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Research Article

Some Notes of Homogeneous Besov–Lorentz Spaces

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In this paper, we consider some properties of homogeneous Besov–Lorentz spaces. First, we get some relationship between $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$ and Besov–Lorentz spaces, and then, we obtain the scaling property of $\dot{B}^{s,q}_{p,r}$ and $\dot{F}_{p,r}{}^{s,q}$.

1. Introduction

In [1], Yang-Cheng-Peng introduced Besov–Lorentz spaces $\dot{B}^{s,q}_{p,r}$ and Triebel–Lizorkin–Lorentz spaces $\dot{F}^{s,q}_{p,r}$ by Littlewood–Paley decomposition and proved that the real interpolation spaces $(\dot{F}^{s,q}_{p_0}, \dot{F}^{s,q}_{p_1})_{\theta,r}$ fall into the Triebel–Lizorkin–Lorentz spaces $\dot{F}^{s,q}_{p,r}$.

For the real interpolation of homogeneous Besov spaces $\dot{B}^{s,q}_{p}$, it is well known that when the index p is fixed, $(\dot{B}^{s_0,q_0}_{p}, \dot{B}^{s_1,q_1}_{p})_{\theta,r}$ are still Besov spaces. If $p_0 \neq p_1$, generally speaking, $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$ will fall outside of the scale of Besov spaces. There are many works which considered the real interpolation, see [1–12]. But does $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$ be the Besov–Lorentz space $\dot{B}^{s,q}_{p,r}$ which is given in [1]? In this paper, we partly answer this question and get some relationship between $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$ and $\dot{B}^{s,q}_{p,r}$. Since the properties of function spaces are significant for PDE, furthermore, we consider the scaling property of $\dot{F}^{s,q}_{p,r}$ and $\dot{B}^{s,q}_{p,r}$.

For homogeneous Besov spaces and Triebel–Lizorkin spaces, we use the characterization based on the homogeneous Littlewood–Paley decomposition, see [11]. Given a nonnegative function $\widehat{\varphi}(\xi) \in \mathscr{D}(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\varphi} = \{\xi \in \mathbb{R}^n\colon |\xi| \leq 2\}$ and $\widehat{\varphi}(\xi) = 1$ if $|\xi| \leq (1/2)$. Define

$$\begin{aligned} \psi(x) &= 2^n \varphi(2x) - \varphi(x), \\ \psi_u(x) &= 2^{nu}\psi(2^u x). \end{aligned} \tag{1}$$

Then $\{\psi_u(x)\}_{u\in\mathbb{Z}}, x \in \mathbb{R}^n$ be a family of function satisfying

$$\begin{cases} \operatorname{supp}\widehat{\psi}_u = \left\{\xi \in \mathbb{R}^n, \dfrac{1}{2} \leq 2^{-u}|\xi| \leq 2\right\}; \\ \left|\widehat{\psi}_u(\xi)\right| \geq C > 0, \text{if } \dfrac{1}{2} < C_1 \leq 2^{-u}|\xi| \leq C_2 < 2; \\ \left|\partial^k \widehat{\psi}_u(\xi)\right| \leq C_k 2^{-u|k|}, \text{for any } k \in \mathbb{N}^n; \\ \displaystyle\sum_{u=-\infty}^{+\infty} \widehat{\psi}_u(\xi) = 1, \text{for all } \xi \in \mathbb{R}^n \smallsetminus \{0\}. \end{cases} \tag{2}$$

Define $f_u = \psi_u * f$, the f_u is called the $u-$ th dyadic block of the Littlewood–Paley decomposition of f. Let $\mathscr{S}$ be the space of all Schwartz functions on $\mathbb{R}^n$. The space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-$*$ topology is denoted by $\mathscr{S}'(\mathbb{R}^n)$. Let $P(\mathbb{R}^n)$ be the space of all polynomials on $\mathbb{R}^n$, and let $\mathscr{S}'(\mathbb{R}^n) \smallsetminus P(\mathbb{R}^n)$ denote by the space of all tempered distributions modulated polynomials equipped with the weak-$*$ topology. For any $f \in \mathscr{S}'(\mathbb{R}^n) \smallsetminus P(\mathbb{R}^n)$, we recollect the definition of $\dot{B}^{s,q}_{p}$ and $\dot{F}^{s,q}_{p}$.

Definition 1. Let $f \in \mathscr{S}'(\mathbb{R}^n) \smallsetminus P(\mathbb{R}^n)$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $u \in \mathbb{Z}$. Then,

(i) for $0<p<\infty$, $f\in \dot{F}_p^{s,q}$, if $\|[\sum_u 2^{usq}|f_u|^q]^{1/q}\|_{L^p}<\infty$,

(ii) for $0<p\le\infty$, $f\in \dot{B}_p^{s,q}$, if $[\sum_u 2^{usq}\|f_u\|_{L^p}^q]^{1/q}<\infty$.

