{\imath=1}{\oplus}}\omega_\imath \overline{Q}\left(\mathbb{k}_{\sigma(\imath)}\right)\right\}$$
$$\left\{\begin{matrix}\left(\begin{matrix}\left\langle\sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\underline{\mu}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\underline{\alpha}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}\right\rangle, \\ \left\langle\prod_{\imath=1}^{n}\left(\underline{\upsilon}_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\underline{\beta}_{\sigma(\imath)}\right)^{\omega_\imath}\right\rangle, \left\langle\prod_{\imath=1}^{n}\left(\underline{\eta}_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\underline{\gamma}_{\sigma(\imath)}\right)^{\omega_\imath}\right\rangle\end{matrix}\right), \\ \left(\begin{matrix}\left\langle\sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\overline{\mu}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\overline{\alpha}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}\right\rangle, \\ \left\langle\prod_{\imath=1}^{n}\left(\overline{\upsilon}_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\overline{\beta}_{\sigma(\imath)}\right)^{\omega_\imath}\right\rangle, \left\langle\prod_{\imath=1}^{n}\left(\overline{\eta}_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\overline{\gamma}_{\sigma(\imath)}\right)^{\omega_\imath}\right\rangle\end{matrix}\right)\end{matrix}\right\}, \tag{37}$$

where $Q(\mathbb{k}_{\sigma(\imath)}) = (\underline{Q}(\mathbb{k}_{\sigma(\imath)}), \overline{Q}(\mathbb{k}_{\sigma(\imath)}))$ *represents the largest value of the permutation from the family of SLDFRNs.*

Proof. Proof is followed from Theorem 17.

The SLDFROWA operator satisfied the following properties. □

Theorem 21. *Let* $Q(\mathbb{k}_\imath) = (\underline{Q}(\mathbb{k}_\imath), \overline{Q}(\mathbb{k}_\imath))\,(\imath = 1,\ldots,n)$ *is a family of SLDFRNs with weight vector* $\omega = (\omega_1,\ldots,\omega_n)^T$, *such as* $\sum_{\imath=1}^{n}\omega_\imath = 1$ *and* $0 \le \omega_\imath \le 1$. *Then, the following properties hold.*

(1) (Idempotency). If $Q(\mathbb{k}_\imath) = Q(\mathbb{k})$ for all $(\imath = 1,\ldots,n)$. Here,

$$\begin{aligned}Q(\mathbb{k}) &= \left(\underline{Q}(\mathbb{k}), \overline{Q}(\mathbb{k})\right) \\ &= \left(\left(\underline{\mu}, \underline{\upsilon}, \underline{\eta}, \underline{\alpha}, \underline{\beta}, \underline{\gamma}\right), \left(\overline{\mu}, \overline{\upsilon}, \overline{\eta}, \overline{\alpha}, \overline{\beta}, \overline{\gamma}\right)\right).\end{aligned} \tag{38}$$

Then,

$$\text{SLDFROWA}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_n)) = Q(\mathbb{k}). \tag{39}$$

(2) (Boundedness). If $Q(\mathbb{k}_\imath)^- = (\min_\imath \underline{Q}(\mathbb{k}_\imath), \max_\imath \overline{Q}(\mathbb{k}_\imath))$ and $Q(\mathbb{k}_\imath)^+ = (\max_\imath \underline{Q}(\mathbb{k}_\imath), \min_\imath \overline{Q}(\mathbb{k}_\imath))$. Then,

$$Q(\mathbb{k}_\imath)^- \le \text{SLDFROWA}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_n)) \le Q(\mathbb{k}_\imath)^+. \tag{40}$$

(3) (Monotonicity). Let $Q(\mathbb{k}_\imath) = (\underline{Q}(\mathbb{k}_\imath), \overline{Q}(\mathbb{k}_\imath))$ and $\mathfrak{J}(\mathscr{L}_\imath) = (\underline{\mathfrak{J}}(\mathscr{L}_\imath), \overline{\mathfrak{J}}(\mathscr{L}_\imath))\,(\imath = 1,\ldots,n)$ be any two family of SLDFRNs, such that $\underline{\mathfrak{J}}(\mathscr{L}_\imath) \le \underline{Q}(\mathbb{k}_\imath)$ and $\overline{\mathfrak{J}}(\mathscr{L}_\imath) \le \overline{Q}(\mathbb{k}_\imath)$. Then,

$$\text{SLDFROWA}(\mathfrak{J}(\mathscr{L}_1),\ldots,\mathfrak{J}(\mathscr{L}_n)) \le \text{SLDFROWA}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_n)). \tag{41}$$

(4) (Commutativity). Let $Q'(\mathbb{k}_\imath) = (\underline{Q}'(\mathbb{k}_\imath), \overline{Q}'(\mathbb{k}_\imath))$ $(\imath = 1,\ldots,n)$. Then,

$$\text{SLDFROWA}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_n)) = \text{SLDFROWA}\left(Q'(\mathbb{k}_1),\ldots,Q'(\mathbb{k}_n)\right). \tag{42}$$

4.3. Spherical Linear Diophantine Fuzzy Rough Hybrid Averaging (SLDFRHA) Aggregation Operator. We introduced the notion of SLDFRHA operators and examined their basic qualities in this section.

Definition 22. Let $Q(\mathbb{k}_\imath) = (\underline{Q}(\mathbb{k}_\imath), \overline{Q}(\mathbb{k}_\imath))(\imath = 1, \ldots, n)$ is a family of SLDFRNs with the weight vector and associated weight vector $\omega = (\omega_1, \ldots, \omega_n)^T, w = (w_1, \ldots, w_n)^T$ such as $\sum_{\imath=1}^{n}\omega_\imath, w_\imath = 1$ and $0 \leq \omega_\imath, w_\imath \leq 1$, respectively. Then, SLDFRHA operator is defined as

$$\mathrm{SLDFRHA}(Q(\mathbb{k}_1), \ldots, Q(\mathbb{k}_n)) = \left\{\overset{n}{\underset{\imath=1}{\oplus}} \omega_\imath \underline{Q}\left(\mathbb{k}^*_{\sigma(\imath)}\right), \overset{n}{\underset{\imath=1}{\oplus}} \omega_\imath \overline{Q}\left(\mathbb{k}^*_{\sigma(\imath)}\right)\right\}. \tag{43}$$

Using the Definition 22, the aggregated value for SLDFRHA operator is given in Theorem 23.

Theorem 23. *Let $Q(\mathbb{k}_\imath) = (\underline{Q}(\mathbb{k}_\imath), \overline{Q}(\mathbb{k}_\imath))(\imath = 1, \ldots, n)$ is a family of SLDFRNs with the weight vector and associated weight vector $\omega = (\omega_1, \ldots, \omega_n)^T, w = (w_1, \ldots, w_n)^T$ such as $\sum_{\imath=1}^{n}\omega_\imath, w_\imath = 1$ and $0 \leq \omega_\imath$, $w_\imath \leq 1$, respectively. Then, SLDFRHA operator is defined as*

$$\begin{aligned}
\mathrm{SLDFRHA}(\mathrm{Q}(\mathbb{k}_1), \ldots, \mathrm{Q}(\mathbb{k}_\mathrm{n})) &= \left\{\overset{\mathrm{n}}{\underset{\imath=1}{\oplus}} \omega_\imath \underline{Q}\left(\mathbb{k}^*_{\sigma(\imath)}\right), \overset{\mathrm{n}}{\underset{\imath=1}{\oplus}} \omega_\imath \overline{Q}\left(\mathbb{k}^*_{\sigma(\imath)}\right)\right\} \\
&= \left\{\begin{array}{l}
\left(\begin{array}{l}
\left\langle \sqrt{1 - \prod_{\imath=1}^{n}\left(1 - \left(\underline{\mu}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n}\left(1 - \left(\underline{\alpha}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}} \right\rangle, \\
\left\langle \prod_{\imath=1}^{n}\left(\underline{\upsilon}^*_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\underline{\beta}^*_{\sigma(\imath)}\right)^{\omega_\imath} \right\rangle, \left\langle \prod_{\imath=1}^{n}\left(\underline{\eta}^*_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\underline{\gamma}^*_{\sigma(\imath)}\right)^{\omega_\imath} \right\rangle
\end{array}\right), \\
\left(\begin{array}{l}
\left\langle \sqrt{1 - \prod_{\imath=1}^{n}\left(1 - \left(\overline{\mu}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n}\left(1 - \left(\overline{\alpha}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}} \right\rangle, \\
\left\langle \prod_{\imath=1}^{n}\left(\overline{\upsilon}^*_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\overline{\beta}^*_{\sigma(\imath)}\right)^{\omega_\imath} \right\rangle, \left\langle \prod_{\imath=1}^{n}\left(\overline{\eta}^*_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\overline{\gamma}^*_{\sigma(\imath)}\right)^{\omega_\imath} \right\rangle
\end{array}\right)
\end{array}\right\},
\end{aligned} \tag{44}$$

where $Q(\mathbb{k}^*_{\sigma(\imath)}) = n w_\imath Q(\mathbb{k}_{\sigma(\imath)}) = (n w_\imath \underline{Q}(\mathbb{k}_{\sigma(\imath)}), n w_\imath \overline{Q}(\mathbb{k}_{\sigma(\imath)}))$ denotes the biggest value of permutation from the family of SLDFRNs and n shows the balancing coefficient.

Proof. Proof is followed from Theorem 17.

Specially, if $w = ((1/n), \ldots, (1/n))^T$, then the proposed SLDFRHA operator becomes the SLDFROWA operator. □

5. Spherical Linear Diophantine Fuzzy Rough Geometric Operator

In this section, we defined spherical linear Diophantine fuzzy rough geometric aggregation operators and their basic properties.

