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Research Article

Topological Complexity and LS-Category of Certain Manifolds

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The Lusternik–Schnirelmann category and topological complexity are important invariants of topological spaces. In this paper, we calculate the Lusternik–Schnirelmann category and topological complexity of products of real projective spaces and their wedge products by using cup and zero-cup length. Also, we will find the topological complexity of $\mathbb{R}P^{2^k+1}$ by using the immersion dimension of $\mathbb{R}P^{2^k+1}$.

1. Introduction

The Lusternik–Schnirelmann category, (LS-category), of a topological space X which introduced in 1920, is an invariant of a manifold which gave a lower bound for the number of critical points of a function on a closed manifold. The topological complexity is a numerical homotopy invariant, introduced by M. Farber in 2001. M. Farber examined the topological complexity of the robotics [1–3]. Topological complexity has close relationship to classical invariant Lusternik–Schnirelmann category.

Definition 1 (see [4]). The Lusternik–Schnirelmann category of a space X is the least integer n such that there exists an open covering $U_1, \ldots, U_{n+1}$ of X with each U_i contractible to a point in the space X. We denote this by $\operatorname{cat}(X) = n$ and we call such a covering $\{U_i\}$ categorical. If no such integer exists, we write $\operatorname{cat}(X) = \infty$.

Definition 2 (see [1]). Let $\pi: PX \longrightarrow X \times X$ be the path fibration. Topological complexity of a topological space X, denoted by $\mathrm{TC}(X)$, is the least nonnegative integer k if there are open subsets $U_0, U_1, \ldots, U_k$ which cover $X \times X$ such that on each U_i there exists a continuous section of π for $i = 0, 1, \ldots, k$.

This paper is organized as follows. In Section 2, we will calculate Lusternik–Schnirelmann category of products of real projective spaces utilizing [5, 6]. In Section 3, we will calculate topological complexity of products of real projective spaces by using the results of [7]. Furthermore, the topological complexity of wedge products of real projective spaces is calculated by using the results of Section 2. Section 4 provides the topological complexity calculation of $\mathbb{R}P^{2^k+1}$ by using [8, 9] and formulates general results from previous sections. Additionally, general examples are given.

Throughout this paper, we denote the immersion dimension of X by $\operatorname{imd}(X)$.

2. LS-Category of the Products of the Real Projective Spaces

This section is devoted to calculating LS-category of the products of real projective spaces by using cup-length.

Definition 3 (see [4]). Let R be a commutative ring and X be a space. The cup-length of X with coefficients in R is the least integer $k\,(\text{or}\,\infty)$ such that all $(k+1)$-fold cup products vanish in the reduced cohomology $\widetilde{H}^*(X; R)$; we denote this integer by $\operatorname{cup}_R(X)$.

To prove the main theorem of this section, we use the following results of [4].

Proposition 4 (see [4]). *The R-cuplength of a space is less than or equal to the category of the space for all coefficients R. In notation, we write* $\text{cup}_R(X) \leq cat(X)$.

Theorem 5 (see [4]). *For a path-connected locally contractible paracompact space,*

$$cat(X) \leq \dim(X). \tag{1}$$

Remark 6. Since $H^*(\mathbb{R}P^n;\mathbb{Z}_2) = \mathbb{Z}_2[a]/\langle a^{n+1}\rangle$ with

$$\deg(a) = 1, n = \text{cup}(\mathbb{R}P^n) \leq \text{cat}(\mathbb{R}P^n) \leq \dim(\mathbb{R}P^n) = n, \tag{2}$$

then $\text{cat}(\mathbb{R}P^n) = n$.

Theorem 7 (see [4]). *Suppose X and Y are path-connected spaces such that $X \times Y$ is completely normal. Then, $cat(X \times Y) \leq cat(X) + cat(Y)$.*

In the sequel, for simplicity, we write P^k for $\mathbb{R}P^k$.

