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Review Article

Results on Solutions for Several Systems of Partial Differential-Difference Equations with Two Complex Variables

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This article is to describe the entire solutions of some partial differential-difference equations and systems. Some theorems about the forms of transcendental entire solutions with finite order for several high-order partial differential-difference equations (or systems) of the Fermat type with two complex variables are obtained. Moreover, some examples are provided to explain that our results are precise to some extent.

1. Introduction

Fermat's last theorem, as everyone knows, states that the Fermat equation $X^n + Y^n = 1$ does not have nontrivial rational solutions for $n \geq 3$ [1]. In 1960s, Gross [2, 3] considered the Fermat-type functional equation

$$G^2(x) + H^2(x) = 1, \tag{1}$$

and they obtained that equation (1) has entire solutions for the form $G = \cos\beta, H = \sin\beta$ and meromorphic solutions for the form $G = 1 - \gamma^2/1 + \gamma^2, H = 2\gamma/1 + \gamma^2$, where β is entire and γ is meromorphic. From then on, many authors further explored these problems when $H(x)$ is the differential or difference operators of $G(x)$. In 2004, Yang and Li [4] discussed the existence of transcendental meromorphic solution for the Fermat-type differential equation $a_1 G(x)^2 + a_2 G'(x)^2 = a_3$, where a_i is the nonzero meromorphic function. In the same year, Li [5] considered the entire solutions for Fermat-type partial differential equations. In recent years, there exist many results for Fermat-type differential-difference equations (see [6–12]) with the aid of the difference Nevanlinna theory with one complex variable. Around 2012, Liu et al. [8, 9, 13] investigated the solutions for differential-difference equations $G'(x)^2 + G(x+c)^2 = 1$ and $G'(x)^2 + [G(x+c) - G(x)]^2 = 1$ and proved that the finite-order transcendental entire solutions for these equations are of the forms $G(x) = \sin(x \pm Bi)$ and $G(x) = 12\sin(2x + Bi)$, respectively, under some conditions on B, c. Gao [14] in 2016 discussed the solutions for a system of differential-difference equations $\begin{cases} [G_1'(x)]^2 + G_2(x+c)^2 = 1 \\ [G_2'(x)]^2 + G_1(x+c)^2 = 1 \end{cases}$, corresponding to the equations mentioned in [8, 13] and obtained that the pair of finite-order transcendental entire solutions (G_1, G_2) of this system is of the forms $(G_1(x), G_2(x)) = (\sin(x - bi), \sin(x - b_1 i))$ or $(G_1(x), G_2(x)) = (\sin(x + bi), \sin(x + b_1 i))$, where b, b_1 are constants, $c = k\pi$, and k is a integer.

Besides, after Li's results [5], Fermat-type partial differential equations, including $\mathfrak{I}_{x_1}^2 + \mathfrak{I}_{x_2}^2 = 1$, $\mathfrak{I}_{x_1}^2 + \mathfrak{I}_{x_2}^2 = e^h$, and $\mathfrak{I}_{x_1}^2 + \mathfrak{I}_{x_2}^2 = p$, have been studied (see [15–17]), and a number of results concerning the existence and forms of solutions for partial differential equations in several complex variables have been studied. In very recent years, with the rapid development of the difference Nevanlinna theory, Xu and Cao [18] discussed the existence of the entire solutions for the Fermat-type partial differential-difference equation:

$$\left(\frac{\partial G(x)}{\partial x_1}\right)^n + G(x+c)^m = 1, \tag{2}$$

in $\mathbb{C}^2$. They pointed out that (i) when m and n are two distinct positive integers, equation (2) cannot have any transcendental entire solution with finite order; (ii) when $m = n = 2$, equation (2) has the transcendental entire solution with finite order. Moreover, Xu and Cao [18] also obtained the following results.

Theorem 1 (see [18]). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. If the partial differential-difference equation is*

$$\left(\frac{\partial G(x)}{\partial x_1}\right)^2 + G(x+c)^2 = 1, \tag{3}$$

there exist transcendental entire solutions with finite order, which have the form:

$$G(x) = \sin\left(B_1 x_1 + B_2 x_2 + h(x_2)\right), \tag{4}$$

where $B_1, B_2 \in \mathbb{C}$ satisfies $B_1^2 = 1$ and $B_1 e^{i(B_1 c_1 + B_2 c_2)} = 1$ and $h(x_2)$ is a polynomial in one variable x_2 such that $h(x_2) \equiv h(x_2 + c_2)$. In the special case whenever $c_2 \neq 0$, there is $G(x) = \sin(B_1 x_1 + B_2 x_2 + Constant)$.

In 2021, Xu et al. [19–21] further investigated the entire solutions for partial differential-difference equations with more general form and obtained the following.

Theorem 2 (see [21]). *Let $c = (c_1, c_2)(\neq (0,0)) \in \mathbb{C}^2$ and $s_0 = c_1 + c_2$. If the partial differential-difference equation is*

$$\left(G(x+c) + \frac{\partial G(x)}{\partial x_1}\right)^2 + \left(G(x+c) + \frac{\partial G(x)}{\partial x_2}\right)^2 = 1, \tag{5}$$

there exist the transcendental entire solution with finite order, and the solution has the following two forms:

(i) $G(x) = \theta(x_1 + x_2)$,
where $\theta(s)$ is a transcendental entire function with finite order in $s := x_1 + x_2$ satisfying

$$\theta(s + s_0) + \theta'(s) = \pm\frac{\sqrt{2}}{2}. \tag{6}$$

(ii) $G(x) = 1 + i/2\ \ (a_1 - a_2)\ \ e^{l(x)+B} - 1 - i/2\ \ (a_1 - a_2)\ \ e^{-l(x)-B} + \theta(x_1 + x_2)$,

where $l(x) = a_1 x_1 + a_2 x_2, a_1, a_2, B \in \mathbb{C}$ and $\theta(s)$ satisfies

$$a_1^2 + a_2^2 = -2, e^{l(c)} = -\frac{ia_1 + a_2}{1+i} = -\frac{1-i}{ia_1 - a_2}, \theta'(s) + \theta(s + s_0) = 0. \tag{7}$$

In 2020, Xu et al. [17] extended these results from equations to Fermat-type systems of partial differential-difference equations and obtained the following.

Theorem 3 (see ([20], Theorem 1.3)). *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then, any pair of transcendental entire solutions with finite order for the system of Fermat-type partial differential-difference equations exists:*

$$\begin{cases} \left(\dfrac{\partial G_1(x)}{\partial x_1}\right)^2 + G_2(x_1 + c_1, x_2 + c_2)^2 = 1, \\ \left(\dfrac{\partial G_2(x)}{\partial x_1}\right)^2 + G_1(x_1 + c_1, x_2 + c_2)^2 = 1, \end{cases} \tag{8}$$

which have the following forms:

$$\left(G_1(x), G_2(x)\right) = \left(\frac{e^{l(x)+B_1} + e^{-(l(x)+B_1)}}{2}, \frac{B_{21} e^{l(x)+B_1} + B_{22} e^{-(l(x)+B_1)}}{2}\right), \tag{9}$$

where $l(x) = a_1 x_1 + a_2 x_2$, B_1 is a constant in $\mathbb{C}$, and a_1, c, B_{21}, B_{22} satisfy one of the following cases

(i) $B_{21} = -i$, $B_{22} = i$, and $a_1 = i$, $l(c) = (2k + 1/2)\pi i$, or $a_1 = -i$, $l(c) = (2k - 1/2)\pi i$

(ii) $B_{21} = i$, $B_{22} = -i$, and $a_1 = i$, $l(c) = (2k - 1/2)\pi i$, or $a_1 = -i$, $l(c) = (2k + 1/2)\pi i$

(iii) $B_{21} = 1$, $B_{22} = 1$, and $a_1 = i$, $l(c) = 2k\pi i$, or $a_1 = -i$, $l(c) = (2k + 1)\pi i$

(iv) $B_{21} = -1$, $B_{22} = -1$, and $a_1 = i$, $l(c) = (2k + 1)\pi i$, or $a_1 = -i$, $l(c) = 2k\pi i$

However, according to above Theorems 1–3, we find that the authors mainly considered first-order partial differential equations and rarely considered second-order or more higher order partial differential equations. In this paper, based on their results, we further consider some more generalized questions, considering mixing higher order partial differential-difference equations (system). Thus, we consider the following two questions.