When $q=\infty$, it should be replaced by the supremum.

Based on the homogeneous Littlewood–Paley decomposition $\{f_u\}_{u\in\mathbb{Z}}$, we recall the definition of Triebel–Lizorkin–Lorentz spaces and Besov–Lorentz spaces which have been studied in [1]. Let $E\subset\mathbb{R}^n$, we denote by $|E|$ the Lebesgue measure of E.

Definition 2. We assume that $f\in\mathcal{S}'(\mathbb{R}^n)\setminus P(\mathbb{R}^n)$, $0<p<\infty$, $0<q,r\le\infty$, $s\in\mathbb{R}$ and $u,v\in\mathbb{Z}$. Then,

(i) $f\in\dot{F}_{p,r}^{s,q}$, if $(\sum_v 2^{vr}|\{\sum_u 2^{qus}|f_u(x)|^q>2^{qv}\}|^{r/p})^{1/r}<\infty$,

(ii) $f\in\dot{B}_{p,r}^{s,q}$, if $[\sum_u 2^{uqs}(\sum_v 2^{rv}|\{|f_u(x)|>2^v\}|^{r/p})^{q/r}]^{1/q}<\infty$.

As $q=\infty$ or $r=\infty$, it should be modified by supremum.

The definition of the above spaces are independent of the choice of ψ_u, see [1, 4, 11, 13, 14].

First, we consider some relationship between $(\dot{B}_{p_0}^{s,q},\dot{B}_{p_1}^{s,q})_{\theta,r}$ and $\dot{B}_{p,r}^{s,q}$. Then, we get the scaling property of $\dot{F}_{p,r}^{s,q}$ and $\dot{B}_{p,r}^{s,q}$. These properties are important in Cauchy problem for nonlinear PDE, such as Navier–Stokes equations.

This paper is organized as follows. In Section 2, we list some background that shall be used in this paper. In Section 3, we give some relationship between $(\dot{B}_{p_0}^{s,q},\dot{B}_{p_1}^{s,q})_{\theta,r}$ and $\dot{B}_{p,r}^{s,q}$. In Section 4, we prove that $\dot{F}_{p,r}^{s,q}$ and $\dot{B}_{p,r}^{s,q}$ have scaling property.

In this paper, we denote by $A\lesssim B$ an estimate of the form $A\le CB$ with some constant C which is independent of the main parameters, but it may vary from line to line, $A\sim B$ means $A\lesssim B$ and $B\lesssim A$.

2. Preliminaries

2.1. Real Interpolation. We recall that if A_0,A_1 is a pair of quasi-normed spaces which are continuously embedded in a Hausdorff space X, then the K – functional

$$K(f,t,A_0,A_1):=\inf_{f=f_0+f_1}\left\{\|f_0\|_{A_0}+t\|f\|_{A_1}\right\}, \tag{3}$$

is defined for all $f\in A_0+A_1$ with $f_0\in A_0$ and $f_1\in A_1$.

Definition 3. Let $0<\theta<1$ and $0<q<\infty$, then

$$(A_0,A_1)_{\theta,q,K}=\left\{f\colon f\in A_0+A_1,\|f\|_{(A_0,A_1)_{\theta,q,K}}=\left\{\int_0^\infty\left[t^{-\theta}K(t,f,A_0,A_1)\right]^q\frac{dt}{t}\right\}^{1/q}<\infty\right\}. \tag{4}$$

If $q=\infty$, then

$$(A_0,A_1)_{\theta,\infty,K}=\left\{f\colon f\in A_0+A_1,\|f\|_{(A_0,A_1)_{\theta,\infty,K}}=\sup_t t^{-\theta}K(t,f,A_0,A_1)<\infty\right\}. \tag{5}$$

In this subsection, we shall replace the continuous t by a discrete variable j. The relationship between t and j is $t=2^j$. This discretization turns out to be a very helpful technical tool. Assume that $f^j=K(f,2^j,A_0,A_1)$. Let us denote by $\lambda^{\theta,q}$ the sequences $\{f^j\}_{j\in\mathbb{Z}}$ such that

$$\left\|\{f^j\}\right\|_{\lambda^{\theta,q}}=\left[\sum_j\left(2^{-j\theta}\left|f^j\right|\right)^q\right]^{1/q}<\infty. \tag{6}$$

The following result implies a discrete representation of the spaces $(A_0,A_1)_{\theta,q,K}$. It was proved in [4].