5.1. Spherical Linear Diophantine Fuzzy Rough Weighted Geometric (SLDFRWG) Aggregation Operator

Definition 24. Let $Q(\mathbb{k}_\imath) = (\underline{Q}(\mathbb{k}_\imath), \overline{Q}(\mathbb{k}_\imath))(\imath = 1, \ldots, n)$ is a family of SLDFRNs with weight vector $\omega = (\omega_1, \ldots, \omega_n)^T$, such as $\sum_{\imath=1}^{n}\omega_\imath = 1$ and $0 \leq \omega_\imath \leq 1$. Then, the SLDFRWG operator is defined as

$$\mathrm{SLDFRWG}(Q(\mathbb{k}_1), \ldots, Q(\mathbb{k}_n)) = \left\{\overset{n}{\underset{\imath=1}{\otimes}} \left(\underline{Q}(\mathbb{k}_\imath)\right)^{\omega_\imath}, \overset{n}{\underset{\imath=1}{\otimes}} \left(\overline{Q}(\mathbb{k}_\imath)\right)^{\omega_\imath}\right\}. \tag{45}$$

Using the Definition 24, the aggregated value for SLDFRWG operator is given in Theorem 25.

Theorem 25. *Let* $Q(\mathbb{k}_\imath) = (\underline{Q}(\mathbb{k}_\imath), \overline{Q}(\mathbb{k}_\imath))\,(\imath = 1, \ldots, n)$ *is a family of SLDFRNs under the weight vector* $\omega = (\omega_1, \ldots, \omega_n)^T$. *Then, SLDFRWG operator is described as*

$$\begin{aligned}
\mathrm{SLDFRWG}(Q(\mathbb{k}_1), \ldots, Q(\mathbb{k}_n)) &= \left\{ \underset{\imath=1}{\overset{n}{\otimes}} \left(\underline{Q}(\mathbb{k}_\imath)\right)^{\omega_\imath}, \underset{\imath=1}{\overset{n}{\otimes}} \left(\overline{Q}(\mathbb{k}_\imath)\right)^{\omega_\imath} \right\} \\
&= \left\{ \begin{array}{c} \left(\begin{array}{c} \left\langle \prod_{\imath=1}^{n} (\underline{\mu}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{n} (\underline{\alpha}_\imath)^{\omega_\imath} \right\rangle, \\ \left\langle \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - (\underline{\upsilon}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - (\underline{\beta}_\imath)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - (\underline{\eta}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - (\underline{\gamma}_\imath)^2\right)^{\omega_\imath}} \right\rangle \end{array} \right), \\ \left(\begin{array}{c} \left\langle \prod_{\imath=1}^{n} (\overline{\mu}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{n} (\overline{\alpha}_\imath)^{\omega_\imath} \right\rangle, \\ \left\langle \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - (\overline{\upsilon}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - (\overline{\beta}_\imath)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - (\overline{\eta}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - (\overline{\gamma}_\imath)^2\right)^{\omega_\imath}} \right\rangle \end{array} \right) \end{array} \right\}.
\end{aligned} \tag{46}$$

Proof. To prove this theorem, using mathematical induction principle as

$$\begin{aligned}
Q(\mathbb{k}_1) \otimes Q(\mathbb{k}_2) &= \left\{ \left(\underline{Q}(\mathbb{k}_1) \otimes \underline{Q}(\mathbb{k}_2)\right), \overline{Q}(\mathbb{k}_1) \otimes \overline{Q}(\mathbb{k}_2) \right\} \\
\omega_1 Q(\mathbb{k}_1) &= \left(\omega_1 \underline{Q}(\mathbb{k}_1), \omega_1 \overline{Q}(\mathbb{k}_1)\right) \\
&= \left\{ \begin{array}{c} \left(\begin{array}{c} \left\langle (\underline{\mu}_1)^{\omega_1}, (\underline{\alpha}_1)^{\omega_1} \right\rangle, \\ \left\langle \sqrt{1 - \left(1 - (\underline{\upsilon}_1)^2\right)^{\omega_1}}, \sqrt{1 - \left(1 - \underline{\beta}_1^q\right)^{\omega_1}} \right\rangle, \\ \left\langle \sqrt{1 - \left(1 - (\underline{\eta}_1)^2\right)^{\omega_1}}, \sqrt{1 - \left(1 - (\underline{\gamma}_1)^2\right)^{\omega_1}} \right\rangle \end{array} \right), \\ \left(\begin{array}{c} \left\langle (\overline{\mu}_1)^{\omega_1}, (\overline{\alpha}_1)^{\omega_1} \right\rangle, \\ \left\langle \sqrt{1 - \left(1 - (\overline{\upsilon}_1)^2\right)^{\omega_1}}, \sqrt{1 - \left(1 - (\overline{\beta}_1)^2\right)^{\omega_1}} \right\rangle, \\ \left\langle \sqrt{1 - \left(1 - (\overline{\eta}_1)^2\right)^{\omega_1}}, \sqrt{1 - \left(1 - (\overline{\gamma}_1)^2\right)^{\omega_1}} \right\rangle \end{array} \right) \end{array} \right\}.
\end{aligned} \tag{47}$$

Let $n = 2$, then

$$\mathrm{SLDFRWG}(Q(\mathbb{k}_1),Q(\mathbb{k}_2)) = \left\{\overset{2}{\underset{\imath=1}{\otimes}}\left(\underline{Q}(\mathbb{k}_\imath)\right)^{\omega_\imath}, \overset{2}{\underset{\imath=1}{\otimes}}\left(\overline{Q}(\mathbb{k}_\imath)\right)^{\omega_\imath}\right\}$$

$$= \left\{\begin{array}{c}\left(\begin{array}{c}\left\langle \prod_{\imath=1}^{2}(\underline{\mu}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{2}(\underline{\alpha}_\imath)^{\omega_\imath}\right\rangle,\\ \left\langle \sqrt{1-\prod_{\imath=1}^{2}\left(1-(\underline{\upsilon}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{2}\left(1-(\underline{\beta}_\imath)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle \sqrt{1-\prod_{\imath=1}^{2}\left(1-(\underline{\eta}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{2}\left(1-(\underline{\gamma}_\imath)^2\right)^{\omega_\imath}}\right\rangle\end{array}\right),\\ \left(\begin{array}{c}\left\langle \prod_{\imath=1}^{2}(\overline{\mu}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{2}(\overline{\alpha}_\imath)^{\omega_\imath}\right\rangle,\\ \left\langle \sqrt{1-\prod_{\imath=1}^{2}\left(1-(\overline{\upsilon}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{2}\left(1-(\overline{\beta}_\imath)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle \sqrt{1-\prod_{\imath=1}^{2}\left(1-(\overline{\eta}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{2}\left(1-(\overline{\gamma}_\imath)^2\right)^{\omega_\imath}}\right\rangle\end{array}\right)\end{array}\right\}. \tag{48}$$

The result holds for $n = 2$.

Now, equation (46) is true for $n = k$,

$$\mathrm{SLDFRWG}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_k)) = \left\{\overset{k}{\underset{\imath=1}{\otimes}}\left(\underline{Q}(\mathbb{k}_\imath)\right)^{\omega_\imath}, \overset{k}{\underset{\imath=1}{\otimes}}\left(\overline{Q}(\mathbb{k}_\imath)\right)^{\omega_\imath}\right\}$$

$$= \left\{\begin{array}{c}\left(\begin{array}{c}\left\langle \prod_{\imath=1}^{k}(\underline{\mu}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{k}(\underline{\alpha}_\imath)^{\omega_\imath}\right\rangle,\\ \left\langle \sqrt{1-\prod_{\imath=1}^{k}\left(1-(\underline{\upsilon}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{k}\left(1-(\underline{\beta}_\imath)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle \sqrt{1-\prod_{\imath=1}^{k}\left(1-(\underline{\eta}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{k}\left(1-(\underline{\gamma}_\imath)^2\right)^{\omega_\imath}}\right\rangle\end{array}\right),\\ \left(\begin{array}{c}\left\langle \prod_{\imath=1}^{k}(\overline{\mu}_\imath)^{\omega_\imath}, \prod_{\imath=1}^{k}(\overline{\alpha}_\imath)^{\omega_\imath}\right\rangle,\\ \left\langle \sqrt{1-\prod_{\imath=1}^{k}\left(1-(\overline{\upsilon}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{k}\left(1-(\overline{\beta}_\imath)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle \sqrt{1-\prod_{\imath=1}^{k}\left(1-(\overline{\eta}_\imath)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{k}\left(1-(\overline{\gamma}_\imath)^2\right)^{\omega_\imath}}\right\rangle\end{array}\right)\end{array}\right\}. \tag{49}$$