Theorem 8. *For any positive integer k_i, $1 \leq i \leq n$, we have*

$$\text{cat}\left(P^{k_1} \times P^{k_2} \times \cdots \times P^{k_n}\right) = k_1 + k_2 + \cdots + k_n. \tag{3}$$

Proof. Since $H^*(P^k;\mathbb{Z}_2) = \mathbb{Z}_2[a]/\langle a^{k+1}\rangle$, by Künneth formulas,

$$H^*\left(P^{k_1}\times\cdots\times P^{k_n}\right) = H^*\left(P^{k_1}\right)\otimes,\ldots,\otimes H^*\left(P^{k_n}\right) = \frac{\mathbb{Z}_2[a_1]}{\langle a_1^{k_1+1}\rangle}\otimes\cdots\otimes\frac{\mathbb{Z}_2[a_n]}{\langle a_n^{k_n+1}\rangle}, \tag{4}$$

where $a_i \in H^1(P^{k_i};\mathbb{Z}_2)$, $1 \leq i \leq n$, is a generator, $a_i^{k_i} \neq 0$, and $a_i^{k_i+1} = 0$. Set

$$\begin{aligned} \alpha_1 &= a_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1, \\ \alpha_2 &= 1 \otimes a_2 \otimes 1 \otimes \cdots \otimes 1, \\ &\vdots \\ \alpha_n &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes a_n. \end{aligned} \tag{5}$$

Thus,

$$\begin{aligned} \alpha_1^{k_1} &= a_1^{k_1} \otimes \cdots \otimes 1, \\ \alpha_2^{k_2} &= 1 \otimes a_2^{k_2} \otimes \cdots \otimes 1, \\ &\vdots \\ \alpha_n^{k_n} &= 1 \otimes 1 \otimes \cdots \otimes a_n^{k_n}. \end{aligned} \tag{6}$$

Therefore,

$$\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_n^{k_n} = a_1^{k_1} \otimes a_2^{k_2} \otimes \cdots \otimes a_n^{k_n} \neq 0. \tag{7}$$

$\text{cup}_{\mathbb{Z}_2}(P^{k_1} \times P^{k_2} \times \cdots \times P^{k_n}) = k_1 + k_2 + \cdots + k_n$. On the other hand, by Theorem 7, $\text{cat}(P^{k_1} \times P^{k_2} \times \cdots \times P^{k_n}) \leq k_1 + k_2 + \ldots + k_n$. Since by Proposition 4, $\text{cup}_{Z_2}(P^{k_1} \times P^{k_2} \times \ldots \times P^{k_n})$ is lower bound for $\text{cat}(P^{k_1} \times P^{k_2} \times \ldots \times P^{k_n})$, then

$$\text{cat}\left(P^{k_1} \times P^{k_2} \times \cdots \times P^{k_n}\right) = k_1 + k_2 + \cdots + k_n. \tag{8}$$

□

Corollary 9. *For any positive integer k, we have*

$$\text{cat}\,\underbrace{\left(P^k \times P^k \times \cdots \times P^k\right)}_{n\text{-times}} = nk. \tag{9}$$

3. Topological Complexity of Products and Wedge Products of Real Projective Spaces

In this section, we will calculate the topological complexity of the products and wedge products of real projective spaces. We also give a lower bound for $\text{TC}(X)$, where X is the product of real projective spaces. The lower bound is quite useful since it allows an effective computation of $\text{TC}(X)$ in many examples. A lower bound for topological complexity is also obtained by using the zero-divisor-cup-length of X.

Definition 10 (see [1]). Let K be a field. The kernel of homomorphism $\cup: H^*(X;K) \otimes H^*(X;K) \longrightarrow H^*(X;K)$ is called the ideal of the zero-divisors of $H^*(X;K)$. The zero-divisors-cup-length of $H^*(X;K)$ is the length of the longest nontrivial product in the ideal of the zero-divisors of $H^*(X;K)$. This number will be denoted by $\text{zcl}(X)$.