Lemma 1. *Let $A_0+A_1=\{f\colon f=f_0+f_1,f_0\in A_0,f_1\in A_1\}$. If $f\in A_0+A_1$, then $f\in(A_0,A_1)_{\theta,q}$ if and only if $\{f^j\}_{j\in\mathbb{Z}}$ belong to $\lambda^{\theta,q}$. Moreover, we have*

$$2^{-\theta}\log 2\left\|f^j\right\|_{\lambda^{\theta,q}}\le\|f\|_{(A_0,A_1)_{\theta,q,K}}\le 2\cdot\log 2\left\|f^j\right\|_{\lambda^{\theta,q}}. \tag{7}$$

By Lemma 1, it is easy to see the following remark:

Remark 1

$$\|f\|_{(A_0,A_1)_{\theta,q,K}}\sim\begin{cases}\left[\sum_j 2^{-jq\theta}K\left(2^j,f,A_0,A_1\right)^q\right]^{1/q}, & 0<q<\infty;\\ \sup_j 2^{-j\theta}K\left(2^j,f,A_0,A_1\right), & q=\infty.\end{cases} \tag{8}$$

Moreover, in this subsection we give some notations. We assume that s is an arbitrary real number. We denote

$$\dot{l}_q^s = \left\{ a\colon a = \{a_j\}, j \in \mathbb{Z}, \|a\|_{\dot{l}_q^s} = \left[\sum_j \left(2^{js} |a_j| \right)^q \right]^{1/q} < \infty \right\}. \tag{9}$$

Let $f_u = \psi_u * f$. Observe the form of Definition 1, it is easy to see that $\dot{F}_p^{s,q}$ is a retract of $L_p(\dot{l}_q^s)$ and $\dot{B}_p^{s,q}$ is a retract of $\dot{l}_q^s(L_p)$. It is

$$\begin{aligned} \|f\|_{\dot{F}_p^{s,q}} &= \left\| \{f_u\} \right\|_{L_p(\dot{l}_q^s)}, \\ \|f\|_{\dot{B}_p^{s,q}} &= \left\| \{f_u\} \right\|_{\dot{l}_q^s(L_p)}. \end{aligned} \tag{10}$$

2.2. Lorentz Spaces. In the following part, we review the definition of Lorentz spaces which are a generalization of Lebesgue spaces. For $x \in \mathbb{R}^n$, the distribution and rearrangement function given by the following formulas:

$$\lambda_f(t) = |\{x\colon |f(x)| \ge t\}| \text{and } f^*(s) = \inf\{t\colon \lambda_f(t) \le s\}. \tag{11}$$

Then, for $1 \le p < \infty$ and $0 < r < \infty$, Lorentz spaces $L_{p,r}$ are defined in the following way

$$L_{p,r} = \left\{ f\colon \|f\|_{p,r} = \left[\frac{r}{p} \int_0^\infty \left(t^{1/p} f^*(t) \right)^r \frac{dt}{t} \right]^{1/r} < \infty \right\}. \tag{12}$$

For $r = \infty$,

$$L_{p,\infty} = \left\{ f\colon \|f\|_{p,\infty} = \sup_t t^{1/p} f^*(t) < \infty \right\}. \tag{13}$$

It is not difficult to see that $L_{p,p} = L_p$. When $r = \infty$, $L_{p,\infty}$ corresponds to the weak $-L_p$ spaces. However, the previous formula is not very useful because it depends on the rearrangement function f^* and we will use an equivalent characterization which has been studied in [1].

Definition 4. Supposed that $1 \le p < \infty$, $0 < r < \infty$, and $u \in \mathbb{Z}$. Then, $f \in L_{p,r}$, if

$$\left(\sum_u 2^{ru} \left| \{|f(x)| > 2^u\} \right|^{r/p} \right)^{1/r} < \infty, \tag{14}$$

as $r = \infty$, it should be replaced by the supremum.

Remark 2. Observe that (14) is a form of the value of a function multiplied by the measure, then in comparison with Definition 4, we can discover the space $\dot{F}_{p,r}^{s,q}$ is a retract of $L_{p,r}(\dot{l}_q^s)$ and $\dot{B}_{p,r}^{s,q}$ is a retract of $\dot{l}_q^s(L_{p,r})$. It can be written as

$$\begin{aligned} \|f\|_{\dot{F}_{p,r}^{s,q}} &= \left\| \{f_u\} \right\|_{L_{p,r}(\dot{l}_q^s)}, \\ \|f\|_{\dot{B}_{p,r}^{s,q}} &= \left\| \{f_u\} \right\|_{\dot{l}_q^s(L_{p,r})}. \end{aligned} \tag{15}$$

So, we can obtain that Triebel–Lizorkin–Lorentz spaces $\dot{F}_{p,r}^{s,q}$ and Besov–Lorentz spaces $\dot{B}_{p,r}^{s,q}$ are the generalized Besov spaces and Triebel–Lizorkin spaces based on the Lorentz spaces $L_{p,r}$. The following lemma is a classical result of real interpolation of Lebesgue spaces, see [4].