Next, equation (46) is true for $n = k + 1$,

$$
\begin{aligned}
&\mathrm{SLDFRWG}\left(Q(\Bbbk_1),\ldots,Q(\Bbbk_k)\oplus Q(\Bbbk_{k+1})\right)\\
&=\left\{\underset{\imath=1}{\overset{k}{\otimes}}\underline{Q}(\Bbbk_\imath)^{\omega_\imath}\otimes\left(\underline{Q}(\Bbbk_{k+1})\right)^{\omega_{k+1}},\underset{\imath=1}{\overset{k}{\otimes}}\overline{Q}(\Bbbk_\imath)^{\omega_\imath}\otimes\left(\overline{Q}(\Bbbk_{k+1})\right)^{\omega_{k+1}}\right\}\\
&=\left\{\begin{array}{c}\left(\begin{array}{c}\left\langle\prod_{\imath=1}^{k}(\underline{\mu}_\imath)^{\omega_\imath},\prod_{\imath=1}^{k}(\underline{\alpha}_\imath)^{\omega_\imath}\right\rangle,\\ \left\langle\sqrt{1-\prod_{\imath=1}^{k}\left(1-(\underline{\upsilon}_\imath)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k}\left(1-(\underline{\beta}_\imath)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\sqrt{1-\prod_{\imath=1}^{k}\left(1-(\underline{\eta}_\imath)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k}\left(1-(\underline{\gamma}_\imath)^2\right)^{\omega_\imath}}\right\rangle\end{array}\right),\\ \left(\begin{array}{c}\left\langle\prod_{\imath=1}^{k}(\overline{\mu}_\imath)^{\omega_\imath},\prod_{\imath=1}^{k}(\overline{\alpha}_\imath)^{\omega_\imath}\right\rangle,\\ \left\langle\sqrt{1-\prod_{\imath=1}^{k}\left(1-(\overline{\upsilon}_\imath)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k}\left(1-(\overline{\beta}_\imath)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\sqrt{1-\prod_{\imath=1}^{k}\left(1-(\overline{\eta}_\imath)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k}\left(1-(\overline{\gamma}_\imath)^2\right)^{\omega_\imath}}\right\rangle\end{array}\right)\end{array}\right\}\\
&\otimes\left\{\begin{array}{c}\left(\begin{array}{c}\left\langle\left(\underline{\mu}_{k+1}\right)^{\omega_\imath},\left(\underline{\alpha}_{k+1}\right)^{\omega_\imath}\right\rangle,\\ \left\langle\sqrt{1-\left(1-(\underline{\upsilon}_{k+1})^2\right)^{\omega_\imath}},\sqrt{1-\left(1-(\underline{\beta}_{k+1})^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\sqrt{1-\left(1-(\underline{\eta}_{k+1})^2\right)^{\omega_\imath}},\sqrt{1-\left(1-(\underline{\gamma}_{k+1})^2\right)^{\omega_\imath}}\right\rangle\end{array}\right),\\ \left(\begin{array}{c}\left\langle(\overline{\mu}_{k+1})^{\omega_\imath},(\overline{\alpha}_{k+1})^{\omega_\imath}\right\rangle,\\ \left\langle\sqrt{1-\left(1-(\overline{\upsilon}_{k+1})^2\right)^{\omega_\imath}},\sqrt{1-\left(1-(\overline{\beta}_{k+1})^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\sqrt{1-\left(1-(\overline{\eta}_{k+1})^2\right)^{\omega_\imath}},\sqrt{1-\left(1-(\overline{\gamma}_{k+1})^2\right)^{\omega_\imath}}\right\rangle\end{array}\right)\end{array}\right\}\\
&=\left\{\begin{array}{c}\left(\begin{array}{c}\left\langle\prod_{\imath=1}^{k+1}(\underline{\mu}_\imath)^{\omega_\imath},\prod_{\imath=1}^{k+1}(\underline{\alpha}_\imath)^{\omega_\imath}\right\rangle,\\ \left\langle\sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\underline{\upsilon}_\imath)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\underline{\beta}_\imath)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\underline{\eta}_\imath)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\underline{\gamma}_\imath)^2\right)^{\omega_\imath}}\right\rangle\end{array}\right),\\ \left(\begin{array}{c}\left\langle\prod_{\imath=1}^{k+1}(\overline{\mu}_\imath)^{\omega_\imath},\prod_{\imath=1}^{k+1}(\overline{\alpha}_\imath)^{\omega_\imath}\right\rangle,\\ \left\langle\sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\overline{\upsilon}_\imath)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\overline{\beta}_\imath)^2\right)^{\omega_\imath}}\right\rangle,\\ \left\langle\sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\overline{\eta}_\imath)^2\right)^{\omega_\imath}},\sqrt{1-\prod_{\imath=1}^{k+1}\left(1-(\overline{\gamma}_\imath)^2\right)^{\omega_\imath}}\right\rangle\end{array}\right)\end{array}\right\}.
\end{aligned}
\tag{50}
$$

This holds true for $n=k+1$. As a result, the above result holds for $n\geq 1$.

From the above result, $\underline{Q}(\Bbbk_\imath)$ and $\overline{Q}(\Bbbk_\imath)$ are SLDFRNs. So, by Definition 13, $\otimes_{\imath=1}^{n}\underline{Q}(\Bbbk_\imath)^{\omega_\imath}$ and $\otimes_{\imath=1}^{n}\overline{Q}(\Bbbk_\imath)^{\omega_\imath}$ are also SLDFRNs. Therefore, $\text{SLDFRWG}(Q(\Bbbk_1),\ldots,Q(\Bbbk_k))$ is still an SLDFRN with SLDF approximation space (N,Q).

The SLDFRWG operator satisfied the following properties. □

Theorem 26. *Let $Q(\Bbbk_\imath)=(\underline{Q}(\Bbbk_\imath),\overline{Q}(\Bbbk_\imath))\,(\imath=1,\ldots,n)$ is a family of SLDFRNs with weight vector $\omega=(\omega_1,\ldots,\omega_n)^T$, such as $\sum_{\imath=1}^{n}\omega_\imath=1$ and $0\le\omega_\imath\le1$. Then, important properties of SLDFRWG operators are*

(1) (Idempotency). If $Q(\Bbbk_\imath)=Q(\Bbbk),\forall(\imath=1,\ldots,n)$. Here,

$$\begin{aligned}Q(\Bbbk)&=\left(\underline{Q}(\Bbbk),\overline{Q}(\Bbbk)\right)\\&=\left(\left(\underline{\mu},\underline{\upsilon},\underline{\eta},\underline{\alpha},\underline{\beta},\underline{\gamma}\right),\left(\overline{\mu},\overline{\upsilon},\overline{\eta},\overline{\alpha},\overline{\beta},\overline{\gamma}\right)\right).\end{aligned}\tag{51}$$

Then,

$$\text{SLDFRWG}(Q(\Bbbk_1),\ldots,Q(\Bbbk_n))=Q(\Bbbk).\tag{52}$$

(2) (Boundedness). If $Q(\Bbbk_\imath)^-=(\min_\imath\underline{Q}(\Bbbk_\imath),\max_\imath\overline{Q}(\Bbbk_\imath))$ and $Q(\Bbbk_\imath)^+=(\max_\imath\underline{Q}(\Bbbk_\imath),\min_\imath\overline{Q}(\Bbbk_\imath))$. Then,

$$Q(\Bbbk_\imath)^-\le\text{SLDFRWG}(Q(\Bbbk_1),\ldots,Q(\Bbbk_n))\le Q(\Bbbk_\imath)^+.\tag{53}$$

(3) (Monotonicity). Let $Q(\Bbbk_\imath)=(\underline{Q}(\Bbbk_\imath),\overline{Q}(\Bbbk_\imath))$ and $\mathfrak{I}(\mathscr{L}_\imath)=(\underline{\mathfrak{I}}(\mathscr{L}_\imath),\overline{\mathfrak{I}}(\mathscr{L}_\imath))\,(\imath=1,\ldots,n)$ be any two families of SLDFRNs, such that $\underline{\mathfrak{I}}(\mathscr{L}_\imath)\le\underline{Q}(\Bbbk_\imath)$ and $\overline{\mathfrak{I}}(\mathscr{L}_\imath)\le\overline{Q}(\Bbbk_\imath)$. Then,

$$\text{SLDFRWG}(\mathfrak{I}(\mathscr{L}_1),\ldots,\mathfrak{I}(\mathscr{L}_n))\le\text{SLDFRWG}(Q(\Bbbk_1),\ldots,Q(\Bbbk_n)).\tag{54}$$

(4) (Commutativity). Let $Q'(\Bbbk_\imath)=(\underline{Q}'(\Bbbk_\imath),\overline{Q}'(\Bbbk_\imath))$ $(\imath=1,\ldots,n)$. Then,

$$\text{SLDFRWG}(Q(\Bbbk_1),\ldots,Q(\Bbbk_n))=\text{SLDFRWG}\left(Q'(\Bbbk_1),\ldots,Q'(\Bbbk_n)\right).\tag{55}$$

5.2. Spherical Linear Diophantine Fuzzy Rough Ordered Weighted Geometric (SLDFROWG) Aggregation Operator. We presented the SLDFROWG operator and reviewed its basic features in this section.

Definition 27. Let $Q(\Bbbk_\imath)=(\underline{Q}(\Bbbk_\imath),\overline{Q}(\Bbbk_\imath))\,(\imath=1,\ldots,n)$ is a family of SLDFRNs with weights $\omega=(\omega_1,\ldots,\omega_n)^T$, such as $\sum_{\imath=1}^{n}\omega_\imath=1$ and $0\le\omega_\imath\le1$. The SLDFROWA operator is defined as;

$$\text{SLDFROWG}(Q(\Bbbk_1),\ldots,Q(\Bbbk_n))=\left\{\mathop{\otimes}_{\imath=1}^{n}\left(\underline{Q}\left(\Bbbk_{\sigma(\imath)}\right)\right)^{\omega_\imath},\mathop{\otimes}_{\imath=1}^{n}\left(\overline{Q}\left(\Bbbk_{\sigma(\imath)}\right)\right)^{\omega_\imath}\right\}.\tag{56}$$

Using the Definition 27, the aggregated value for SLDFROWG operator is given in Theorem 28.

Theorem 28. *Let* $Q(\Bbbk_\imath) = (\underline{Q}(\Bbbk_\imath), \overline{Q}(\Bbbk_\imath))\,(\imath = 1, \ldots, n)$ *is a family of SLDFRNs with weights* $\omega = (\omega_1, \ldots, \omega_n)^T$. *Then, SLDFROWG operator is determined as*

$$\mathrm{SLDFROWG}(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)) = \left\{ \bigotimes_{\imath=1}^{n} \left(\underline{Q}(\Bbbk_{\sigma(\imath)})\right)^{\omega_\imath}, \bigotimes_{\imath=1}^{n} \left(\overline{Q}(\Bbbk_{\sigma(\imath)})\right)^{\omega_\imath} \right\}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left\langle \prod_{\imath=1}^{n} \left(\underline{\mu}_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n} \left(\underline{\alpha}_{\sigma(\imath)}\right)^{\omega_\imath} \right\rangle, \\ \left\langle \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - \left(\underline{\upsilon}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - \left(\underline{\beta}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - \left(\underline{\eta}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - \left(\underline{\gamma}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}} \right\rangle \end{array} \right), \\ \left(\begin{array}{c} \left\langle \prod_{\imath=1}^{n} \left(\overline{\mu}_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n} \left(\overline{\alpha}_{\sigma(\imath)}\right)^{\omega_\imath} \right\rangle, \\ \left\langle \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - \left(\overline{\upsilon}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - \left(\overline{\beta}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}} \right\rangle, \\ \left\langle \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - \left(\overline{\eta}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1 - \prod_{\imath=1}^{n} \left(1 - \left(\overline{\gamma}_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}} \right\rangle \end{array} \right) \end{array} \right\}. \tag{57}$$

where $Q(\Bbbk_{\sigma(\imath)}) = (\underline{Q}(\Bbbk_{\sigma(\imath)}), \overline{Q}(\Bbbk_{\sigma(\imath)}))$ *represents the biggest value of permutation from the family of SLDFRNs.*

Proof. Proof is followed from Theorem 25.