Question 4. How to describe the solution for PDDEs when first-order partial differential-difference equations in Theorems 1 and 2 are replaced by some high-order partial differential-difference equations?

Question 5. What happens about the solutions of the system of PDDEs when the system in Theorem 3 includes the second-order partial derivative and second-order mixed partial derivative?

Based on the above questions, we will investigate the entire solutions for some second-order and high-order partial differential-difference equations (systems) in this paper and obtain some results which will be listed in Section 2 by using the methods of previous articles [17, 19–21]. Now, we first introduce some lemmas to prove our main results.

Lemma 6 (see [22, 23]). *For an entire function F on $\mathbb{C}^m$, $F(0)\neq 0$, we put $\rho(n_F)=\rho<\infty$. Then, there exist a canonical function f_F and a function $g_F\in\mathbb{C}^m$ such that $F(x)=f_F(x)e^{gF(x)}$. For the special case, $m=1$, f_F is the canonical product of Weierstrass.*

Lemma 7. *(see [24]). If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases:*

(a) The internal function h is a polynomial, and the external function g is of finite order.

(b) Or else the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.

Lemma 8 (see [25]). *Let $f_j(\neq 0)$, $j=1,2,3$ be meromorphic functions on $\mathbb{C}^m$ such that f_1 is not constant, and $f_1+f_2+f_3=1$, and such that,*

$$\sum_{j=1}^{3}\left\{N_2\left(r,\frac{1}{f_j}\right)+2\overline{N}\left(r,f_j\right)\right\}<\lambda T(r,f_1)+O(\log^+T(r,f_1)), \tag{10}$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda<1$ is a positive number. Then, either $f_2=1$ or $f_3=1$.

2. Main Results and Some Examples

First, we assume that readers are very familiar with some basic notations and theorems of Nevanlinna value distribution theory (see [7, 26, 27]). Besides, let $\rho(G)$ denote the order of growth of G, where

$$\rho(G)=\lim\sup_{r\longrightarrow\infty}\frac{\log^+T(r,G)}{\log r}, \tag{11}$$

and let $x+y=(x_1+y_1,x_2+y_2)$ for any $x=(x_1,x_2)$, $y=(y_1,y_2)\in\mathbb{C}^2$. In this paper, based on the results of Xu and Cao [18] and Liu and Xu [15, 17], we further obtain some more generalized results, considering mixing higher order partial differential-difference equations (system). We obtain the following.

Theorem 9. *Let $c=(c_1,c_2)\in\mathbb{C}^2-\{(0,0)\}$ and $s_0=c_1+c_2$. If the partial differential equation,*

$$\left(G(x+c)+\frac{\partial^2G(x)}{\partial x_1^2}\right)^2+\left(G(x+c)+\frac{\partial^2G(x)}{\partial x_1\partial x_2}\right)^2=1, \tag{12}$$

has the transcendental entire solution with $\rho(G)<+\infty$, then $G(x)$ has the following two forms:

(i) $G(x)=\theta(x_1+x_2)$,

where $\theta(s)$ is a transcendental entire function with $\rho(\theta)<+\infty$ in $s:=x_1+x_2$ satisfying

$$\theta(s+s_0)+\theta''(s)=\pm\frac{\sqrt{2}}{2}. \tag{13}$$

(ii) $G(x)=a_1-ia_2/2i\ (a_1-a_2)e^{a_1x_1+a_2x_2+B'}-a_1+ia_2/2\ i(a_1-a_2)e^{-(a_1x_1+a_2x_2+B')}+\theta(s)$,

where $\theta(s)$ is a transcendental entire function with $\rho(\theta)<+\infty$ in $s:=x_1+x_2$ satisfying

$$a_1^4+a_1^2a_2^2=2, i\left(a_1^4-1\right)+a_1^3a_2=e^{2l(c)}, \theta(s+s_0)+\theta''(s)=0. \tag{14}$$

Furthermore, when the second-order partial derivative is replaced with the n-th order partial derivative in equation (12), we obtain the following.

Theorem 10. *Let $c=(c_1,c_2)\in\mathbb{C}^2-\{(0,0)\}$ and $s_0=c_1+c_2$. If the partial differential equation,*

$$\left(G(x+c)+\frac{\partial^nG(x)}{\partial x_1^n}\right)^2+\left(G(x+c)+\frac{\partial^nG(x)}{\partial x_1^{n-1}\partial x_2}\right)^2=1, \tag{15}$$

has the transcendental entire solution with $\rho(G)<+\infty$, then $G(x)$ has the following two forms:

(i) $G(x)=\theta(x_1+x_2)$,

where $\theta(s)$ is a transcendental entire function of finite order in $s:=x_1+x_2$ satisfying

$$\theta(s+s_0)+\theta^{(n)}(s)=\pm\frac{\sqrt{2}}{2}. \tag{16}$$

(ii) $G(x)=(a_1-ia_2)/2i(a_1-a_2)e^{l(x)+B-l(c)}-(a_1+ia_2)/2i(a_1-a_2)e^{-(l(x)+B+l(c))}+\theta(s)$,

where $l(x_1,x_2)=a_1x_1+a_2x_2$, $a_1,a_2,B\in\mathbb{C}$, and $\theta(s)$ is a transcendental entire function with $\rho(\theta)<+\infty$, in $s:=x_1+x_2$ satisfying

$$\theta(s+s_0)+\theta^{(n)}(s)=0, a_1^{2n}+a_1^{2n-2}a_2^2=2, i\left(a_1^{2n}-1\right)+a_1^{2n-1}a_2=e^{2l(c)}. \tag{17}$$

Remark 11. Based on the above two results, we easily see that Theorem 9 is especially an example of Theorem 10 for the case $n=2$.

The following two examples are sufficient to show that our results are correct.

Example 1. Let $G(x)=\pm\sqrt{2}/2+e^{x_1+x_2}$. Then, $G(x)$ is a transcendental entire solution with $\rho(G)<+\infty$, of (i) of equations (12) and (15) with $(c_1,c_2)=(\pi/2i,\pi/2i)$.

Example 2. Let $a_1=1, a_2=-1$ and

$$G(x)=\frac{(-i-1)}{4}e^{x_1-x_2+B}+\frac{(i-1)}{4}e^{-(x_1-x_2+B)}+e^{x_1+x_2}. \tag{18}$$

Then, $G(x)$ is a transcendental entire solution with $\rho(G)<+\infty$, of (ii) of equations (12) and (15) with $(c_1,c_2)=(3\pi/4i,\pi/4i)$.

Theorem 12. *Let $c=(c_1,c_2)\in\mathbb{C}^2-\{(0,0)\}$. If the partial differential equation,*

$$[G(x+c)]^2+\left[G(x)+\frac{\partial^2 G(x)}{\partial x_1\partial x_2}\right]^2=1, \tag{19}$$

has the transcendental entire solution with $\rho(G)<+\infty$, then $G(x)$ has the following form:

$$G(x)=\frac{e^{l(x)+B-l(c)}+e^{-l(x)-B+l(c)}}{2}, \tag{20}$$

where $l(x)=a_1x_1+a_2x_2$ and B, a_1, a_2 are constants satisfying $(1+a_1a_2)^2=1$ and $e^{2l(c)}=-1$.

The following example shows the existence of the finite-order transcendental entire solutions for equation (19).

Example 3. Let $a_1=\sqrt{2}, a_2=-\sqrt{2}$ and

$$G(x)=\frac{e^{\sqrt{2}x_2-\sqrt{2}x_1}-e^{\sqrt{2}x_1-\sqrt{2}x_2}}{2i}. \tag{21}$$

Then, $G(x)$ is a transcendental entire solution of equation (19) with $\rho(G)<+\infty$ and $(c_1,c_2)=(\sqrt{2}\pi/8i, -\sqrt{2}\pi/8i)$.