Lemma 2. *Given* $0 < p_0 < p_1 \le \infty$, $0 < r \le \infty$, $0 < \theta < 1$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$. *We have*

$$\left(L_{p_0}, L_{p_1} \right)_{\theta,r} = L_{p,r}. \tag{16}$$

Furthermore, we recall the vector valued version of Minkowski's inequality.

Lemma 3. *Let* $k, j \in \mathbb{Z}$, $p_j, q_k > 0$, $1 \le p < \infty$ *and* $a_{jk} \in \mathbb{R}$, *then*

$$\left[\sum_k q_k \left(\sum_j p_j |a_{jk}| \right)^p \right]^{1/p} \le \sum_j p_j \left(\sum_k q_k |a_{jk}|^p \right)^{1/p}. \tag{17}$$

2.3. Triebel–Lizorkin–Lorentz Spaces. Fixed the indices s and q, the real interpolation spaces of Triebel–Lizorkin spaces $(\dot{F}_{p_0}^{s,q}, \dot{F}_{p_1}^{s,q})_{\theta,r}$ are Triebel–Lizorkin–Lorentz spaces. The following theorem has been proved in [1].

Theorem 1. *Let* $0 < p_0, p_1 < \infty$, $0 < q_0, q_1, r \le \infty$ and $0 < \theta < 1$. *Then,*

(i) $(\dot{F}_{p_0}^{s,q}, \dot{F}_{\infty}^{s,q})_{\theta,r} = \dot{F}_{p,r}^{s,q}$, *if* $1/p = (1-\theta)/p_0$,

(ii) $(\dot{F}_{p_0,q_0}^{s,q}, \dot{F}_{p_1,q_1}^{s,q})_{\theta,r} = \dot{F}_{p,r}^{s,q}$, *if* $1/p = (1-\theta)/p_0 + \theta/p_1$.

By (10) and (15), we can rewrite Theorem 1 as given in the following:

Remark 3

$$\left(L_{p_0}(\dot{l}_q^s), L_{p_1}(\dot{l}_q^s) \right)_{\theta,r} = L_{p,r}(\dot{l}_q^s), \text{if } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{18}$$

Especially, for $L_{p,p} = L_p$, we have

Corollary 1. *Let* $1/p = (1-\theta)/p_0 + \theta/p_1$. *If* $p = r$, *then*

$$\left(L_{p_0}(\dot{l}_q^s), L_{p_1}(\dot{l}_q^s) \right)_{\theta,p} = L_p(\dot{l}_q^s), \tag{19}$$

or equivalently,

$$\left(\dot{F}_{p_0}^{s,q}, \dot{F}_{p_1}^{s,q} \right)_{\theta,p} = \dot{F}_p^{s,q}. \tag{20}$$

3. Relationship between $(\dot{B}_{p_0}^{s,q}, \dot{B}_{p_1}^{s,q})_{\theta,r}$ and $\dot{B}_{p,r}^{s,q}$

In this section, we give the relationship between the real interpolation spaces $(\dot{B}_{p_0}^{s,q}, \dot{B}_{p_1}^{s,q})_{\theta,r}$ and Besov–Lorentz spaces $\dot{B}_{p,r}^{s,q}$.

Theorem 2. *Let $\theta \in (0,1)$, $s \in \mathbb{R}$, $0 < q, r \le \infty$, $1 \le p_0 < p_1 < \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then,*

(i) $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r} \hookrightarrow \dot{B}^{s,q}_{p,r}$, *if* $0 < r \le q \le \infty$,

(ii) $\dot{B}^{s,q}_{p,r} \hookrightarrow (\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,r}$, *if* $0 < q \le r \le \infty$,

(iii) $(\dot{B}^{s,q}_{p_0}, \dot{B}^{s,q}_{p_1})_{\theta,q} = \dot{B}^{s,q}_{p,q}$, *if* $0 < r = q \le \infty$.

Before the proof of Theorem 2, we first list several useful lemmas. Define the functional K_p by

$$K_p = K_p(t, f, A_0, A_1) = \inf_{f=f_0+f_1} \left(\|f_0\|^p_{A_0} + t^p \|f_1\|^p_{A_1} \right)^{1/p}, \tag{21}$$

$$\|f\|_{(A_0,A_1)_{\theta,q,K_p}} = \left[\int_0^\infty \left[t^{-\theta} K_p(t, f, A_0, A_1) \right]^q \frac{dt}{t} \right]^{1/q}. \tag{22}$$

First, we recall an important lemma about $K_p(t, f, A_0, A_1)$, see [8].