The SLDFROWG operator satisfied the following properties. □

Theorem 29. *Let* $Q(\Bbbk_\imath) = (\underline{Q}(\Bbbk_\imath), \overline{Q}(\Bbbk_\imath))\,(\imath = 1, \ldots, n)$ *is a family of SLDFRNs with weights* $\omega = (\omega_1, \ldots, \omega_n)^T$, *such that* $\sum_{\imath=1}^{n} \omega_\imath = 1$ *and* $0 \le \omega_\imath \le 1$. *Then, important properties of SLDFROWG operator are given as*

(1) *(Idempotency). If* $Q(\Bbbk_\imath) = Q(\Bbbk)$ *for all* $(\imath = 1, \ldots, n)$. *Here,*

$$\begin{aligned} Q(\Bbbk) &= \left(\underline{Q}(\Bbbk), \overline{Q}(\Bbbk)\right) \\ &= \left(\left(\underline{\mu}, \underline{\upsilon}, \underline{\eta}, \underline{\alpha}, \underline{\beta}, \underline{\gamma}\right), \left(\overline{\mu}, \overline{\upsilon}, \overline{\eta}, \overline{\alpha}, \overline{\beta}, \overline{\gamma}\right)\right). \end{aligned} \tag{58}$$

Then,

$$\mathrm{SLDFROWG}(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)) = Q(\Bbbk). \tag{59}$$

(2) *(Boundedness). If* $Q(\Bbbk_\imath)^- = (\min_\imath \underline{Q}(\Bbbk_\imath), \max_\imath \overline{Q}(\Bbbk_\imath))$ *and* $Q(\Bbbk_\imath)^+ = (\max_\imath \underline{Q}(\Bbbk_\imath), \min_\imath \overline{Q}(\Bbbk_\imath))$. *Then,*

$$Q(\Bbbk_\imath)^- \le \mathrm{SLDFROWG}(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)) \le Q(\Bbbk_\imath)^+. \tag{60}$$

(3) *(Monotonicity). Let* $Q(\Bbbk_\imath) = (\underline{Q}(\Bbbk_\imath), \overline{Q}(\Bbbk_\imath))$ *and* $\mathfrak{I}(\mathscr{L}_\imath) = (\underline{\mathfrak{I}}(\mathscr{L}_\imath), \overline{\mathfrak{I}}(\mathscr{L}_\imath))\,(\imath = 1, \ldots, n)$ *be any two families of SLDFRNs, such that* $\underline{\mathfrak{I}}(\mathscr{L}_\imath) \le \underline{Q}(\Bbbk_\imath)$ *and* $\overline{\mathfrak{I}}(\mathscr{L}_\imath) \le \overline{Q}(\Bbbk_\imath)$. *Then,*

$$\mathrm{SLDFROWG}(\mathfrak{I}(\mathscr{L}_1), \ldots, \mathfrak{I}(\mathscr{L}_n)) \le \mathrm{SLDFROWG}(Q(\Bbbk_1), \ldots, Q(\Bbbk_n)). \tag{61}$$

(4) *(Commutativity). Let* $Q'(\Bbbk_\imath) = (\underline{Q}'(\Bbbk_\imath), \overline{Q}'(\Bbbk_\imath))$ $(\imath = 1, \ldots, n)$. *Then,*

$$\text{SLDFROWG}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_n)) = \text{SLDFROWG}\left(Q'(\mathbb{k}_1),\ldots,Q'(\mathbb{k}_n)\right). \tag{62}$$

5.3. Spherical Linear Diophantine Fuzzy Rough Hybrid Geometric (SLDFRHG) Aggregation Operator. We introduced the notion of SLDFRHG operators and examined their basic qualities in this section.

Definition 30. Let $Q(\mathbb{k}_\imath) = (\underline{Q}(\mathbb{k}_\imath),\overline{Q}(\mathbb{k}_\imath))\,(\imath = 1,\ldots,n)$ is a family of SLDFRNs with the weight vector and associated weight vector $\omega = (\omega_1,\ldots,\omega_n)^T, w = (w_1,\ldots,w_n)^T$ such as $\sum_{\imath=1}^n \omega_\imath, w_\imath = 1$ and $0 \le \omega_\imath, w_\imath \le 1$, respectively. Then, SLDFRHG operator is defined as

$$\text{SLDFRHG}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_n)) = \left\{\bigotimes_{\imath=1}^{n}\left(\underline{Q}\left(\mathbb{k}^*_{\sigma(\imath)}\right)\right)^{\omega_\imath}, \bigotimes_{\imath=1}^{n}\left(\overline{Q}\left(\mathbb{k}^*_{\sigma(\imath)}\right)\right)^{\omega_\imath}\right\}. \tag{63}$$

Based on Definition 30, the aggregated value for SLDFRHG operator is given in Theorem 31.

Theorem 31. *Let $Q(\mathbb{k}_\imath) = (\underline{Q}(\mathbb{k}_\imath),\overline{Q}(\mathbb{k}_\imath))\,(\imath = 1,\ldots,n)$ be a family of SLDFRNs with weight vector and associated weight vector $\omega = (\omega_1,\ldots,\omega_n)^T, w = (w_1,\ldots,w_n)^T$ such as $\sum_{\imath=1}^n \omega_\imath, w_\imath = 1$ and $0 \le \omega_\imath, w_\imath \le 1$, respectively. Then, the SLDFRHG operator is given as*

$$\begin{aligned}
\text{SLDFRHG}(Q(\mathbb{k}_1),\ldots,Q(\mathbb{k}_n)) &= \left\{\bigotimes_{\imath=1}^{n}\left(\underline{Q}\left(\mathbb{k}^*_{\sigma(\imath)}\right)\right)^{\omega_\imath}, \bigotimes_{\imath=1}^{n}\left(\overline{Q}\left(\mathbb{k}^*_{\sigma(\imath)}\right)\right)^{\omega_\imath}\right\}\\
&= \left\{\begin{array}{c}
\left(\begin{array}{c}
\left\langle \prod_{\imath=1}^{n}\left(\underline{\mu}^*_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\underline{\alpha}^*_{\sigma(\imath)}\right)^{\omega_\imath}\right\rangle,\\
\left\langle \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\underline{\upsilon}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\underline{\beta}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}\right\rangle,\\
\left\langle \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\underline{\eta}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\underline{\gamma}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}\right\rangle
\end{array}\right),\\
\left(\begin{array}{c}
\left\langle \prod_{\imath=1}^{n}\left(\overline{\mu}^*_{\sigma(\imath)}\right)^{\omega_\imath}, \prod_{\imath=1}^{n}\left(\overline{\alpha}^*_{\sigma(\imath)}\right)^{\omega_\imath}\right\rangle,\\
\left\langle \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\overline{\upsilon}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\overline{\beta}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}\right\rangle,\\
\left\langle \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\overline{\eta}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}, \sqrt{1-\prod_{\imath=1}^{n}\left(1-\left(\overline{\gamma}^*_{\sigma(\imath)}\right)^2\right)^{\omega_\imath}}\right\rangle
\end{array}\right)
\end{array}\right\}.
\end{aligned} \tag{64}$$

where $Q(\mathbb{k}^*_{\sigma(\imath)}) = (Q(\mathbb{k}_{\sigma(\imath)}))^{nw_\imath} = ((\underline{Q}(\mathbb{k}_{\sigma(\imath)}))^{nw_\imath}, (\overline{Q}(\mathbb{k}_{\sigma(\imath)}))^{nw_\imath})$ denoted the largest value of permutation from the family of SLDFRNs and n shows the balancing coefficient.

Proof. Proof is followed from Theorem 25.

Specially, if $w = ((1/n),\ldots,(1/n))^T$, then the proposed SLDFRHG operator becomes the SLDFROWG operator. □

6. Approach for MAGDM Based on Spherical Linear Diophantine Fuzzy Rough Information

In this section, we design an algorithm based on the proposed spherical linear diophantine fuzzy rough EDAS method to address the classical MAGDM problem.

This part shall develop an MAGDM method using the defined operator with spherical linear diophantine fuzzy rough information. For the conventional MAGDM problem, assume that $\wp = \{\wp_1, \ldots, \wp_m\}$ is a set of m alternatives and let $\mathscr{C} = \{\mathscr{C}_1, \ldots, \mathscr{C}_n\}$ be a set of n attribute. Let an expert $E = (E_1, \ldots, E_l)$, who gives his evaluation assessment for every alternative $\wp_\iota\,(\iota = 1, \ldots, n)$ against their attribute $\mathscr{C}_j\,(1, \ldots, n)$. Let $\omega = (\omega_1, \ldots, \omega_t)^T$ be the weights for attributes, such as $\sum_{i=1}^{n} \omega_l$ and $0 \le \omega_i \le 1$. The main steps of the algorithm are stated as follows:

Step 1: The collective expert data are arranged in a matrix.

$$\mathscr{M} = \left[Q\left(\mathbb{k}_{\iota j}^{l}\right)\right]_{m\times n}. \tag{65}$$

Step 2: The following transformation can be used to normalize the decision matrix:

$$\mathscr{M} = \begin{cases} Q\left(\mathbb{k}_{\iota j}\right) = \left(\underline{Q}\,(\mathbb{k}_\iota), \overline{Q}\,(\mathbb{k}_\iota)\right) \text{ for benefit,} \\ Q\left(\mathbb{k}_{\iota j}\right) = \left(\overline{Q}\,(\mathbb{k}_\iota), \underline{\mathfrak{R}}\,(\mathbb{k}_\iota)\right) \text{ for cost.} \end{cases} \tag{66}$$

Step 3: Aggregated the information given by experts using the proposed operator.