Theorem 11 (see [1]). *The number TC(X) is greater than the zero-divisors-cup-length of $H^*(X;K)$.*

Theorem 12 (see [1]). *For any path-connected metric spaces X and Y,*

$$\text{TC}(X \times Y) \leq \text{TC}(X) + \text{TC}(Y) - 1. \tag{10}$$

Theorem 13 (see [10]). *For any $n \neq 1, 3, 7$, the number $TC(\mathbb{R}P^n)$ equals the smallest k such that the projective space $\mathbb{R}P^n$ admits an immersion into $\mathbb{R}^{k-1}$. Also, for $n = 1, 3, 7$, $TC(\mathbb{R}P^n) = n + 1$.*

Remark 14. Note that by Theorem 13,

$$\text{TC}(P^n) = \text{imd}(P^n) + 1 \text{ and } \text{zcl}(P^n) < \text{TC}(P^n) = \text{imd}(P^n) + 1. \tag{11}$$

So, $\text{zcl}(P^n) \leq \text{imd}(P^n)$, that is, $\text{zcl}(P^n)$ is lower bound for $\text{imd}(P^n)$.

Lemma 15. *For any positive integers k_i for $i = 1, 2, \ldots, n$, we have*

$$\operatorname{zcl}\left(P^{2^{k_1}}\times P^{2^{k_2}}\times\cdots\times P^{2^{k_n}}\right)\geq\left(2^{k_1+1}-1\right)+\left(2^{k_2+1}-1\right)+\cdots+\left(2^{k_n+1}-1\right). \tag{12}$$

Proof. Let $a_i \in H^1(P^{2^{k_i}};\mathbb{Z}_2)$, $1\leq i\leq n$, be a generator. Clearly, $a_i^{2^{k_i}}\neq 0$ and $a_i^{2^{k_i}+1}=0$. For $1\leq i\leq n$, let

$$\alpha_i\in H^*\left(P^{2^{k_1}}\times P^{2^{k_2}}\times,\ldots,\times P^{2^{k_n}}\right)\otimes H^*\left(P^{2^{k_1}}\times P^{2^{k_2}}\times,\ldots,\times P^{2^{k_n}}\right), \tag{13}$$

be defined by

$$\begin{aligned}\alpha_1&=(a_1\otimes 1\otimes,\ldots,\otimes 1)\otimes(1\otimes,\ldots,\otimes 1)+(1\otimes,\ldots,\otimes 1)\otimes(a_1\otimes 1\otimes,\ldots,\otimes 1),\\ \alpha_2&=(1\otimes a_2\otimes\cdots\otimes 1)\otimes(1\otimes\cdots\otimes 1)+(1\otimes\cdots\otimes 1)\otimes(1\otimes a_2\otimes\cdots\otimes 1),\\ &\vdots\\ \alpha_n&=(1\otimes 1\otimes\cdots\otimes a_n)\otimes(1\otimes\cdots\otimes 1)+(1\otimes\cdots\otimes 1)\otimes(1\otimes 1\otimes\cdots\otimes a_n).\end{aligned} \tag{14}$$

We may show by easy calculation that α_i's are in the kernel of

$$\cup: H^*(X)\otimes H^*(X)\longrightarrow H^*(X). \tag{15}$$

Clearly,

$$\alpha_1^{2^{k_1}}=\left(a_1^{2^{k_1}}\otimes 1\otimes\cdots\otimes 1\right)\otimes(1\otimes,\ldots,\otimes 1)+(1\otimes\cdots\otimes 1)\otimes\left(a_1^{2^{k_1}}\otimes 1\otimes\cdots\otimes 1\right), \tag{16}$$

and calculation reveals that

$$\alpha_1^{2^{k_1}}\left(\alpha_1^{2^{k_1}-1}\right)=\alpha_1^{2^{k_1+1}-1}=a_1^{2^{k_1}-1}\otimes a_1^{2^{k_1}}+\cdots+a_1^{2^{k_1}}\otimes a_1^{2^{k_1}-1}\neq 0. \tag{17}$$

Similarly, we can show that $\alpha_2^{2^{k_2+1}-1}\neq 0,\ldots,\alpha_m^{2^{k_n+1}-1}\neq 0$, and so $\alpha_1^{2^{k_1+1}-1}\alpha_2^{2^{k_2+1}-1}\cdots\alpha_n^{2^{k_n+1}-1}\neq 0$. Consequently,

$$\operatorname{zcl}\left(P^{2^{k_1}}\times P^{2^{k_2}}\times\cdots\times P^{2^{k_n}}\right)\geq\left(2^{k_1+1}-1\right)+\left(2^{k_2+1}-1\right)+\cdots+\left(2^{k_n+1}-1\right). \tag{18}$$

□

Lemma 16. *For any positive integers k_i, $i=1,2,\ldots,n$, we have*

$$TC\left(P^{2^{k_1}}\times P^{2^{k_2}}\times\cdots\times P^{2^{k_n}}\right)\leq\left(\left(2^{k_1+1}-1\right)+\left(2^{k_2+1}-1\right)+\cdots+\left(2^{k_n+1}-1\right)\right)+1. \tag{19}$$

.