Lemma 4. *Let $\overline{A} = (A_0, A_1)$ be a couple of quasi-normed spaces. For any $p > 0$, we have*

$$\|f\|_{(A_0,A_1)_{\theta,q,K}} \sim \|f\|_{(A_0,A_1)_{\theta,q,K_p}}. \tag{23}$$

We do not prove Theorem 2 directly. We prove the following equivalent theorem:

Theorem 3. *Let $\theta \in (0,1)$, $s \in \mathbb{R}$, $0 < q, r \le \infty$ and $1 \le p_0 < p_1 < \infty$. Then,*

(i) $(\dot{l}^s_q(L_{p_0}), \dot{l}^s_q(L_{p_1}))_{\theta,r} \hookrightarrow \dot{l}^s_q((L_{p_0}, L_{p_1})_{\theta,r})$, *if* $0 < r \le q \le \infty$,

(ii) $\dot{l}^s_q((L_{p_0}, L_{p_1})_{\theta,r}) \hookrightarrow (\dot{l}^s_q(L_{p_0}), \dot{l}^s_q(L_{p_1}))_{\theta,r}$, *if* $0 < q \le r \le \infty$,

(iii) $(\dot{l}^s_q(L_{p_0}), \dot{l}^s_q(L_{p_1}))_{\theta,q} = \dot{l}^s_q((L_{p_0}, L_{p_1})_{\theta,q})$, *if* $0 < q = r \le \infty$.

It is easy to see that Theorem 3 implies Theorem 2.

Remark 4. Denote $1/p = (1-\theta)/p_0 + \theta/p_1$. For function f and $u \in \mathbb{Z}$, denote $f_u = \psi_u * f$. We have

$$\begin{aligned} \|f\|_{\dot{B}^{s,q}_{p_0}} &= \|\{f_u\}\|_{\dot{l}^s_q(L_{p_0})} \\ \|f\|_{\dot{B}^{s,q}_{p_1}} &= \|\{f_u\}\|_{\dot{l}^s_q(L_{p_1})}, \\ \|f\|_{\dot{B}^{s,q}_{p,r}} &= \|\{f_u\}\|_{\dot{l}^s_q(L_{p,r})} = \|\{f_u\}\|_{\dot{l}^s_q\left(\left(L_{p_0}, L_{p_1}\right)_{\theta,r}\right)}. \end{aligned} \tag{24}$$

Hence, Theorem 3 is equivalent to Theorem 2.

Since $L_{p,p} = L_p$, by Theorem 3, we can see that a part of real interpolation spaces are still Besov spaces. In fact, we have

Remark 5. Let $1/p = (1-\theta)/p_0 + \theta/p_1$. If $q = p$, then

$$\left(\dot{l}^s_p(L_{p_0}), \dot{l}^s_p(L_{p_1}) \right)_{\theta,p} = \dot{l}^s_p(L_p). \tag{25}$$

Consequently,

$$\left(\dot{B}^{s,p}_{p_0}, \dot{B}^{s,p}_{p_1} \right)_{\theta,p} = \dot{B}^{s,p}_p. \tag{26}$$

Now, we come to prove Theorem 3.

Proof. Let a denote the sequence $\{a_j\}_{j\in\mathbb{Z}}$, $\|f\|_p := \|f\|_{L_p}$ and

$$\|f\|_{(A_0,A_1)_{\theta,q}} := \|f\|_{(A_0,A_1)_{\theta,q,K}}. \tag{27}$$

For $j \in \mathbb{Z}$ and $0 < q < \infty$, we deduce

$$\begin{aligned} K_q(t,a) &:= K_q\left(t, a, \dot{l}^s_q(L_{p_0}), \dot{l}^s_q(L_{p_1})\right) = \left[\sum_j 2^{jsq} \inf_{a_j = a^0_j + a^1_j} \left(\left\|a^0_j\right\|^q_{p_0} + t^q \left\|a^1_j\right\|^q_{p_1} \right) \right]^{1/q} \\ &\sim \left\{ \sum_j 2^{jsq} \left[\inf_{a_j = a^0_j + a^1_j} \left(\left\|a^0_j\right\|_{p_0} + t \left\|a^1_j\right\|_{p_1} \right) \right]^q \right\}^{1/q}. \end{aligned} \tag{28}$$

For $q=\infty$, we have

$$K_\infty(t,a)=\sup_j\left[2^{js}\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+t\left\|a_j^1\right\|_{p_1}\right)\right]. \quad (29)$$

As $0<r<\infty$, by (28), Lemma 4 and using a discrete representation of the space $\|f\|_{(A_0,A_1)_{\theta,q}}$ which is described in Lemma 1, we deduce that

$$\begin{aligned}\|a\|_{\left(\tilde{l}_q^s\left(L_{p_0}\right),\tilde{l}_q^s\left(L_{p_1}\right)\right)_{\theta,r}}&\sim\left\{\int_0^\infty\left[t^{-\theta}K_q(t,a)\right]^r\frac{dt}{t}\right\}^{1/r}\sim\left[\sum_k 2^{-kr\theta}K_q\left(2^k,a\right)^r\right]^{1/r}\\&\sim\left(\sum_k 2^{-kr\theta}\left\{\sum_j 2^{jsq}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^q\right\}^{r/q}\right)^{1/r}.\end{aligned} \quad (30)$$