Step 4: Using the proposed method, calculate the value of AvS for all possibilities.

$$\begin{aligned} \mathrm{AvS} &= \left[\mathrm{AvS}_j\right]_{1\times n} \\ &= \left\{\frac{1}{m}\sum_{\iota=1}^{n} Q\left(\mathbb{k}_{\iota j}^{n}\right)\right\}_{1\times n} \\ &= \left\{\begin{array}{l} \left(\begin{array}{c} \left\langle \sqrt{1-\prod_{\iota=1}^{m}\left(1-\left(\underline{\mu}_\iota\right)^{qn}\right)^{1/m}}, \sqrt{1-\prod_{\iota=1}^{n}\left(1-\left(\underline{\alpha}_\iota\right)^{qn}\right)^{1/m}} \right\rangle, \\ \left\langle \prod_{\iota=1}^{n}\left(\left(\underline{\upsilon}_\iota\right)^{n}\right)^{1/m}, \prod_{\iota=1}^{n}\left(\left(\underline{\beta}_\iota\right)^{n}\right)^{1/m} \right\rangle, \left\langle \prod_{\iota=1}^{n}\left(\left(\underline{\eta}_\iota\right)^{n}\right)^{1/m}, \prod_{\iota=1}^{n}\left(\left(\underline{\gamma}_\iota\right)^{n}\right)^{1/m} \right\rangle \end{array}\right), \\ \left(\begin{array}{c} \left\langle \sqrt{1-\prod_{\iota=1}^{n}\left(1-\left(\overline{\mu}_\iota\right)^{qn}\right)^{1/m}}, \sqrt{1-\prod_{\iota=1}^{n}\left(1-\left(\overline{\alpha}_\iota\right)^{qn}\right)^{1/m}} \right\rangle, \\ \left\langle \prod_{\iota=1}^{n}\left(\left(\overline{\upsilon}_\iota\right)^{n}\right)^{1/m}, \prod_{\iota=1}^{n}\left(\left(\overline{\beta}_\iota\right)^{n}\right)^{1/m} \right\rangle, \left\langle \prod_{\iota=1}^{n}\left(\left(\overline{\eta}_\iota\right)^{n}\right)^{1/m}, \prod_{\iota=1}^{n}\left(\left(\overline{\gamma}_\iota\right)^{n}\right)^{1/m} \right\rangle \end{array}\right) \end{array}\right\}_{1\times n}. \end{aligned} \tag{67}$$

Step 5: Using the determined value of Av S, find the value of PDA S and NDA S, using the following formula:

$$\begin{aligned} \mathrm{PDAS}_{\iota j} &= \left[\mathrm{PDAS}_{\iota j}\right]_{m\times n} \\ &= \frac{\max\left(0, \left[\mathrm{Sco}^*\left(Q\left(\mathbb{k}_{\iota j}^{n}\right)\right) - \mathrm{Sco}^*\left(\mathrm{AvS}_j\right)\right]\right)}{\mathrm{Sco}^*\left(\mathrm{AvS}_j\right)}, \\ \mathrm{NDAS}_{\iota j} &= \left[\mathrm{NDAS}_{\iota j}\right]_{m\times n} \\ &= \frac{\max\left(0, \left[Sco^*\left(\mathrm{AvS}_j\right) - \mathrm{Sco}^*\left(Q\left(\mathbb{k}_{\iota j}^{n}\right)\right)\right]\right)}{\mathrm{Sco}^*\left(\mathrm{AvS}_j\right)}. \end{aligned} \tag{68}$$

Step 6: In this step, find the positive (SP_ι) and negative weight distance (SN_ι).

$$\begin{aligned} \mathrm{SP}_\iota &= \sum_{j=1}^{n} \omega_j \mathrm{PDAS}_{\iota j}, \\ \mathrm{SN}_\iota &= \sum_{j=1}^{n} \omega_j \mathrm{NDAS}_{\iota j}. \end{aligned} \tag{69}$$

Step 7: Using the following formulas to normalize the SP_ι and SN_ι,

$$\begin{aligned} \mathrm{NSP}_\iota &= \frac{SP_\iota}{\max_\iota\,(\mathrm{SP}_\iota)}, \\ \mathrm{NSN}_\iota &= 1 - \frac{\mathrm{SN}_\iota}{\max_\iota\,(\mathrm{SN}_\iota)}. \end{aligned} \tag{70}$$

Step 8: Using the values of NSP_ι and NSN_ι, find AS (appraisal score value) by the given formula:

$$\mathrm{AS}_\imath = \frac{1}{2}\left(\mathrm{NSP}_\imath + \mathrm{NSN}_\imath\right). \tag{71}$$

Step 9: Based on the value of $\mathrm{AS}_\imath$, give ranking to alternatives.

Figure 1 shows the flowchart of the proposed algorithm.

7. Illustrative Example

Using a numerical example to choose the best company out of six options, the proposed MAGDM method is illustrated (adapted from [62]).

In February 2019, a recent storm generated severe rainfall in the Lasbella district and nearby portions of Baluchistan, Pakistan, were devastated with unprecedented flash floods. This flood ruined a considerable number of roadways connecting the Lasbella district to other regions of Baluchistan. In this setting, the Pakistani government must undertake a significant number of road construction projects, either to conserve existing highways or to construct new roads.

These projects were completed by a small number of well-known contractors, and the selection process was based solely on bid pricing. Increased project complexity, technical capabilities, improved performance, safety, and budgetary constraints have necessitated the employment of multi-attribute decision-making methodologies in recent years. The Pakistani government has put a notice in the newspapers, and one construction business has been tasked with selecting the best construction company from a list of five probable possibilities, such as $\wp_1$: Ahmed Construction, $\wp_2$: MATRACON Pakistan Private (Pvt) Limited (Ltd), $\wp_3$: Eastern Highway Company, $\wp_4$: Banu Mukhtar Concrete Pvt. Ltd. $\wp_5$: Khyber Grace Pvt. Ltd. on the basis of the attributes, $\mathscr{C}_1$: technical capability, $\mathscr{C}_2$: higher performance, $\mathscr{C}_3$: safety, $\mathscr{C}_4$: financial requirements, $\mathscr{C}_5$: time saving, that is bid for these projects, and all the attributes are of the benefit type, so there is no need to normalize it. The government's goal is then to select the best construction company among them for the job. To fulfil it, let two experts give their assessment whose weights are $w = (0.5, 0.5)^T$ and $\omega = (0.20, 0.18, 0.22, 0.25, 0.15)^T$ be the weight vectors corresponding to the five attributes, such that they evaluated each company and gave their preferences in terms of spherical fuzzy information and thus constructed the following decision matrices shown in Tables 1 and 2:

Step 1: Tables 1 and 2 provide the cumulative information provided by each expert for each alternative $\wp_i$ under the attribute $\mathscr{C}_j$.

Step 2: As all the attributes are of the same type (benefit), so normalization is not required.

Step 3: Table 3 shows the results of aggregating the information provided by experts against their weights using the SLDFRWA operators.

Step 4: Find the Av S value by using the proposed approached for each alternative with each attribute given in Table 4.

Step 5: Using the determined value of Av S from Table 5, find the scores of Av $S_\imath\,(\imath = 1, \ldots, 5)$ as

$$\begin{aligned}
\mathrm{AvS}_1 &= 0.3751,\\
\mathrm{AvS}_2 &= 0.4801,\\
\mathrm{AvS}_3 &= 0.3150,\\
\mathrm{AvS}_4 &= 0.2893,\\
\mathrm{AvS}_5 &= 0.3725.
\end{aligned} \tag{72}$$

Now, find the PDA S and NDA S value as given in Tables 5 and 6.

Step 6: Using the attribute weights $\omega = (0.20, 0.18, 0.22, 0.25, 0.15)^T$, to find the value of $\mathrm{SP}_\imath$ and $\mathrm{SN}_\imath$, is given in Table 7.

Step 7: Normalize $\mathrm{SP}_\imath$ and $\mathrm{SN}_\imath$ is given by

$$\begin{aligned}
\mathrm{NSP}_1 &= 0.974,\\
\mathrm{NSP}_2 &= 0.651,\\
\mathrm{NSP}_3 &= 0.563,\\
\mathrm{NSP}_4 &= 0.796,\\
\mathrm{NSP}_5 &= 1.000,\\
\mathrm{NSN}_1 &= 0.438,\\
\mathrm{NSN}_2 &= 0.359,\\
\mathrm{NSN}_3 &= 0.243,\\
\mathrm{NSN}_4 &= 0.000,\\
\mathrm{NSN}_5 &= 0.477.
\end{aligned} \tag{73}$$

Step 8: Find the appraisal score (AS) value using $\mathrm{NSP}_\imath$ and $\mathrm{NSN}_\imath$;

$$\begin{aligned}
\mathrm{AS}_1 &= 0.706,\\
\mathrm{AS}_2 &= 0.505,\\
\mathrm{AS}_3 &= 0.403,\\
\mathrm{AS}_4 &= 0.398,\\
\mathrm{AS}_5 &= 0.738.
\end{aligned} \tag{74}$$

Step 9: Table 8 shows the score values and ranking of the defined model based on the EDAS approach. As a result, the optimal option is $\wp_5$.

Figure 2 shows the alternatives ranking graphically.

7.1. Comparative Study.

A comparative study was conducted in the context of various other existing methods in order to establish the superiority of our specified SLDFR-EDAS method.