Finally, we continue to investigate the transcendental entire solutions of a certain system of Fermat-type second-order partial differential-difference equations and obtain the following.

Theorem 13. *Let $c=(c_1,c_2)\in\mathbb{C}^2-\{(0,0)\}$. If the system of the partial differential equation,*

$$\begin{cases} G_1^2(x+c)+\left(G_2(x)+\dfrac{\partial^2 G_2(x)}{\partial x_1\partial x_2}\right)^2=1, \\ G_2^2(x+c)+\left(G_1(x)+\dfrac{\partial^2 G_1(x)}{\partial x_1^2}\right)^2=1, \end{cases} \tag{22}$$

has the transcendental entire solution with $\rho(G)<+\infty$, then (G_1,G_2) must be of the following forms:

$$\begin{aligned}(G_1(x),G_2(x))&=(\pm\cos(l(x)+B_1),-\sin(l(x)+B_1)),\\ (G_1(x),G_2(x))&=(\pm\sin(l(x)+B_1),-\sin(l(x)+B_1)),\end{aligned} \tag{23}$$

where $l(x)=a_1x_1+a_2x_2, a_1\neq 0, a_2\neq 0, a_1^2=2, e^{2il(c)}=\pm 1$ and B_1 is a constant in $\mathbb{C}$.

Example 4. Let $a_1=\sqrt{2}, a_2=\sqrt{2}$; thus, we have following four cases:

Case 1: let $(c_1,c_2)=(\sqrt{2}\pi/2,\sqrt{2}\pi/2)$, $B_1=0$, and

$$G=(G_1,G_2)=(\cos(\sqrt{2}x_1+\sqrt{2}x_2),-\sin(\sqrt{2}x_1+\sqrt{2}x_2)). \tag{24}$$

Thus, (G_1,G_2) is a pair of the transcendental entire solution of equation (22) with $\rho(G)<+\infty$.

Case 2: let $(c_1,c_2)=(\sqrt{2}\pi/4,\sqrt{2}\pi/4)$, $B_1=0$, and

$$G=(G_1,G_2)=(-\cos(\sqrt{2}x_1+\sqrt{2}x_2),-\sin(\sqrt{2}x_1+\sqrt{2}x_2)). \tag{25}$$

Thus, (G_1,G_2) is a pair of the transcendental entire solution of equation (22) with $\rho(G)<+\infty$.

Case 3: let $(c_1,c_2)=(\sqrt{2}\pi/8,\sqrt{2}\pi/8)$, $B_1=0$, and

$$G=(G_1,G_2)=(\sin(\sqrt{2}x_1+\sqrt{2}x_2),-\sin(\sqrt{2}x_1+\sqrt{2}x_2)). \tag{26}$$

Thus, (G_1, G_2) is a pair of the transcendental entire solution of equation (22) with $\rho(G) < +\infty$.

Case 4: let $(c_1, c_2) = (3\sqrt{2}\pi/8, 3\sqrt{2}\pi/8)$, $B_1 = 0$, and

$$G = (G_1, G_2) = (-\sin(\sqrt{2}x_1 + \sqrt{2}x_2), -\sin(\sqrt{2}x_1 + \sqrt{2}x_2)). \tag{27}$$

Thus, (G_1, G_2) is a pair of the transcendental entire solution of equation (22) with $\rho(G) < +\infty$.

3. Proof of Main Results

3.1. The Proof of Theorem 9. Let $G(x)$ be a transcendental entire solution with $\rho(G) < +\infty$ of equation (12). Two cases will be considered below.

Case 14. If $G(x+c) + \partial^2 G(x)/\partial^2 x_1$ is a constant, we set

$$G(x+c) + \frac{\partial^2 G(x)}{\partial x_1^2} = K_1, \quad K_1 \in \mathbb{C}. \tag{28}$$

On the basis of equation (12), we can see that $G(x+c) + \partial^2 G(x)/\partial x_1 \partial x_2$ is also a constant, and thus, we set

$$G(x+c) + \frac{\partial^2 G(x)}{\partial x_1 \partial x_2} = K_2, \quad K_2 \in \mathbb{C}. \tag{29}$$

This leads to $K_1^2 + K_2^2 = 1$. In view of (28) and (29), and taking partial derivations on both two sides of equations (28) and (29) for the variables x_2, x_1, respectively, and noting that the fact $\partial^3 G(x)/\partial x_2 \partial x_1^2 = \partial^3 G(x)/\partial x_1^2 \partial x_2$, it yields that

$$\begin{aligned} &\frac{\partial^2 G(x)}{\partial x_1^2} - \frac{\partial^2 G(x)}{\partial x_1 \partial x_2} = K_1 - K_2, \\ &\frac{\partial G(x+c)}{\partial x_1} - \frac{\partial G(x+c)}{\partial x_2} = 0. \end{aligned} \tag{30}$$

These mean that

$$\frac{\partial G(x)}{\partial x_1} - \frac{\partial G(x)}{\partial x_2} = 0, K_1 = K_2 = \pm \frac{\sqrt{2}}{2}. \tag{31}$$

The characteristic equations of (31) are

$$\frac{dx_1}{\mathrm{d}t} = 1, \frac{dx_2}{\mathrm{d}t} = -1, \frac{dG}{dt} = K_1 - K_2. \tag{32}$$

Let the initial conditions be $x_1 = 0, x_2 = s$ and $G = G(0, s) \coloneqq \theta(s)$ with a parameter s, and we then can obtain the following parametric representation for solutions of the characteristic equations: $G_1 = t, G_2 = -t + s$,

$$\begin{aligned} G(t,s) &= \int_0^t (K_1 - K_2)\mathrm{d}t + \theta(s) \\ &= (K_1 - K_2)t + \theta(s). \end{aligned} \tag{33}$$

Noting that $K_1 = K_2$, we have

$$G(x) = \theta(x_1 + x_2). \tag{34}$$

Substituting (34) into (28) or (29), we obtain that

$$\theta(s + s_0) + \theta''(s) = \pm \frac{\sqrt{2}}{2}. \tag{35}$$

Therefore, this completes the proof of (i) of Theorem 9.

Case 15. If $G(x+c) + \partial^2 G(x)/\partial x_1^2$ is not a constant, we first write equation (12) as the following form:

$$\begin{aligned} &\left[G(x+c) + \frac{\partial^2 G(x)}{\partial x_1^2} + i\left(G(x+c) + \frac{\partial^2 G(x)}{\partial x_1 \partial x_2}\right)\right] \\ &\times \left[G(x+c) + \frac{\partial^2 G(x)}{\partial x_1^2} - i\left(G(x+c) + \frac{\partial^2 G(x)}{\partial x_1 \partial x_2}\right)\right] = 1, \end{aligned} \tag{36}$$

which indicates that both,

$$\begin{aligned} &G(x+c) + \frac{\partial^2 G(x)}{\partial x_1^2} + i\left(G(x+c) + \frac{\partial^2 G(x)}{\partial x_1 \partial x_2}\right), \\ &G(x+c) + \frac{\partial^2 G(x)}{\partial x_1^2} - i\left(G(x+c) + \frac{\partial^2 G(x)}{\partial x_1 \partial x_2}\right), \end{aligned} \tag{37}$$

have no poles and zeros. Thus, by Lemmas 6 and 8, there exists a polynomial $p(x)$ such that

$$\begin{aligned} &G(x+c) + \frac{\partial^2 G(x)}{\partial x_1^2} + i\left(G(x+c) + \frac{\partial^2 G(x)}{\partial x_1 \partial x_2}\right) = e^{p(x)}, \\ &G(x+c) + \frac{\partial^2 G(x)}{\partial x_1^2} - i\left(G(x+c) + \frac{\partial^2 G(x)}{\partial x_1 \partial x_2}\right) = e^{-p(x)}, \end{aligned} \tag{38}$$

which lead to

$$G(x+c) + \frac{\partial^2 G(x)}{\partial x_1^2} = \frac{e^{p(x)} + e^{-p(x)}}{2}, \tag{39}$$

$$G(x+c) + \frac{\partial^2 G(x)}{\partial x_1 \partial x_2} = \frac{e^{p(x)} - e^{-p(x)}}{2i}. \tag{40}$$