If $r=\infty$, by (29), then

$$\|a\|_{\left(\tilde{l}_q^s\left(L_{p_0}\right),\tilde{l}_q^s\left(L_{p_1}\right)\right)_{\theta,\infty}}\sim\sup_k 2^{-k\theta}\left\{\sum_j 2^{jsq}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^q\right\}^{1/q}. \quad (31)$$

When $q=\infty$,

$$\|a\|_{\left(\tilde{l}_\infty^s\left(L_{p_0}\right),\tilde{l}_\infty^s\left(L_{p_1}\right)\right)_{\theta,r}}\sim\left(\sum_k 2^{-kr\theta}\left\{\sup_j 2^{js}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]\right\}^r\right)^{1/r}. \quad (32)$$

As $0<q<\infty$, applying a discrete representation of the space $\|a_j\|_{(L_{p_0},L_{p_1})_{\theta,r}}$, we have

$$\|a_j\|_{\left(L_{p_0},L_{p_1}\right)_{\theta,r}}\sim\left\{\sum_k 2^{-kr\theta}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^r\right\}^{1/r}. \quad (33)$$

Applying (33), we know that

$$\begin{aligned}\|a\|_{\tilde{l}_q^s\left(\left(L_{p_0},L_{p_1}\right)_{\theta,r}\right)}&=\left(\sum_j 2^{jsq}\|a_j\|^q_{\left(L_{p_0},L_{p_1}\right)_{\theta,r}}\right)^{1/q}\\&\sim\left(\sum_j 2^{jsq}\left\{\sum_k 2^{-kr\theta}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^r\right\}^{q/r}\right)^{1/q}.\end{aligned} \quad (34)$$

When $q=\infty$,

$$\|a\|_{\tilde{l}_\infty^s\left(\left(L_{p_0},L_{p_1}\right)_{\theta,r}\right)}\sim\sup_j 2^{js}\left\{\sum_k 2^{-kr\theta}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^r\right\}^{1/r}. \quad (35)$$

For $r=\infty$,

$$\|a\|_{\tilde{l}_q^s\left(\left(L_{p_0},L_{p_1}\right)_{\theta,\infty}\right)}\sim\left(\sum_j 2^{sjq}\left\{\sup_k 2^{-k\theta}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]\right\}^q\right)^{1/q}. \quad (36)$$

(i) If $0<r\le q<\infty$, then $q/r\ge 1$. From (30), (34) and Minkowski's inequality (17), it follows that

$$\begin{aligned}\|a\|_{\dot{l}_q^s\left(\left(L_{p_0},L_{p_1}\right)_{\theta,r}\right)} &\sim \left(\sum_j 2^{jsq}\left\{\sum_k 2^{-kr\theta}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^r\right\}^{q/r}\right)^{1/q}\\ &=\left[\left(\sum_j 2^{jsq}\left\{\sum_k 2^{-kr\theta}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^r\right\}^{q/r}\right)^{r/q}\right]^{1/r}\\ &\le\left(\sum_k 2^{-kr\theta}\left\{\sum_j 2^{jsq}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^q\right\}^{r/q}\right)^{1/r}\\ &\sim\|a\|_{\left(\dot{l}_q^s\left(L_{p_0}\right),\dot{l}_q^s\left(L_{p_1}\right)\right)_{\theta,r}}.\end{aligned}\tag{37}$$

As $q=\infty$, by (32) and (35), we obtain

$$\begin{aligned}\|a\|_{\dot{l}_\infty^s\left(\left(L_{p_0},L_{p_1}\right)_{\theta,r}\right)} &\sim \sup_j 2^{js}\left\{\sum_k 2^{-kr\theta}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^r\right\}^{1/r}\\ &\le\left(\sum_k 2^{-kr\theta}\left\{\sup_j 2^{js}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]\right\}^r\right)^{1/r}\\ &\sim\|a\|_{\left(\dot{l}_\infty^s\left(L_{p_0}\right),\dot{l}_\infty^s\left(L_{p_1}\right)\right)_{\theta,r}}.\end{aligned}\tag{38}$$

So

$$\left(\dot{l}_q^s\left(L_{p_0}\right),\dot{l}_q^s\left(L_{p_1}\right)\right)_{\theta,r}\hookrightarrow\dot{l}_q^s\left(\left(L_{p_0},L_{p_1}\right)_{\theta,r}\right),\text{ if } 0<r\le q\le\infty.\tag{39}$$