Table 9 summarizes the aggregated findings of our comparative analysis of existing models using Table 4. Table 9 shows that the other available methods are unable to solve the defined illustrated case of Section 6 using SLDF rough values. Also, Table 9 shows that there is a shortage of rough information on the current methods, and these

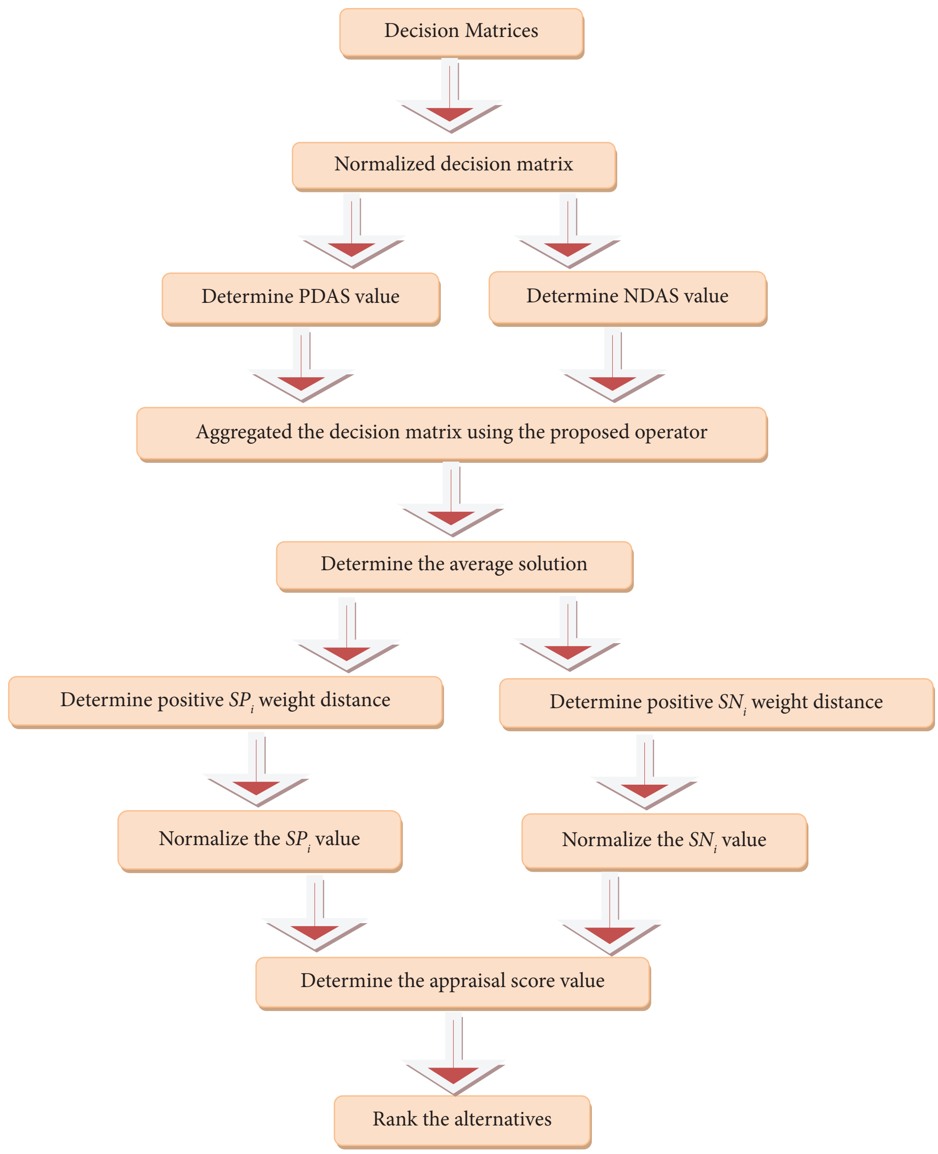

Figure 1: Flowchart of the proposed algorithm.

Table 1: SLDFR evaluation information given by expert E_1.

	$\mathscr{C}_1$
$\wp_1$	$((\langle 0.5, 0.6, 0.4\rangle, \langle 0.2, 0.3, 0.5\rangle), (\langle 0.4, 0.2, 0.7\rangle, \langle 0.3, 0.5, 0.2\rangle))$
$\wp_2$	$((\langle 0.8, 0.3, 0.5\rangle, \langle 0.4, 0.2, 0.3\rangle), (\langle 0.6, 0.4, 0.8\rangle, \langle 0.4, 0.4, 0.1\rangle))$
$\wp_3$	$((\langle 0.4, 0.7, 0.2\rangle, \langle 0.6, 0.2, 0.2\rangle), (\langle 0.5, 0.5, 0.3\rangle, \langle 0.2, 0.5, 0.3\rangle))$
$\wp_4$	$((\langle 0.6, 0.5, 0.7\rangle, \langle 0.4, 0.1, 0.5\rangle), (\langle 0.7, 0.3, 0.7\rangle, \langle 0.3, 0.6, 0.1\rangle))$
$\wp_5$	$((\langle 0.3, 0.6, 0.4\rangle, \langle 0.3, 0.4, 0.3\rangle), (\langle 0.4, 0.6, 0.5\rangle, \langle 0.1, 0.2, 0.6\rangle))$
	$\mathscr{C}_2$
$\wp_1$	$((\langle 0.3, 0.8, 0.3\rangle, \langle 0.6, 0.3, 0.1\rangle), (\langle 0.8, 0.6, 0.5\rangle, \langle 0.6, 0.1, 0.3\rangle))$
$\wp_2$	$((\langle 0.7, 0.4, 0.5\rangle, \langle 0.2, 0.5, 0.3\rangle), (\langle 0.9, 0.1, 0.6\rangle, \langle 0.4, 0.3, 0.2\rangle))$
$\wp_3$	$((\langle 0.2, 0.9, 0.6\rangle, \langle 0.5, 0.1, 0.4\rangle), (\langle 0.6, 0.7, 0.3\rangle, \langle 0.2, 0.5, 0.3\rangle))$
$\wp_4$	$((\langle 0.5, 0.7, 0.4\rangle, \langle 0.4, 0.3, 0.2\rangle), (\langle 0.3, 0.8, 0.5\rangle, \langle 0.1, 0.3, 0.6\rangle))$
$\wp_5$	$((\langle 0.6, 0.4, 0.5\rangle, \langle 0.3, 0.2, 0.5\rangle), (\langle 0.4, 0.6, 0.3\rangle, \langle 0.2, 0.4, 0.1\rangle))$

Table 1: Continued.

	$\mathscr{C}_3$
$\wp_1$	$(((\langle 0.4, 0.7, 0.3\rangle, \langle 0.2, 0.3, 0.4\rangle), (\langle 0.5, 0.2, 0.8\rangle, \langle 0.1, 0.6, 0.2\rangle))$
$\wp_2$	$(((\langle 0.6, 0.5, 0.7\rangle, \langle 0.4, 0.1, 0.3\rangle), (\langle 0.7, 0.3, 0.4\rangle, \langle 0.2, 0.5, 0.3\rangle))$
$\wp_3$	$(((\langle 0.5, 0.3, 0.6\rangle, \langle 0.7, 0.2, 0.1\rangle), (\langle 0.6, 0.4, 0.6\rangle, \langle 0.3, 0.4, 0.2\rangle))$
$\wp_4$	$(((\langle 0.5, 0.6, 0.2\rangle, \langle 0.5, 0.3, 0.2\rangle), (\langle 0.4, 0.2, 0.8\rangle, \langle 0.3, 0.1, 0.5\rangle))$
$\wp_5$	$(((\langle 0.5, 0.9, 0.4\rangle, \langle 0.4, 0.3, 0.1\rangle), (\langle 0.3, 0.7, 0.4\rangle, \langle 0.3, 0.5, 0.2\rangle))$
	$\mathscr{C}_4$
$\wp_1$	$(((\langle 0.7, 0.4, 0.5\rangle, \langle 0.1.0.5, 0.2\rangle), (\langle 0.9, 0.1, 0.3\rangle, \langle 0.1, 0.5, 0.4\rangle))$
$\wp_2$	$(((\langle 0.3, 0.8, 0.4\rangle, \langle 0.2, 0.3, 0.5\rangle), (\langle 0.8, 0.6, 0.5\rangle, \langle 0.6, 0.2, 0.1\rangle))$
$\wp_3$	$(((\langle 0.5, 0.7, 0.6\rangle, \langle 0.4, 0.3, 0.2\rangle), (\langle 0.3, 0.8, 0.4\rangle, \langle 0.2, 0.4, 0.3\rangle))$
$\wp_4$	$(((\langle 0.2, 0.9, 0.3\rangle, \langle 0.5, 0.1, 0.3\rangle), (\langle 0.6, 0.7, 0.6\rangle, \langle 0.5, 0.3, 0.2\rangle))$
$\wp_5$	$(((\langle 0.4, 0.4, 0.8\rangle, \langle 0.1, 0.5, 0.4\rangle), (\langle 0.8, 0.1, 0.6\rangle, \langle 0.4, 0.2, 0.4\rangle))$
	$\mathscr{C}_5$
$\wp_1$	$(((\langle 0.5, 0.7, 0.4\rangle, \langle 0.4, 0.1, 0.3\rangle), (\langle 0.3, 0.8, 0.4\rangle, \langle 0.5, 0.3, 0.2\rangle))$
$\wp_2$	$(((\langle 0.6, 0.4, 0.7\rangle, \langle 0.5, 0.2, 0.1\rangle), (\langle 0.8, 0.6, 0.3\rangle, \langle 0.2, 0.6, 0.1\rangle))$
$\wp_3$	$(((\langle 0.2, 0.9, 0.5\rangle, \langle 0.3, 0.4, 0.2\rangle), (\langle 0.2, 0.9, 0.7\rangle, \langle 0.1, 0.3, 0.5\rangle))$
$\wp_4$	$(((\langle 0.7, 0.4, 0.6\rangle, \langle 0.1, 0.5, 0.4\rangle), (\langle 0.9, 0.1, 0.6\rangle, \langle 0.4, 0.1, 0.4\rangle))$
$\wp_5$	$(((\langle 0.6, 0.5, 0.2\rangle, \langle 0.3, 0.4, 0.2\rangle), (\langle 0.2, 0.9, 0.7\rangle, \langle 0.1, 0.3, 0.5\rangle))$

Table 2: SLDFR evaluation information given by expert E_2.