In view of (39) and (40), we get

$$\frac{\partial^2 G(x)}{\partial x_1^2} - \frac{\partial^2 G(x)}{\partial x_1 \partial x_2} = \frac{1+i}{2}e^{p(x)} + \frac{1-i}{2}e^{-p(x)}. \tag{41}$$

Taking partial derivations on both sides of equations (39) and (40) for the variables x_2, x_1, respectively, and combining with the fact $\partial^3 G(x)/\partial x_2 \partial x_1^2 = \partial^3 G(x)/\partial x_1^2 \partial x_2$, it yields that

$$\frac{\partial G(x+c)}{\partial x_1}-\frac{\partial G(x+c)}{\partial x_2}=\left(\frac{1}{2i}\frac{\partial p}{\partial x_1}-\frac{1}{2}\frac{\partial p}{\partial x_2}\right)e^{p(x)}+\left(\frac{1}{2i}\frac{\partial p}{\partial x_1}+\frac{1}{2}\frac{\partial p}{\partial x_2}\right)e^{-p(x)}. \tag{42}$$

Thus, it follows from (41) and (42) that

$$(1+i)Q_1(x)e^{p(x+c)+p(x)}+(1+i)Q_2(x)e^{p(x+c)-p(x)}-ie^{2p(x+c)}=1, \tag{43}$$

$$(1-i)Q_1(x)e^{p(x)-p(x+c)}+(1-i)Q_2(x)e^{-p(x+c)-p(x)}+ie^{-2p(x+c)}=1, \tag{44}$$

where

$$Q_1(x)=\frac{1}{2i}\frac{\partial^2 p}{\partial x_1^2}+\frac{1}{2i}\left(\frac{\partial p}{\partial x_1}\right)^2-\frac{1}{2}\frac{\partial^2 p}{\partial x_1\partial x_2}-\frac{1}{2}\frac{\partial p}{\partial x_1}\frac{\partial p}{\partial x_2},$$
$$Q_2(x)=\frac{1}{2i}\frac{\partial^2 p}{\partial x_1^2}-\frac{1}{2i}\left(\frac{\partial p}{\partial x_1}\right)^2+\frac{1}{2}\frac{\partial^2 p}{\partial x_1\partial x_2}-\frac{1}{2}\frac{\partial p}{\partial x_1}\frac{\partial p}{\partial x_2}. \tag{45}$$

Next, we will prove that $Q_1(x)$ and $Q_2(x)$ cannot be equal to 0. Obviously, $Q_1(x)\equiv 0$ and $Q_2(x)\equiv 0$ cannot hold at the same time. Otherwise, we have $-ie^{2p(x+c)}=1$, and this is a contradiction. If $Q_1(x)\equiv 0$ and $Q_2(x)\equiv 0$, then it follows from (43) that

$$e^{2p(x+c)}=i+(1-i)Q_2(x)e^{p(x+c)-p(x)}. \tag{46}$$

Due to the fact that $p(x)$ is a nonconstant polynomial, we have that $N(r,e^{2p(x+c)})=0$, $N(r,1/e^{2p(x+c)})=0$, and $N(r,1/(1-i)Q_2(x)e^{p(x+c)-p(x)})=o(r,e^{2p(x+c)})$. In view of the Nevanlinna second main theorem in several complex variables and (46), we have

$$\begin{aligned}T\left(r,e^{2p(x+c)}\right)&\le N\left(r,e^{2p(x+c)}\right)+N\left(r,\frac{1}{e^{2p(x+c)}}\right)+N\left(r,\frac{1}{e^{2p(x+c)}-i}\right)\\&\le N\left(r,\frac{1}{(1-i)Q_2(x)e^{p(x+c)-p(x)}}\right)+S\left(r,e^{2p(x+c)}\right)=o\left(T\left(r,e^{2p(x+c)}\right)\right).\end{aligned} \tag{47}$$

This is impossible. If $Q_1(x)\equiv 0$ and $Q_2(x)\equiv 0$, similarly, we can get a contradiction. Consequently, these cases can conclude that $Q_1(x)\equiv 0$ and $Q_2(x)\equiv 0$. Due to the fact that $p(x)+p(x+c)$ and $2p(x+c)$ cannot be constant and by Lemma 8, it follows from (43) and (44) that

$$(1+i)Q_2(x)e^{p(x+c)-p(x)}=1, \tag{48}$$

and

$$(1-i)Q_1(x)e^{p(x)-p(x+c)}=1. \tag{49}$$

Therefore, (48) and (49) indicate that $p(x+c)-p(x)$ is a constant (let $p(x+c)-p(x)=\zeta,\zeta\in\mathbb{C}$). Thus, it means that $p(x)=l(x)+B$, where l is a linear function as the form $l(x)=a_1x_1+a_2x_2, a_1(\neq 0), a_2,$ and B_1, B_2 are constants. Thus, we have

$$(1-i)\left[\frac{1}{2i}a_1^2-\frac{1}{2}a_1a_2\right]e^{-l(c)}=1,\ (1+i)\left[-\frac{1}{2i}a_1^2-\frac{1}{2}a_1a_2\right]e^{l(c)}=1, \tag{50}$$

where $l(c)=a_1c_1+a_2c_2$. This leads to

$$a_1^4+a_1^2a_2^2=2, i\left(a_1^4-1\right)+a_1^3a_2=e^{2l(c)}. \tag{51}$$

Thus, we have from (42) that

$$\frac{\partial G(x+c)}{\partial x_1}-\frac{\partial G(x+c)}{\partial x_2}=\left(\frac{a_1}{2i}-\frac{a_2}{2}\right)e^{a_1x_1+a_2x_2+B}+\left(\frac{a_1}{2i}+\frac{a_2}{2}\right)e^{-\left(a_1x_1+a_2x_2+B\right)}, \tag{52}$$

that is,

$$\frac{\partial G(x)}{\partial x_1}-\frac{\partial G(x)}{\partial x_2}=\left(\frac{a_1}{2i}-\frac{a_2}{2}\right)e^{a_1x_1+a_2x_2+B'}+\left(\frac{a_1}{2i}+\frac{a_2}{2}\right)e^{-\left(a_1x_1+a_2x_2+B'\right)}, \tag{53}$$

where $B' = B - a_1c_1 - a_2c_2$. The characteristic equations of (53) are

$$\frac{dx_1}{dt}=1,\frac{dx_2}{dt}=-1,$$
$$\frac{dG}{dt}=\left(\frac{a_1}{2i}-\frac{a_2}{2}\right)e^{a_1x_1+a_2x_2+B'}+\left(\frac{a_1}{2i}+\frac{a_2}{2}\right)e^{-\left(a_1x_1+a_2x_2+B'\right)}. \tag{54}$$

By using the initial conditions $x_1 = 0, x_2 = s$ and $G = G(0,s) := \theta_0(s)$ with a parameters, we can obtain the following parametric representation for solutions of the characteristic equations: $x_1 = t, x_2 = -t + s$,

$$\begin{aligned}G(t,s)&=\int_0^t\left[\left(\frac{a_1-ia_2}{2i}\right)e^{(a_1-a_2)t+a_2s+B'}+\left(\frac{a_1+ia_2}{2i}\right)e^{-\left((a_1-a_2)t+a_2s+B'\right)}\right]dt+\theta_0(s)\\&=\frac{a_1-ia_2}{2i(a_1-a_2)}e^{(a_1-a_2)t+a_2s+B'}-\frac{a_1+ia_2}{2i(a_1-a_2)}e^{-\left((a_1-a_2)t+a_2s+B'\right)}+\theta(s),\end{aligned} \tag{55}$$

where $\theta(s)$ is an entire function with finite order in s satisfying

$$\theta(s)=\theta_0(s)-\frac{a_1-ia_2}{2i(a_1-a_2)}e^{a_2s+B'}+\frac{a_1+ia_2}{2i(a_1-a_2)}e^{-\left(a_2s+B'\right)}. \tag{56}$$

Thus, it yields that.

$$G(x)=\frac{a_1-ia_2}{2i(a_1-a_2)}e^{a_1x_1+a_2x_2+B'}-\frac{a_1+ia_2}{2i(a_1-a_2)}e^{-\left(a_1x_1+a_2x_2+B'\right)}+\theta(s). \tag{57}$$

Substituting (57) into (39) and combining with (51), we can deduce that $\theta(s)$ satisfies

$$\theta(s+s_0)+\theta''(s)=0. \tag{58}$$

Therefore, from Cases 14 and 15, the proof of the theorem is completed.