(ii) Conversely, if $0<q\le r<\infty$, then $r/q\ge 1$. In a similar way, by (30), (34) with Minkowski's inequality (17), we obtain

$$\begin{aligned}&\|a\|_{\left(\dot{l}_q^s\left(L_{p_0}\right),\dot{l}_q^s\left(L_{p_1}\right)\right)_{\theta,r}}\\ &\sim\left(\sum_k 2^{-kr\theta}\left\{\sum_j 2^{jsq}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^q\right\}^{r/q}\right)^{1/r}\\ &=\left[\left(\sum_k 2^{-kr\theta}\left\{\sum_j 2^{jsq}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^q\right\}^{r/q}\right)^{q/r}\right]^{1/q}\\ &\le\left(\sum_j 2^{jsq}\left\{\sum_k 2^{-kr\theta}\left[\inf_{a_j=a_j^0+a_j^1}\left(\left\|a_j^0\right\|_{p_0}+2^k\left\|a_j^1\right\|_{p_1}\right)\right]^r\right\}^{q/r}\right)^{1/q}\\ &\sim\|a\|_{\dot{l}_q^s\left(\left(L_{p_0},L_{p_1}\right)_{\theta,r}\right)}.\end{aligned}\tag{40}$$

When $r=\infty$, by (31) and (36) and the same as we did in (i), we omit the details. Hence,

$$\dot{l}_q^s\Big(\big(L_{p_0}, L_{p_1}\big)_{\theta,r}\Big) \hookrightarrow \big(\dot{l}_q^s\big(L_{p_0}\big), \dot{l}_q^s\big(L_{p_1}\big)\big)_{\theta,r}, \text{if } 0 < q \le r \le \infty. \quad (41)$$

(iii) For $0 < q = r \le \infty$, from (i) and (ii), it is easy to see that

$$\Big(\dot{l}_q^s\big(L_{p_0}\big), \dot{l}_q^s\big(L_{p_1}\big)\Big)_{\theta,q} = \dot{l}_q^s\Big(\big(L_{p_0}, L_{p_1}\big)_{\theta,q}\Big). \quad (42)$$

We finish the proof of Theorem 3. □

By Remark 3, as contrasted with the real interpolation of Triebel–Lizorkin spaces, we conclude the following result:

Remark 6. Fixed the indices s and q, the real interpolation of Triebel–Lizorkin spaces satisfy

$$\Big(L_{p_0}\big(\dot{l}_q^s\big), L_{p_1}\big(\dot{l}_q^s\big)\Big)_{\theta,r} = L_{p,r}\big(\dot{l}_q^s\big), \text{with } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \quad (43)$$

The real interpolation of Triebel–Lizorkin spaces for index p can be directly interpolated as $L_{p,r}$, but the real interpolation of Besov spaces for index p can not be directly interpolated as $L_{p,r}$, the relationship $(\dot{l}_q^s(L_{p_0}), \dot{l}_q^s(L_{p_1}))_{\theta,r}$ between $\dot{l}_q^s(L_{p,r})$ depends on whether $q \le r$ or $q \ge r$.

Moreover, we need to point out that for the inhomogeneous spaces $F_p^{s,q}$ and $B_p^{s,q}$, all the results of real interpolation are also true.

4. Scaling Property

In this section, we get the scaling property of $\dot{F}_{p,r}^{s,q}$ and $\dot{B}_{p,r}^{s,q}$. Homogeneous spaces can be the critical spaces of many nonlinear partial differential equations. Critical spaces hold an important status in nonlinear partial differential equations. First, we recall the scaling property of Lorentz spaces, see [15].

Lemma 5. *The Lorentz space $L_{p,r}(\mathbb{R}^n)$ is homogeneous, for any strictly positive number Λ and any f belonging to $L_{p,r}(\mathbb{R}^n)$:*

$$\|f(\Lambda\cdot)\|_{L_{p,r}} = \Lambda^{-n/p}\|f\|_{L_{p,r}}. \quad (44)$$

Suppose that f is a tempered distribution, and consider the tempered distribution f_N defined by $f_N \coloneqq f(2^N\cdot)$, then, we have the following proposition.