	$\mathscr{C}_1$
$\wp_1$	$(((\langle 0.8, 0.3, 0.5\rangle, \langle 0.4, 0.2, 0.3\rangle), (\langle 0.6, 0.4, 0.8\rangle, \langle 0.4, 0.4, 0.1\rangle))$
$\wp_2$	$(((\langle 0.3, 0.8, 0.3\rangle, \langle 0.6, 0.3, 0.1\rangle), (\langle 0.8, 0.6, 0.5\rangle, \langle 0.6, 0.1, 0.3\rangle))$
$\wp_3$	$(((\langle 0.4, 0.7, 0.2\rangle, \langle 0.6, 0.2, 0.2\rangle), (\langle 0.5, 0.5, 0.3\rangle, \langle 0.2, 0.5, 0.3\rangle))$
$\wp_4$	$(((\langle 0.3, 0.6, 0.4\rangle, \langle 0.3, 0.4, 0.3\rangle), (\langle 0.4, 0.6, 0.5\rangle, \langle 0.1, 0.2, 0.6\rangle))$
$\wp_5$	$(((\langle 0.6, 0.5, 0.7\rangle, \langle 0.4, 0.1, 0.5\rangle), (\langle 0.7, 0.3, 0.7\rangle, \langle 0.3, 0.6, 0.1\rangle))$
	$\mathscr{C}_2$
$\wp_1$	$(((\langle 0.2, 0.9, 0.6\rangle, \langle 0.5, 0.1, 0.4\rangle), (\langle 0.6, 0.7, 0.3\rangle, \langle 0.2, 0.5, 0.3\rangle))$
$\wp_2$	$(((\langle 0.7, 0.4, 0.5\rangle, \langle 0.2, 0.5, 0.3\rangle), (\langle 0.9, 0.1, 0.6\rangle, \langle 0.4, 0.3, 0.2\rangle))$
$\wp_3$	$(((\langle 0.2, 0.9, 0.3\rangle, \langle 0.5, 0.1, 0.3\rangle), (\langle 0.6, 0.7, 0.6\rangle, \langle 0.5, 0.3, 0.2\rangle))$
$\wp_4$	$(((\langle 0.6, 0.4, 0.5\rangle, \langle 0.3, 0.2, 0.5\rangle), (\langle 0.4, 0.6, 0.3\rangle, \langle 0.2, 0.4, 0.1\rangle))$
$\wp_5$	$(((\langle 0.5, 0.6, 0.2\rangle, \langle 0.5, 0.3, 0.2\rangle), (\langle 0.4, 0.2, 0.9\rangle, \langle 0.3, 0.1, 0.5\rangle))$
	$\mathscr{C}_3$
$\wp_1$	$(((\langle 0.4, 0.7, 0.3\rangle, \langle 0.2, 0.3, 0.4\rangle), (\langle 0.5, 0.5, 0.8\rangle, \langle 0.1, 0.6, 0.2\rangle))$
$\wp_2$	$(((\langle 0.6, 0.5, 0.7\rangle, \langle 0.4, 0.1, 0.3\rangle), (\langle 0.7, 0.3, 0.4\rangle, \langle 0.2, 0.5, 0.3\rangle))$
$\wp_3$	$(((\langle 0.8, 0.3, 0.6\rangle, \langle 0.7, 0.2, 0.1\rangle), (\langle 0.6, 0.4, 0.6\rangle, \langle 0.3, 0.4, 0.2\rangle))$
$\wp_4$	$(((\langle 0.5, 0.7, 0.6\rangle, \langle 0.4, 0.3, 0.2\rangle), (\langle 0.3, 0.8, 0.4\rangle, \langle 0.2, 0.4, 0.3\rangle))$
$\wp_5$	$(((\langle 0.5, 0.9, 0.4\rangle, \langle 0.4, 0.3, 0.1\rangle), (\langle 0.3, 0.7, 0.4\rangle, \langle 0.3, 0.5, 0.2\rangle))$
	$\mathscr{C}_4$
$\wp_1$	$(((\langle 0.7, 0.4, 0.6\rangle, \langle 0.1, 0.5, 0.4\rangle), (\langle 0.9, 0.1, 0.6\rangle, \langle 0.4, 0.1, 0.4\rangle))$
$\wp_2$	$(((\langle 0.3, 0.8, 0.4\rangle, \langle 0.2, 0.3, 0.5\rangle), (\langle 0.8, 0.6, 0.5\rangle, \langle 0.6, 0.2, 0.1\rangle))$
$\wp_3$	$(((\langle 0.5, 0.6, 0.4\rangle, \langle 0.2, 0.3, 0.5\rangle), (\langle 0.4, 0.2, 0.7\rangle, \langle 0.3, 0.5, 0.2\rangle))$
$\wp_4$	$(((\langle 0.5, 0.7, 0.4\rangle, \langle 0.4, 0.1, 0.3\rangle), (\langle 0.3, 0.8, 0.4\rangle, \langle 0.5, 0.3, 0.2\rangle))$
$\wp_5$	$(((\langle 0.4, 0.4, 0.8\rangle, \langle 0.1, 0.5, 0.4\rangle), (\langle 0.8, 0.1, 0.6\rangle, \langle 0.4, 0.2, 0.4\rangle))$
	$\mathscr{C}_5$
$\wp_1$	$(((\langle 0.5, 0.7, 0.4\rangle, \langle 0.4, 0.3, 0.2\rangle), (\langle 0.3, 0.8, 0.5\rangle, \langle 0.1, 0.3, 0.6\rangle))$
$\wp_2$	$(((\langle 0.6, 0.5, 0.2\rangle, \langle 0.3, 0.4, 0.2\rangle), (\langle 0.2, 0.9, 0.7\rangle, \langle 0.1, 0.3, 0.5\rangle))$
$\wp_3$	$(((\langle 0.2, 0.9, 0.5\rangle, \langle 0.3, 0.4, 0.2\rangle), (\langle 0.2, 0.9, 0.7\rangle, \langle 0.1, 0.3, 0.5\rangle))$
$\wp_4$	$(((\langle 0.6, 0.4, 0.7\rangle, \langle 0.5, 0.2, 0.1\rangle), (\langle 0.8, 0.6, 0.3\rangle, \langle 0.2, 0.6, 0.1\rangle))$
$\wp_5$	$(((\langle 0.7, 0.4, 0.5\rangle, \langle 0.1.0.5, 0.2\rangle), (\langle 0.9, 0.1, 0.3\rangle, \langle 0.1, 0.5, 0.4\rangle))$

Table 3: Aggregated decision matrix using the SLDFRWA operator.

$\mathscr{C}_1$	
$\wp_1$	$\begin{pmatrix} (\langle 0.361, 0.182, 0.485\rangle, \langle 0.321, 0.265, 0.146\rangle), \\ (\langle 0.265, 0.374, 0.244\rangle, \langle 0.277, 0.310, 0.331\rangle) \end{pmatrix}$
$\wp_2$	$\begin{pmatrix} (\langle 0.259, 0.179, 0.142\rangle, \langle 0.341, 0.238, 0.193\rangle), \\ (\langle 0.209, 0.373, 0.208\rangle, \langle 0.322, 0.250, 0.361\rangle) \end{pmatrix}$
$\wp_3$	$\begin{pmatrix} (\langle 0.416, 0.411, 0.162\rangle, \langle 0.185, 0.290, 0.211\rangle), \\ (\langle 0.285, 0.217, 0.381\rangle, \langle 0.216, 0.241, 0.482\rangle) \end{pmatrix}$
$\wp_4$	$\begin{pmatrix} (\langle 0.361, 0.263, 0.434\rangle, \langle 0.317, 0.385, 0.303\rangle), \\ (\langle 0.276, 0.154, 0.272\rangle, \langle 0.121, 0.217, 0.199\rangle) \end{pmatrix}$
$\wp_5$	$\begin{pmatrix} (\langle 0.417, 0.236, 0.144\rangle, \langle 0.373, 0.233, 0.253\rangle), \\ (\langle 0.126, 0.414, 0.223\rangle, \langle 0.191, 0.278, 0.314\rangle) \end{pmatrix}$
$\mathscr{C}_2$	
$\wp_1$	$\begin{pmatrix} (\langle 0.376, 0.213, 0.421\rangle, \langle 0.371, 0.317, 0.290\rangle), \\ (\langle 0.308, 0.275, 0.167\rangle, \langle 0.285, 0.214, 0.268\rangle) \end{pmatrix}$
$\wp_2$	$\begin{pmatrix} (\langle 0.356, 0.246, 0.410\rangle, \langle 0.361, 0.288, 0.319\rangle), \\ (\langle 0.275, 0.387, 0.264\rangle, \langle 0.217, 0.186, 0.307\rangle) \end{pmatrix}$
$\wp_3$	$\begin{pmatrix} (\langle 0.482, 0.271, 0.269\rangle, \langle 0.381, 0.317, 0.174\rangle), \\ (\langle 0.260, 0.331, 0.401\rangle, \langle 0.198, 0.271, 0.294\rangle) \end{pmatrix}$
$\wp_4$	$\begin{pmatrix} (\langle 0.327, 0.218, 0.394\rangle, \langle 0.163, 0.290, 0.420\rangle), \\ (\langle 0.374, 0.276, 0.266\rangle, \langle 0.231, 0.314, 0.326\rangle) \end{pmatrix}$
$\wp_5$	$\begin{pmatrix} (\langle 0.423, 0.323, 0.334\rangle, \langle 0.213, 0.320, 0.246\rangle), \\ (\langle 0.134, 0.426, 0.126\rangle, \langle 0.265, 0.234, 0.365\rangle) \end{pmatrix}$
$\mathscr{C}_3$	
$\wp_1$	$\begin{pmatrix} (\langle 0.432, 0.380, 0.219\rangle, \langle 0.241, 0.324, 0.354\rangle), \\ (\langle 0.274, 0.297, 0.360\rangle, \langle 0.397, 0.198, 0.309\rangle) \end{pmatrix}$
$\wp_2$	$\begin{pmatrix} (\langle 0.419, 0.328, 0.217\rangle, \langle 0.320, 0.297, 0.287\rangle), \\ (\langle 0.223, 0.271, 0.255\rangle, \langle 0.177, 0.265, 0.212\rangle) \end{pmatrix}$
$\wp_3$	$\begin{pmatrix} (\langle 0.264, 0.227, 0.116\rangle, \langle 0.427, 0.384, 0.227\rangle), \\ (\langle 0.279, 0.318, 0.339\rangle, \langle 0.431, 0.290, 0.273\rangle) \end{pmatrix}$
$\wp_4$	$\begin{pmatrix} (\langle 0.412, 0.217, 0.283\rangle, \langle 0.163, 0.330, 0.219\rangle), \\ (\langle 0.337, 0.398, 0.187\rangle, \langle 0.215, 0.239, 0.335\rangle) \end{pmatrix}$
$\wp_5$	$\begin{pmatrix} (\langle 0.378, 0.329, 0.261\rangle, \langle 0.328, 0.277, 0.291\rangle), \\ (\langle 0.177, 0.331, 0.262\rangle, \langle 0.429, 0.207, 0.224\rangle) \end{pmatrix}$
$\mathscr{C}_4$	
$\wp_1$	$\begin{pmatrix} (\langle 0.317, 0.384, 0.280\rangle, \langle 0.251, 0.364, 0.153\rangle), \\ (\langle 0.233, 0.172, 0.318\rangle, \langle 0.384, 0.217, 0.220\rangle) \end{pmatrix}$
$\wp_2$	$\begin{pmatrix} (\langle 0.143, 0.340, 0.209\rangle, \langle 0.214, 0.239, 0.313\rangle), \\ (\langle 0.275, 0.160, 0.138\rangle, \langle 0.234, 0.427, 0.209\rangle) \end{pmatrix}$
$\wp_3$	$\begin{pmatrix} (\langle 0.273, 0.337, 0.291\rangle, \langle 0.307, 0.223, 0.109\rangle), \\ (\langle 0.132, 0.327, 0.195\rangle, \langle 0.307, 0.210, 0.431\rangle) \end{pmatrix}$
$\wp_4$	$\begin{pmatrix} (\langle 0.142, 0.266, 0.337\rangle, \langle 0.412, 0.328, 0.211\rangle), \\ (\langle 0.223, 0.270, 0.119\rangle, \langle 0.251, 0.270, 0.266\rangle) \end{pmatrix}$
$\wp_5$	$\begin{pmatrix} (\langle 0.379, 0.185, 0.332\rangle, \langle 0.241, 0.188, 0.224\rangle), \\ (\langle 0.279, 0.404, 0.275\rangle, \langle 0.385, 0.220, 0.281\rangle) \end{pmatrix}$
$\mathscr{C}_5$	
$\wp_1$	$\begin{pmatrix} (\langle 0.411, 0.291, 0.218\rangle, \langle 0.210, 0.308, 0.177\rangle), \\ (\langle 0.263, 0.316, 0.297\rangle, \langle 0.326, 0.233, 0.283\rangle) \end{pmatrix}$
$\wp_2$	$\begin{pmatrix} (\langle 0.168, 0.374, 0.338\rangle, \langle 0.207, 0.315, 0.216\rangle), \\ (\langle 0.193, 0.217, 0.284\rangle, \langle 0.263, 0.183, 0.279\rangle) \end{pmatrix}$
$\wp_3$	$\begin{pmatrix} (\langle 0.327, 0.364, 0.211\rangle, \langle 0.217, 0.308, 0.254\rangle), \\ (\langle 0.250, 0.316, 0.289\rangle, \langle 0.346, 0.283, 0.316\rangle) \end{pmatrix}$