3.2. The Proof of Theorem 10. Similar to the proof of Theorem 9, we will consider two cases below.

Case 16. If $G(x+c)+\partial^nG(x)/\partial x_1^n$ and $G(x+c)+\partial^nG(x)/\partial x_1^{n-1}\partial x_2$ are constants, by the same discussion of Theorem 9, we have

$$G(x+c)+\frac{\partial^nG(x)}{\partial x_1^n}=K_1,\quad K_1\in\mathbb{C}, \tag{59}$$

$$G(x+c)+\frac{\partial^nG(x)}{\partial x_1^{n-1}\partial x_2}=K_2,\quad K_2\in\mathbb{C}. \tag{60}$$

Thus, this leads to $K_1^2 + K_2^2 = 1$. In view of (59) and (60), and taking partial derivations on both two sides of equations (61) and (62) for the variables x_2, x_1, respectively, and combining with the fact $\partial^{n+1}G(x)/\partial x_2\partial x_1^n = \partial^{n+1}G(x)/\partial x_1^n\partial x_2$, it yields that

$$\frac{\partial^nG(x)}{\partial x_1^n}-\frac{\partial^nG(x)}{\partial x_1^{n-1}\partial x_2}=K_1-K_2, \tag{61}$$

$$\frac{\partial G(x+c)}{\partial x_1}-\frac{\partial G(x+c)}{\partial x_2}=0, \tag{62}$$

which implies that

$$\frac{\partial G(x)}{\partial x_1}-\frac{\partial G(x)}{\partial x_2}=0, K_1=K_2=\pm\frac{\sqrt{2}}{2}. \tag{63}$$

The characteristic equations of (63) are

$$\frac{dx_1}{dt}=1,$$
$$\frac{dx_2}{dt}=-1, \tag{64}$$
$$\frac{dG}{dt}=K_1-K_2.$$

By using the initial conditions $x_1 = 0, x_2 = s$, and $G = G(0,s) := \theta(s)$ with a parameter s, we can obtain the following parametric representation for solutions of the characteristic equations: $x_1 = t, x_2 = -t + s$,

$$\begin{aligned} G(t,s) &= \int_0^t (K_1 - K_2)\mathrm{d}t + \theta(s) \\ &= (K_1 - K_2)t + \theta(s). \end{aligned} \tag{65}$$

In view of the above conditions, we have

$$G(x) = \theta(x_1 + x_2). \tag{66}$$

Substituting (66) into (59) or (60), we obtain

$$\theta(s + s_0) + \theta^{(n)}(s) = \pm \frac{\sqrt{2}}{2}, \tag{67}$$

which completes the proof of (i) of Theorem 10.

Case 17. If $G(x+c) + \partial^n G(x)/\partial^n x_1$ and $G(x+c) + \partial^n G(x)/\partial^{n-1} x_1 \partial x_2$ are not constants, similar to the proof of Theorem 9, we can get

$$G(x+c) + \frac{\partial^n G(x)}{\partial x_1^n} = \frac{e^{p(x)} + e^{-p(x)}}{2}, \tag{68}$$

$$G(x+c) + \frac{\partial^n G(x)}{\partial x_1^{n-1}\partial x_2} = \frac{e^{p(x)} - e^{-p(x)}}{2i}, \tag{69}$$

which lead to

$$\frac{\partial^n G(x)}{\partial x_1^n} - \frac{\partial^n G(x)}{\partial x_1^{n-1}\partial x_2} = \frac{1+i}{2} e^{p(x)} + \frac{1-i}{2} e^{-p(x)}. \tag{70}$$

Taking the partial derivations on both two sides of equations (68) and (69) for the variables x_2, x_1, respectively, by combining with the fact $\partial^{n+1} G(x)/\partial x_2 \partial x_1^n = \partial^{n+1} G(x)/\partial x_1^n \partial x_2$, it yields that

$$\frac{\partial G(x+c)}{\partial x_1} - \frac{\partial G(x+c)}{\partial x_2} = \left(\frac{1}{2i}\frac{\partial p}{\partial x_1} - \frac{1}{2}\frac{\partial p}{\partial x_2}\right) e^{p(x)} + \left(\frac{1}{2i}\frac{\partial p}{\partial x_1} + \frac{1}{2}\frac{\partial p}{\partial x_2}\right) e^{-p(x)}. \tag{71}$$

Thus, it follows from (70) and (71) that

$$(1+i)Q_1(x) e^{p(x+c)+p(x)} + (1+i)Q_2(x) e^{p(x+c)-p(x)} - i e^{2p(x+c)} = 1, \tag{72}$$

$$(1-i)Q_1(x) e^{p(x)-p(x+c)} + (1-i)Q_2(x) e^{-p(x+c)-p(x)} + i e^{-2p(x+c)} = 1, \tag{73}$$

where

$$\begin{aligned} Q_1(x) &= C_{n-1}^k \left(\frac{1}{2i}\frac{\partial p}{\partial x_1} - \frac{1}{2}\frac{\partial p}{\partial x_2}\right)^{(k)} \left(\frac{\partial p}{\partial z_1}\right)^{(n-1-k)}, \\ Q_2(x) &= C_{n-1}^k \left(\frac{1}{2i}\frac{\partial p}{\partial x_1} + \frac{1}{2}\frac{\partial p}{\partial x_2}\right)^{(k)} \left(-\frac{\partial p}{\partial x_1}\right)^{(n-1-k)}. \end{aligned} \tag{74}$$

Next, we will prove that $Q_1(x)$ and $Q_2(x)$ cannot be equal to 0. Obviously, $Q_1(x) \equiv 0$ and $Q_2(x) \equiv 0$ cannot hold at the same time. Otherwise, we have $-ie^{2p(x+c)} = 1$, and this is impossible. Similar to the cases about $Q_1(x)$ and $Q_2(x)$ in the proof of Theorem 9, we can conclude that $Q_1(x) \equiv 0$ and $Q_2(x) \equiv 0$. Since $p(x) + p(x+c)$ and $2p(x+c)$ cannot be constant, by Lemma 8, it follows from (72) and (73) that

$$(1+i)Q_2(x) e^{p(x+c)-p(x)} = 1, \tag{75}$$

and

$$(1-i)Q_1(x) e^{p(x)-p(x+c)} = 1. \tag{76}$$

Then, (75) and (76) imply that $p(x+c) - p(x) = \zeta$, where ζ is a constant in $\mathbb{C}$. Thus, we get that $p(x) = l(x) + B$, where l is a linear function as the form $l(x) = a_1 x_1 + a_2 x_2, a_1 (\neq 0), a_2$, and B_1, B_2 are constants. Thus, we have

$$(1-i)\left[\frac{1}{2i}a_1^n - \frac{1}{2}a_1^{n-1}a_2\right] e^{-l(c)} = 1, \ (1+i)\left[-\frac{1}{2i}a_1^n - \frac{1}{2}a_1^{n-1}a_2\right] e^{l(c)} = 1, \tag{77}$$

where $l(c) = a_1 c_1 + a_2 c_2$, and this leads to

$$a_1^{2n} + a_1^{2n-2} a_2^2 = 2, i\left(a_1^{2n} - 1\right) + a_1^{2n-1} a_2 = e^{2l(c)}. \tag{78}$$

In view of (71), we have

$$\begin{aligned}\frac{\partial G(x+c)}{\partial x_1} - \frac{\partial G(x+c)}{\partial x_2} &= \left(\frac{a_1}{2i} - \frac{a_2}{2}\right) e^{a_1 x_1 + a_2 x_2 + B} \\ &+ \left(\frac{a_1}{2i} + \frac{a_2}{2}\right) e^{-\left(a_1 x_1 + a_2 x_2 + B\right)}.\end{aligned} \tag{79}$$