Proof. By Theorems 12 and 13,

$$
\begin{aligned}
&TC\left(P^{2^{k_1}} \times P^{2^{k_2}} \times \cdots \times P^{2^{k_n}}\right)\\
&\le TC\left(P^{2^{k_1}}\right) + TC\left(P^{2^{k_2}}\right)+, \ldots, +TC\left(P^{2^{k_n}}\right) - (n-1)\\
&\le 2^{k_1+1} + 2^{k_2+1} + \cdots + 2^{k_n+1} - (n-1)\\
&= \left(\left(2^{k_1+1} - 1\right) + \left(2^{k_2+1} - 1\right) + \cdots + \left(2^{k_n+1} - 1\right)\right) + 1.
\end{aligned}
\tag{20}
$$

Clearly, Lemmas 15 and 16 imply the following result. □

Theorem 17. *For any positive integers k_i, $i = 1, 2, \ldots, n$, we have*

$$
\mathrm{TC}\left(P^{2^{k_1}} \times P^{2^{k_2}} \times \cdots \times P^{2^{k_n}}\right) = \left(2^{k_1+1} - 1\right) + \left(2^{k_2+1} - 1\right) + \cdots + \left(2^{k_n+1} - 1\right) + 1. \tag{21}
$$

Corollary 18. *For any positive integer k, we have*

$$
\mathrm{TC}\left(\underbrace{P^{2^k} \times P^{2^k} \times \ldots \times P^{2^k}}_{n-\text{times}}\right) = n\left(2^{k+1} - 1\right) + 1. \tag{22}
$$

Remark 19. By Theorem 17, clearly Theorem 7.1 in [10] is true for $P^{2^{k_1}} \times P^{2^{k_2}} \times \cdots \times P^{2^{k_n}}$. Note that we do not know if this is true for arbitrary products.

To calculate topological complexity of the wedge products of real projective spaces, we use the next theorem from [11].

Theorem 20 (see [11]). *Let X, Y be Hausdorff normal topological spaces and path connected with nondegenerate basepoints, such that $X \times X$, $Y \times Y$ and $X \times Y$ are normal. Then,*

$$
\mathrm{TC}(X \vee Y) = \max\{\mathrm{TC}(X), \mathrm{TC}(Y), \mathrm{cat}(X \times Y)\}. \tag{23}
$$

Theorem 21. *For any positive integers, $n_1 \ge n_2 \ge \cdots \ge n_k$, we have*

$$
\mathrm{TC}\left(P^{n_1} \vee P^{n_2} \vee \ldots \vee P^{n_k}\right) = \max\{\mathrm{imd}(P^{n_1}) + 1, n_1 + n_2\} = \max\{\mathrm{TC}(P^{n_1}), n_1 + n_2\}. \tag{24}
$$

Proof. Proof follows by induction on k. If $k = 2$, then by Theorem 20, we have

$$
\begin{aligned}
\mathrm{TC}\left(P^{n_1} \vee P^{n_2}\right) &= \max\{\mathrm{TC}(P^{n_1}), \mathrm{TC}(P^{n_2}), \mathrm{cat}(P^{n_1} \times P^{n_2})\}\\
&= \max\{\mathrm{imd}(P^{n_1}) + 1, \mathrm{imd}(P^{n_2}) + 1, n_1 + n_2\}\\
&= \{\mathrm{imd}(P^{n_1}) + 1, n_1 + n_2\}.
\end{aligned}
\tag{25}
$$