Table 3: Continued.

$\wp_4$	$\begin{pmatrix} (\langle 0.327, 0.324, 0.186\rangle, \langle 0.270, 0.284, 0.210\rangle), \\ (\langle 0.295, 0.381, 0.219\rangle, \langle 0.217, 0.332, 0.193\rangle) \end{pmatrix}$
$\wp_5$	$\begin{pmatrix} (\langle 0.362, 0.432, 0.328\rangle, \langle 0.396, 0.209, 0.402\rangle), \\ (\langle 0.370, 0.236, 0.221\rangle, \langle 0.174, 0.324, 0.258\rangle) \end{pmatrix}$

Table 4: The value of the average solution (AvS).

$\mathscr{C}_1$	$\begin{pmatrix} (\langle 0.362, 0.193, 0.375\rangle, \langle 0.411, 0.262, 0.209\rangle), \\ (\langle 0.484, 0.276, 0.328\rangle, \langle 0.280, 0.375, 0.184\rangle) \end{pmatrix}$
$\mathscr{C}_2$	$\begin{pmatrix} (\langle 0.376, 0.210, 0.428\rangle, \langle 0.409, 0.246, 0.397\rangle), \\ (\langle 0.253, 0.298, 0.132\rangle, \langle 0.328, 0.510, 0.284\rangle) \end{pmatrix}$
$\mathscr{C}_3$	$\begin{pmatrix} (\langle 0.374, 0.332, 0.372\rangle, \langle 0.184, 0.406, 0.330\rangle), \\ (\langle 0.518, 0.219, 0.411\rangle, \langle 0.369, 0.310, 0.204\rangle) \end{pmatrix}$
$\mathscr{C}_4$	$\begin{pmatrix} (\langle 0.541, 0.360, 0.317\rangle, \langle 0.280, 0.411, 0.361\rangle), \\ (\langle 0.346, 0.221, 0.362\rangle, \langle 0.289, 0.252, 0.322\rangle) \end{pmatrix}$
$\mathscr{C}_5$	$\begin{pmatrix} (\langle 0.485, 0.219, 0.408\rangle, \langle 0.264, 0.322, 0.298\rangle), \\ (\langle 0.269, 0.372, 0.377\rangle, \langle 0.310, 0.353, 0.415\rangle) \end{pmatrix}$

Table 5: The results of the PDA S_{ij} matrix.

	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$
$\wp_1$	0.0372	0.0499	0.0434	0.0000	0.0697
$\wp_2$	0.000	0.0000	0.0000	0.0641	0.0000
$\wp_3$	0.0418	0.0602	0.0113	0.0000	0.0000
$\wp_4$	0.000	0.000	0.0825	0.0000	0.0424
$\wp_5$	0.0289	0.0745	0.0000	0.0352	0.0000

Table 6: The results of the NDA S_{ij} matrix.

	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$
$\wp_1$	0.0000	0.0000	0.0000	0.0373	0.0000
$\wp_2$	0.1062	0.0231	0.0792	0.0000	0.0871
$\wp_3$	0.0000	0.0000	0.000	0.0099	0.0435
$\wp_4$	0.0581	0.0255	0.000	0.0428	0.0000
$\wp_5$	0.0000	0.0000	0.0482	0.0000	0.0326

Table 7: The results of SP_i and SN_i.

$SP_1 = 0.0616$	$SN_1 = 0.0464$
$SP_2 = 0.0412$	$SN_2 = 0.0530$
$SP_3 = 0.0356$	$SN_3 = 0.0626$
$SP_4 = 0.0503$	$SN_4 = 0.0827$
$SP_5 = 0.0632$	$SN_5 = 0.0432$

Table 8: Proposed AOs and their score values.

Operators	Appraisal score values of alternatives					Ranking
	AS_1	AS_2	AS_3	AS_4	AS_5	
SLDFRWA	0.706	0.505	0.403	0.398	0.738	$\wp_5 > \wp_1 > \wp_2 > \wp_3 > \wp_4$
SLDFROWA	0.731	0.553	0.438	0.419	0.762	$\wp_5 > \wp_1 > \wp_2 > \wp_3 > \wp_4$
SLDFRHA	0.775	0.619	0.480	0.464	0.815	$\wp_5 > \wp_1 > \wp_2 > \wp_3 > \wp_4$
SLDFRWG	0.737	0.542	0.469	0.437	0.750	$\wp_5 > \wp_1 > \wp_2 > \wp_3 > \wp_4$
SLDFROWG	0.764	0.558	0.463	0.454	0.777	$\wp_5 > \wp_1 > \wp_2 > \wp_3 > \wp_4$
SLDFRHG	0.779	0.581	0.476	0.490	0.792	$\wp_5 > \wp_1 > \wp_2 > \wp_3 > \wp_4$

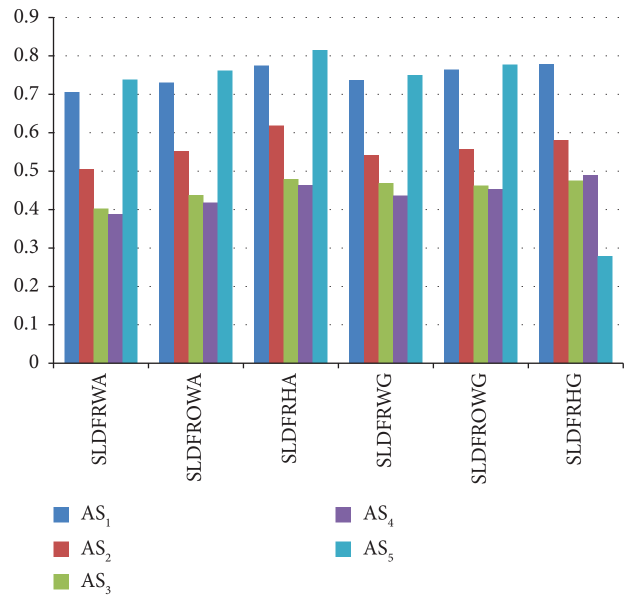

Figure 2: Alternative ranking graphically.

Table 9: Comparative study of the proposed method with existing methods.

Methods	Appraisal scores of alternatives	Ranking
Ashraf et al. [36]	Inaccessible	×
Ashraf et al. [35]	Inaccessible	×
Deli and Çagman [39]	Inaccessible	×
Riaz et al. [48]	Inaccessible	×
Qiyas et al. [40]	Inaccessible	×
Qiyas et al. [41]	Inaccessible	×
Zeng et al. [62]	Inaccessible	×

methods are incapable of solving and ranking the given case. As a result, the established methodology is more capable and dependable than the other present methods.

8. Conclusion

The SLDFR-EDAS method is described in this paper to solve a DM problem in the SLDF environment. When there are more competing needs for MAGDM difficulties, the EDAS approach plays a vital role. This procedure is based on the average solution: PDAS and NDAS methods. The greater PDAS value and lower NDAS value are known to be the optimal alternatives. The SLDFR-EDAS method is used to analyze the hybrid structure of the EDAS method with SLDFRNs. Some aggregation operators are proposed, including SLDFRWA, SLDFROWA, SLDFRHA, SLDFRWG, SLDFROWG, and SLDFRHG. The fundamental features of the proposed operator are thoroughly addressed. For the defined operators, score and accuracy functions are created. After that, the SLDFR-EDAS model for MAGDM and their step-by-step procedure are shown utilising the defined operators. A numerical case study is also provided to validate the effectiveness of the developed strategy. A validity test is also used to demonstrate the validity of the suggested approach. Finally, we conduct a contrastive and comparative analysis of existing approaches and our proposed method.

Our future work will be focused on solving other real-life problems with spherical linear diophantine hesitant fuzzy set (SLDHFS), spherical linear diophantine fuzzy graphs (SLDF-graphs), and interval-valued spherical linear diophantine fuzzy set (IVSLDFS). SLDFSs may be extended to any other aggregation operators, such as prioritized AOs, power mean AOs, Dombi's AOs, Bonferroni mean AOs, Heronian mean AOs, and so on.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors participated in every stage of the research, and all authors read and approved the final manuscript.

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