Proposition 1. *We assumed that $1 \le p < \infty$, $0 < q, r < \infty$, $s \in \mathbb{R}$, $x \in \mathbb{R}^n$, $u \in \mathbb{Z}$, an integer N, and a distribution f of $S'(\mathbb{R}^n)\setminus P(\mathbb{R}^n)$, then, we have*

$$\begin{aligned} \|f_N\|_{\dot{F}_{p,r}^{s,q}} &= 2^{N(s-(n/p))}\|f\|_{\dot{F}_{p,r}^{s,q}}, \\ \|f_N\|_{\dot{B}_{p,r}^{s,q}} &= 2^{N(s-(n/p))}\|f\|_{\dot{B}_{p,r}^{s,q}}. \end{aligned} \quad (45)$$

Proof. Let us recall:

$$\|f\|_{\dot{F}_{p,r}^{s,q}} = \big\|\{f_u\}\big\|_{L_{p,r}(\dot{l}_q^s)}, \|f\|_{\dot{B}_{p,r}^{s,q}} = \big\|\{f_u\}\big\|_{\dot{l}_q^s(L_{p,r})}, \quad (46)$$

and then by definition of $(f_N)_u$ and the change of variable $t = 2^N x$, we obtain

$$\begin{aligned} (f_N)_u &= f_u\big(2^N\cdot\big) = \psi_u * f_N \\ &= 2^{un}\int_{\mathbb{R}^n}\psi\big(2^u(x-y)\big)f_N(y)dy \\ &= 2^{un}\int_{\mathbb{R}^n}\psi\big(2^u(x-y)\big)f\big(2^N y\big)dy \\ &= 2^{(u-N)n}\int_{\mathbb{R}^n}\psi\big(2^u x - 2^{u-N}t\big)f(t)dt \\ &= 2^{(u-N)n}\int_{\mathbb{R}^n}\psi\big(2^{j-N}\big(2^N x - t\big)\big)f(t)dt \\ &= f_{u-N}\big(2^N x\big). \end{aligned} \quad (47)$$

By Lemma 5, we have

$$\big\|(f_N)_u\big\|_{L_{p,r}} = 2^{-N(n/p)}\big\|f_{u-N}\big\|_{L_{p,r}}. \quad (48)$$

Then,

$$\begin{aligned} \|f_N\|_{\dot{B}_{p,r}^{s,q}} &= \big\|f\big(2^N\cdot\big)\big\|_{\dot{B}_{p,r}^{s,q}} = \Big\{\big\|f_u\big(2^N\cdot\big)\big\|_{L_{p,r}}\Big\}_{\dot{l}_q^s} = \Big\{\big\|(f_N)_u\big\|_{L_{p,r}}\Big\}_{\dot{l}_q^s} \\ &= \left[\sum_u\Big(2^{su}2^{-Nn/p}\big\|f_{u-N}\big\|_{L_{p,r}}\Big)^q\right]^{1/q} \\ &= \left[\sum_u\Big(2^{s(u-N)}2^{N(s-n/p)}\big\|f_{u-N}\big\|_{L_{p,r}}\Big)^q\right]^{1/q} \\ &= 2^{N(s-n/p)}\left[\sum_{u-N}\Big(2^{s(u-N)}\big\|f_{u-N}\big\|_{L_{p,r}}\Big)^q\right]^{1/q} \\ &= 2^{N(s-n/p)}\|f\|_{\dot{B}_{p,r}^{s,q}}, \end{aligned} \quad (49)$$

for the homogeneous Triebel–Lizorkin–Lorentz spaces, the proof is similar, we omit the details, see [16]. □

More generally, there exists a constant C, depending only on s, such that for all positive Λ, we have

$$C^{-1}\Lambda^{s-(n/p)}\|f\|_{\dot{B}_{p,r}^{s,q}} \le \|f(\Lambda\cdot)\|_{\dot{B}_{p,r}^{s,q}} \le C\Lambda^{s-(n/p)}\|f\|_{\dot{B}_{p,r}^{s,q}}, \quad (50)$$

and the similar for $\dot{F}_{p,r}^{s,q}$. Then we have the following corollary.

Corollary 2. *Let $1 \le p < \infty$, $0 < q, r < \infty$, $s \in \mathbb{R}$, $x \in \mathbb{R}^n$ and a distribution f of S', for any strictly positive number Λ, we have*

$$\|f(\Lambda\cdot)\|_{\dot{F}_{p,r}^{s,q}} \sim \Lambda^{(s-(n/p))}\|f\|_{\dot{F}_{p,r}^{s,q}}, \|f(\Lambda\cdot)\|_{\dot{B}_{p,r}^{s,q}} \sim \Lambda^{(s-(n/p))}\|f\|_{\dot{B}_{p,r}^{s,q}}. \quad (51)$$

Remark 7. There have been a lot of results about the properties of these two types spaces, such as the wavelet decomposition characterization of $\dot{B}_{p,r}^{s,q}$ and $\dot{F}_{p,r}^{s,q}$ is already obtained, see [1], the boundedness of operators in the generalized Besov-type was considered in [17], and so on. These properties are important in Cauchy problem for nonlinear PDE. For instance, based on these properties, we can consider the well-posedness of the Navier–Stokes equations in $\dot{B}^{s,q}_{p,r}$ and $\dot{F}^{s,q}_{p,r}$.

Data Availability

The data used to support the findings of this study are cited at relevant places within the text as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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