This implies

$$\begin{aligned}\frac{\partial G(x)}{\partial x_1} - \frac{\partial G(x)}{\partial x_2} &= \left(\frac{a_1}{2i} - \frac{a_2}{2}\right) e^{a_1 x_1 + a_2 x_2 + B'} \\ &+ \left(\frac{a_1}{2i} + \frac{a_2}{2}\right) e^{-\left(a_1 x_1 + a_2 x_2 + B'\right)},\end{aligned} \tag{80}$$

where $B' = B - a_1 c_1 - a_2 c_2$. The characteristic equations of (80) are

$$\begin{aligned}&\frac{dx_1}{dt} = 1, \\ &\frac{dx_2}{dt} = -1, \\ &\frac{dG}{dt} = \left(\frac{a_1}{2i} - \frac{a_2}{2}\right) e^{a_1 x_1 + a_2 x_2 + B'} + \left(\frac{a_1}{2i} + \frac{a_2}{2}\right) e^{-\left(a_1 x_1 + a_2 x_2 + B'\right)}.\end{aligned} \tag{81}$$

By using the initial conditions $x_1 = 0, x_2 = s$ and $G = G(0, s) := \theta_0(s)$ with parameters, we can obtain the following parametric representation for solutions of the characteristic equations: $x_1 = t, x_2 = -t + s$,

$$\begin{aligned}G(t, s) &= \frac{a_1 - i a_2}{2i\left(a_1 - a_2\right)} e^{\left(a_1 - a_2\right)t + a_2 s + B'} \\ &- \frac{a_1 + i a_2}{2i\left(a_1 - a_2\right)} e^{-\left(\left(a_1 - a_2\right)t + a_2 s + B'\right)} + \theta(s),\end{aligned} \tag{82}$$

where $\theta(s)$ is an entire function with finite order in s such that

$$\theta(s) = \theta_0(s) - \frac{a_1 - i a_2}{2i\left(a_1 - a_2\right)} e^{a_2 s + B'} + \frac{a_1 + i a_2}{2i\left(a_1 - a_2\right)} e^{-\left(a_2 s + B'\right)}. \tag{83}$$

Thus, it yields that

$$\begin{aligned}G(x) &= \frac{a_1 - i a_2}{2i\left(a_1 - a_2\right)} e^{a_1 x_1 + a_2 x_2 + B'} \\ &- \frac{a_1 + i a_2}{2i\left(a_1 - a_2\right)} e^{-\left(a_1 x_1 + a_2 x_2 + B'\right)} + \theta(s).\end{aligned} \tag{84}$$

Substituting (84) into (68) and combining with (76), we can deduce that $\theta(s)$ satisfies

$$\theta\left(s + s_0\right) + \theta^{(n)}(s) = 0. \tag{85}$$

Therefore, from Cases 16 and 17, the proof of this theorem is completed.

3.3. *The Proof of Theorem 12.*

We suppose that $G(x)$ is a finite-order transcendental entire solution of equation (19). Apparently, $G(x+c)$ and $G(x) + \partial^2 G(x)/\partial x_1 \partial x_2$ are not constants; otherwise, there is a contradiction with the assumption. In view of the proof of Theorem 9, we get

$$G(x+c) = \frac{e^{p(x)} + e^{-p(x)}}{2}, \tag{86}$$

$$G(x) + \frac{\partial^2 G(x)}{\partial x_1 \partial x_2} = \frac{e^{p(x)} - e^{-p(x)}}{2i}. \tag{87}$$

Differentiating both sides of equation (86) for the variables x_2, x_1, respectively, and combining (86) and (87), this leads to

$$\frac{1}{2}\left(1 + \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{\partial p}{\partial x_1}\frac{\partial p}{\partial x_2}\right) e^{p(x)} + \frac{1}{2}\left(1 - \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{\partial p}{\partial x_1}\frac{\partial p}{\partial x_2}\right) e^{-p(x)} = \frac{e^{p(x+c)} - e^{-p(x+c)}}{2i}. \tag{88}$$

Thus, it follows from (88) that

$$-iQ_1(x) e^{p(x+c)+p(x)} - iQ_2(x) e^{p(x+c)-p(x)} - e^{2p(x+c)} = 1, \tag{89}$$

$$iQ_1(x) e^{-p(x+c)+p(x)} + iQ_2(x) e^{-p(x+c)-p(x)} + e^{-2p(x+c)} = 1, \tag{90}$$

where

$$Q_1(x) = 1 + \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{\partial p}{\partial x_1}\frac{\partial p}{\partial x_2},$$
$$Q_2(x) = 1 - \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{\partial p}{\partial x_1}\frac{\partial p}{\partial x_2}. \tag{91}$$

Next, we will prove that $Q_1(x)$ and $Q_2(x)$ cannot be equal to 0. Obviously, $Q_1(x) \equiv 0$ and $Q_2(x) \equiv 0$ cannot hold at the same time. Otherwise, we have $e^{2p(x+c)} = 1$, and this is a contradiction. If $Q_1(x) \equiv 0$ and $Q_2(x) \equiv 0$, then it follows from (89) that

$$-iQ_2(x)e^{p(x+c)-p(x)} - e^{2p(x+c)} = 1. \tag{92}$$

Similar to the cases about $Q_1(x) \equiv 0$ and $Q_2(x) \equiv 0$ in the proof of Theorem 9, we can get that $Q_1(x) \equiv 0$ and $Q_2(x) \equiv 0$. Since $p(x) + p(x+c)$ and $2p(x+c)$ cannot be constant, and by Lemma 8, it follows from (89) and (90) that

$$-iQ_2(x)e^{p(x+c)-p(x)} = 1, \tag{93}$$

and

$$iQ_1(x)e^{p(x)-p(x+c)} = 1. \tag{94}$$

Hence, (93) and (94) imply that $p(x+c) - p(x) = \zeta$, where ζ is a constant in $\mathbb{C}$. Thus, it means that $p(x) = l(x) + B$, where l is a linear function as the form $l(x) = a_1x_1 + a_2x_2, a_1(\neq 0), a_2(\neq 0), B$ are constants. Thus, we have

$$-i(1 + a_1a_2)e^{l(c)} = 1, i(1 + a_1a_2)e^{-l(c)} = 1, \tag{95}$$

where $l(c) = a_1c_1 + a_2c_2$, and this leads to

$$a_1a_2 = -2, -1 = e^{2l(c)}. \tag{96}$$

Therefore, the proof of this theorem is completed.

4. The Proof of Theorem 13

We suppose that (G_1, G_2) is a pair of finite-order transcendental entire functions satisfying system (22). First, system (22) can be rewritten as

$$\begin{cases} \left[G_1(x+c) + i\left(G_2 + \dfrac{\partial^2 G_2}{\partial x_1 \partial x_2}\right)\right]\left[G_1(x+c) - i\left(G_2 + \dfrac{\partial^2 g_2}{\partial x_1 \partial x_2}\right)\right] = 1, \\ \left[G_2(x+c) + i\left(G_1 + \dfrac{\partial^2 G_1}{\partial x_1^2}\right)\right]\left[G_2(x+c) - i\left(G_1 + \dfrac{\partial^2 G_1}{\partial x_1^2}\right)\right] = 1. \end{cases} \tag{97}$$

Since G_1, G_2 are finite-order transcendental entire functions, then there exist two nonconstant polynomials $p_1(x), p_2(x)$ such that

$$\begin{cases} G_1(x+c) + i\left(G_2(x) + \dfrac{\partial^2 G_2}{\partial x_1 \partial x_2}\right) = e^{ip_1(x)}, \\ G_1(x+c) - i\left(G_2(x) - \dfrac{\partial^2 G_2}{\partial x_1 \partial x_2}\right) = e^{-ip_1(x)}, \\ G_2(x+c) + i\left(G_1(x) + \dfrac{\partial^2 G_1}{\partial x_1^2}\right) = e^{ip_2(x)}, \\ G_2(x+c) - i\left(G_1(x) + \dfrac{\partial^2 G_1}{\partial x_1^2}\right) = e^{-ip_2(x)}. \end{cases} \tag{98}$$

In view of (98), it yields that.

$$\begin{cases} G_1(x+c) = \dfrac{e^{ip_1(x)} + e^{-ip_1(x)}}{2}, \\ G_2(x) + \dfrac{\partial^2 G_2}{\partial x_1 \partial x_2} = \dfrac{e^{ip_1(x)} - e^{-ip_1(x)}}{2i}, \\ G_2(x+c) = \dfrac{e^{ip_2(x)} + e^{-ip_2(x)}}{2}, \\ G_1(x) + \dfrac{\partial^2 G_1}{\partial x_1^2} = \dfrac{e^{ip_2(x)} - e^{-ip_2(x)}}{2i}, \end{cases} \tag{99}$$

which implies

$$\left(-i + \frac{\partial^2 p_1}{\partial z_1^2} + i\left(\frac{\partial p_1}{\partial z_1}\right)^2\right)e^{ip_1(x)+ip_2(x+c)} + \left(-i - \frac{\partial^2 p_1}{\partial z_1^2} + i\left(\frac{\partial p_1}{\partial z_1}\right)^2\right)e^{ip_2(x+c)-ip_1(x)} + e^{2ip_2(x+c)} \equiv 1, \tag{100}$$

or

$$\left(-i+\frac{\partial^2 p_2}{\partial x_1\partial x_2}+i\frac{\partial p_2}{\partial x_1}\frac{\partial p_2}{\partial x_2}\right)e^{ip_2(x)+ip_1(x+c)}+\left(-i-\frac{\partial^2 p_2}{\partial x_1\partial x_2}+i\frac{\partial p_2}{\partial x_1}\frac{\partial p_2}{\partial x_2}\right)e^{ip_1(x+c)-ip_2(x)}+e^{2ip_1(x+c)}\equiv 1. \tag{101}$$

By using the Nevanlinna second main theorem in several complex variables, we can deduce that $-i+\partial^2 p_1/\partial x_1^2+i(\partial p_1/\partial x_1)^2$ and $-i-\partial^2 p_1/\partial x_1^2+i(\partial p_1/\partial x_1)^2$ cannot be equal to 0. Similarly, $-i+\partial^2 p_2/\partial x_1\partial x_2+i\partial p_2/\partial x_1\partial p_2/\partial x_2$ and $-i-\partial^2 p_2/\partial x_1\partial x_2+i\partial p_2/\partial x_1\partial p_2/\partial x_2$ cannot also be equal to 0. Thus, by Lemma 8, (100) and (101), we have.

$$\begin{aligned}&\left(-i+\frac{\partial^2 p_1}{\partial x_1^2}+i\left(\frac{\partial p_1}{\partial x_1}\right)^2\right)e^{ip_1(x)+ip_2(x+c)}\equiv 1,\\&\left(-i-\frac{\partial^2 p_1}{\partial x_1^2}+i\left(\frac{\partial p_1}{\partial x_1}\right)^2\right)e^{ip_2(x+c)-ip_1(x)}\equiv 1,\\&\left(-i+\frac{\partial^2 p_2}{\partial x_1\partial x_2}+i\frac{\partial p_2}{\partial x_1}\frac{\partial p_2}{\partial x_2}\right)e^{ip_2(x)+ip_1(x+c)}\equiv 1,\\&\left(-i-\frac{\partial^2 p_2}{\partial x_1\partial x_2}+i\frac{\partial p_2}{\partial x_1}\frac{\partial p_2}{\partial x_2}\right)e^{ip_1(x+c)-ip_2(x)}\equiv 1.\end{aligned} \tag{102}$$

Now, we will discuss four cases below.

Case 18

$$\begin{cases}\left(-i+\dfrac{\partial^2 p_1}{\partial x_1^2}+i\left(\dfrac{\partial p_1}{\partial x_1}\right)^2\right)e^{ip_1(x)+ip_2(x+c)}\equiv 1,\\ \left(-i+\dfrac{\partial^2 p_2}{\partial x_1\partial x_2}+i\dfrac{\partial p_2}{\partial x_1}\dfrac{\partial p_2}{\partial x_2}\right)e^{ip_2(x)+ip_1(x+c)}\equiv 1.\end{cases} \tag{103}$$

Due to the fact that $p_1(x), p_2(x)$ are polynomials, it follows from (103) that $p_1(x)+p_2(x+c)\equiv C_1$ and $p_1(x+c)+p_2(x)\equiv C_2$, and here and below, C_1, C_2 are constants. Thus, it yields that $p_1(x+2c)-p_1(x)\equiv C_2-C_1$ and $p_2(z+2c)-p_2(x)\equiv C_1-C_2$. This leads to $p_1(x)=l(x)+w(x)+B_1$ and $p_2(x)=-l(x)-w(x)+B_2$, where $l(x)=a_1x_1+a_2x_2$ $(a_1(\neq 0), a_2(\neq 0), B_1, B_2\in\mathbb{C})$ and $w(x):=w(s)$, $w(s)$ is a polynomial in s in $\mathbb{C}$, $s=c_2x_1-c_1x_2$.

Following, we will illustrate that $w(x)\equiv 0$. Supposing that $\deg_s w=n (n\geq 0)$, equation (103) implies

$$c_1^2\frac{d^2w}{ds^2}+ic_1^2\left(\frac{dw}{ds}\right)^2+2ic_1a_1\frac{dw}{ds}\equiv\zeta_0, \tag{104}$$

that is,

$$c_1^2\frac{d^2w}{ds^2}\equiv\zeta_0-ic_1^2\left(\frac{dw}{ds}\right)^2-2ic_1a_1\frac{dw}{ds}, \tag{105}$$

where $\zeta_0\in\mathbb{C}$. By comparing the degree of s on both sides of the above equation, we have $n-2=2(n-1)$, that is, $n=0$. Thus, the form of $l(x)+w(x)+B$ is still the linear form of $A_1x_1+A_2x_2+B$, which means that $w(x)\equiv 0$. Thus, this means that $\partial^2 p_1/\partial x_1\partial x_2\equiv\partial^2 p_2/\partial x_1\partial x_2\equiv 0$. Substituting these into (103), we have

$$\begin{cases}\left(-i+ia_1^2\right)e^{-il(c)+i\left(B_1+B_2\right)}\equiv 1,\\ \left(-i+ia_1a_2\right)e^{il(c)+i\left(B_1+B_2\right)}\equiv 1,\end{cases} \tag{106}$$

Moreover, based on (100)-(103), we can see that

$$\begin{cases}\left(i+\dfrac{\partial^2 p_1}{\partial x_1^2}-i\left(\dfrac{\partial p_1}{\partial x_1}\right)^2\right)e^{-ip_1(x)-ip_2(x+c)}\equiv 1,\\ \left(i+\dfrac{\partial^2 p_2}{\partial x_1\partial x_2}-i\dfrac{\partial p_2}{\partial x_1}\dfrac{\partial p_2}{\partial x_2}\right)e^{-ip_2(x)-ip_1(x+c)}\equiv 1,\end{cases} \tag{107}$$

which means that

$$\begin{cases}\left(i-ia_1^2\right)e^{il(c)-i\left(B_1+B_2\right)}\equiv 1,\\ \left(i-ia_1a_2\right)e^{-il(c)-i\left(B_1+B_2\right)}\equiv 1.\end{cases} \tag{108}$$

Thus, we can deduce from (106) and (108) that

$$-2a_1^2+a_1^4=0, -2a_1a_2+a_1^2a_2^2=0. \tag{109}$$

Noting that $a_1\neq 0$ and $a_2\neq 0$, then we have $a_1^2=2$ and $a_1a_2=2$. In view of (106) and (108), we get $e^{2il(c)}=1$. Thus, it follows from (99) that

$$G_1(x)=\frac{e^{il(x)+iB_1-il(c)}+e^{-il(x)-iB_1+il(c)}}{2}, \tag{110}$$

and

$$G_2(x)=\frac{e^{-il(x)+iB_2+il(c)}+e^{il(x)-iB_2-il(c)}}{2}. \tag{111}$$

If $e^{il(c)}=1$, then $l(c)=2k\pi$ and $e^{i(B_1+B_2)}\equiv -i$. Thus, it follows from (110) and (111) that

$$\begin{aligned}G_1(x)&=\frac{e^{i\left(l(x)+B_1\right)}+e^{-i\left(l(x)+B_1\right)}}{2}=\cos\left(l(x)+B_1\right),\\G_2(x)&=\frac{e^{i\left(l(x)-B_2\right)}+e^{-i\left(l(x)-B_2\right)}}{2}=\cos\left(l(x)-B_2\right)\\&=-\sin\left(l(x)+B_1\right).\end{aligned} \tag{112}$$

If $e^{il(c)}=-1$, then $l(c)=(2k+1)\pi$ and $e^{i(B_1+B_2)}=i$. Thus, it follows from (110) and (111) that

$$\begin{aligned} G_1(x) &= -\cos(l(x)+B_1),\\ G_2(x) &= -\sin(l(x)+B_1). \end{aligned} \tag{113}$$

Case 19

$$\begin{cases} \left(-i+\dfrac{\partial^2 p_1}{\partial x_1^2}+i\left(\dfrac{\partial p_1}{\partial x_1}\right)^2\right)e^{ip_1(x)+ip_2(x+c)} \equiv 1,\\ \left(-i-\dfrac{\partial^2 p_2}{\partial x_1\partial x_2}+i\dfrac{\partial p_2}{\partial x_1}\dfrac{\partial p_2}{\partial x_2}\right)e^{ip_1(x+c)-ip_2(x)} \equiv 1, \end{cases} \tag{114}$$

From (114), we get that $p_2(x+c)+p_1(x)\equiv C_1$ and $p_1(x+c)-p_2(x)\equiv C_2$, which deduce that $p_1(x+2c)+p_1(x)\equiv C_1+C_2$. Noting that $p_1(x), p_2(x)$ are nonconstant polynomials, we obtain a contradiction.

Case 20

$$\begin{cases} \left(-i-\dfrac{\partial^2 p_1}{\partial x_1^2}+i\left(\dfrac{\partial p_1}{\partial x_1}\right)^2\right)e^{ip_2(x+c)-ip_1(x)} \equiv 1,\\ \left(-i+\dfrac{\partial^2 p_2}{\partial x_1\partial x_2}+i\dfrac{\partial p_2}{\partial x_1}\dfrac{\partial p_2}{\partial x_2}\right)e^{ip_2(x)+ip_1(x+c)} \equiv 1, \end{cases} \tag{115}$$

From (115), it follows that $p_2(x+c)-p_1(x)\equiv C_1$ and $p_1(x+c)+p_2(x)\equiv C_2$, which leads to $p_2(x+2c)+p_2(x)\equiv C_1+C_2$. Noting that $p_1(x), p_2(x)$ are nonconstant polynomials, we obtain a contradiction.

Case 21

$$\begin{cases} l\left(-i-\dfrac{\partial^2 p_1}{\partial x_1^2}+i\left(\dfrac{\partial p_1}{\partial x_1}\right)^2\right)e^{ip_2(x+c)-ip_1(x)} \equiv 1,\\ \left(-i-\dfrac{\partial^2 p_2}{\partial x_1\partial x_2}+i\dfrac{\partial p_2}{\partial x_1}\dfrac{\partial p_2}{\partial x_2}\right)e^{ip_1(x+c)-ip_2(x)} \equiv 1. \end{cases} \tag{116}$$

Similar to the argument as in Case 18, we can deduce that $p_1(x)=l(x)+B_1, p_2(x)=l(x)+B_2$, where $l(x)=a_1x_1+a_2x_2$ $(a_1(\neq 0), a_2(\neq 0), B_1, B_2\in\mathbb{C})$. Hence, it follows that $(\partial^2 p_1/\partial x_1^2)\equiv(\partial^2 p_2/\partial x_1\partial x_2)\equiv 0$. Substituting these into (116), we have

$$\begin{cases} \left(-i+ia_1^2\right)e^{il(c)-i(B_1-B_2)} \equiv 1,\\ \left(-i+ia_1a_2\right)e^{il(c)+i(B_1-B_2)} \equiv 1. \end{cases} \tag{117}$$

Moreover, based on (100)–(103), we can see that

$$\begin{cases} \left(i-\dfrac{\partial^2 p_1}{\partial x_1^2}-i\left(\dfrac{\partial p_1}{\partial x_1}\right)^2\right)e^{ip_1(x)-ip_2(x+c)} \equiv 1,\\ \left(i-\dfrac{\partial^2 p_2}{\partial x_1\partial x_2}-i\dfrac{\partial p_2}{\partial x_1}\dfrac{\partial p_2}{\partial x_2}\right)e^{ip_2(x)-ip_1(x+c)} \equiv 1, \end{cases} \tag{118}$$

which means that

$$\begin{cases} \left(i-ia_1^2\right)e^{-il(c)+i(B_1-B_2)} \equiv 1,\\ \left(i-ia_1a_2\right)e^{-il(c)-i(B_1-B_2)} \equiv 1. \end{cases} \tag{119}$$

Thus, we can deduce from (117) and (119) that

$$-2a_1^2+a_1^4=0, -2a_1a_2+a_1^2a_2^2=0. \tag{120}$$

In view of $a_1\neq 0$ and $a_2\neq 0$, then we have $a_1^2=2$ and $a_1a_2=2$. By combining with (117) and (119), we also get $e^{2il(c)}=-1$. In view of (99), we have

$$G_1(x)=\frac{e^{il(x)+iB_1-il(c)}+e^{-il(x)-iB_1+il(c)}}{2}, \tag{121}$$

$$G_2(x)=\frac{e^{il(x)+iB_2-il(c)}+e^{-il(x)-iB_2+il(c)}}{2}. \tag{122}$$

If $e^{il(c)}=i$, then $l(c)=(2k+1/2)\pi$ and $e^{i(B_1-B_2)}=-1$ $(B_1-B_2=2k\pi+\pi)$. Thus, it follows from (121) and (122) that

$$\begin{aligned} G_1(x) &= \frac{e^{i(l(x)+B_1)}-e^{-i(l(x)+B_1)}}{2i}=\sin(l(x)+B_1),\\ G_2(x) &= \frac{-ie^{i(l(x)+B_2)}+ie^{-i(l(x)+B_2)}}{2}=\frac{e^{i(l(x)+B_2)}-e^{-i(l(x)+B_2)}}{2i}=-\sin(l(x)+B_1). \end{aligned} \tag{123}$$

If $e^{il(c)}=-i$, then $l(c)=(2k-1/2)\pi$ and $e^{i(B_1-B_2)}=1$ $(B_1-B_2=2k\pi)$. Thus, it follows from (121) and (122) that.

$$\begin{aligned} G_1(x) &= -\frac{e^{i(l(x)+B_1)}-e^{-i(l(x)+B_1)}}{2i}=-\sin(l(x)+B_1),\\ G_2(x) &= \frac{ie^{i(l(x)+B_2)}-ie^{-i(l(x)+B_2)}}{2}=-\frac{e^{i(l(x)+B_2)}-e^{-i(l(x)+B_2)}}{2i}=-\sin(l(x)+B_1). \end{aligned} \tag{124}$$

Therefore, the proof of Theorem 12 is completed.

5. Conclusion

In view of Theorems 9, 10, 12, and 13, we give the exact form of entire solutions for some second-order and higher order (mixed) partial differential equations (systems). These results are some improvements of the previous results, which are mainly concerning with first-order partial differential equations. Meantime, some examples (including Examples 2, 3, and 4) show that our results are precise to some extent.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Y. X. Chen and H. Y. Xu were responsible for conceptualization. Y. X. Chen and L. B. Xie were responsible for writing the original draft. Y. X. Chen, L. B. Xie, and H. Y. Xu were responsible for writing, reviewing, and editing. Y. X. Chen and L. B. Xie were responsible for funding acquisition.

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