We recall that $\mathrm{cat}(X \vee Y) = \max\{\mathrm{cat}(X), \mathrm{cat}(Y)\}$, so by induction, we have

$$
\begin{aligned}
&\mathrm{TC}\left(P^{n_1} \vee P^{n_2} \vee \cdots \vee P^{n_k}\right)\\
&= \max\{\mathrm{TC}(P^{n_1}), \mathrm{TC}(P^{n_2} \vee \cdots \vee P^{n_k}), \mathrm{cat}(P^{n_1} \times (P^{n_2} \vee \cdots \vee P^{n_k}))\}\\
&= \max\{\mathrm{imd}(P^{n_1}) + 1,\ \max\{\mathrm{imd}(P^{n_2}) + 1, n_2 + n_3\}, n_1 + n_2\}\\
&= \max\{\mathrm{imd}(P^{n_1}) + 1, \mathrm{imd}(P^{n_2}) + 1, n_2 + n_3, n_1 + n_2\}\\
&= \max\{\mathrm{imd}(P^{n_1}) + 1, n_1 + n_2\}\\
&= \max\{\mathrm{TC}(P^{n_1}), n_1 + n_2\}.
\end{aligned}
\tag{26}
$$

The next corollary follows from Theorem 21. □

Corollary 22. *If $n_1 = 2^m \ge n_2 \ge n_3 \ge \ldots \ge n_k$, then*

$$\mathrm{TC}\left(P^{2^m} \vee P^{n_2} \vee, \ldots, \vee P^{n_k}\right) = 2^{m+1}. \quad (27)$$

4. More Examples on Topological Complexity

First, we calculate topological complexity of P^{2^k+1} by immersion dimension of P^{2^k+1}.

Theorem 23 (see [9]). *If $n \equiv 1 \mod 4$ and $n > 8$, then n imm $2n - 3$.*

Using Remark 14 and Lemma 15 gives us the following lemma.

Lemma 24. *$zcl(P^{2^k+1}) = 2^{k+1} - 1$, for any $k \geq 3$.*

Lemma 25. *For $k \geq 3$, if $imd(P^{2^k+1}) = 2^{k+1} - 1$, then $TC(P^{2^k+1}) = 2^{k+1}$.*

Proof. Using Lemma 24, Remark 14, and Theorem 23, we have

$$\mathrm{zcl}\left(P^{2^k+1}\right) = 2^{k+1} - 1 \leq \mathrm{imd}\left(p^{2^k+1}\right) \leq 2^{k+1} - 1. \quad (28)$$

This implies that $\mathrm{imd}(P^{2^k+1}) = 2^{k+1} - 1$. Therefore, $\mathrm{TC}(P^{2^k+1}) = 2^{k+1}$. □

Example 1. If $n = 2^5 + 1 = 33$, $33 \equiv 1$ mod, then $\mathrm{imd}(P^{33}) = 63$ and $\mathrm{TC}(P^{33}) = 64$.

From Example 1, we may fill nonimmersion and immersion parts for $m = 2^k + 1$ in the Don Davis table of immersion and embedding of real projective spaces.

By calculating the zero cup-length of product and Theorem 12, we have the next proposition.

Proposition 26. *Let $A = \{6, 7\} \cup \{2^k \mid k = 0, 1, 2, 3, \cdots\} \cup \{2^k + 1 \mid k = 1, 2, 3, \cdots\}$. If $n_i \in A - \{6, 7\}$, in which $n_i = 2^{k_i}$ or $n_i = 2^{k_i} + 1$, then*

$$\begin{aligned}
\mathrm{TC}\left(P^6 \times P^7 \times P^{n_1} \times P^{n_2} \times \cdots \times P^{n_m}\right) &= 14 + \left(2^{k_1+1} - 1\right) + \left(2^{k_2+1} - 1\right) + \cdots + \left(2^{k_m+1} - 1\right) + 1, \\
\mathrm{TC}\left(P^6 \times P^{n_1} \times P^{n_2} \times \cdots \times P^{n_m}\right) &= 7 + \left(2^{k_1+1} - 1\right) + \left(2^{k_2+1} - 1\right) + \cdots + \left(2^{k_m+1} - 1\right) + 1, \\
\mathrm{TC}\left(P^7 \times P^{n_1} \times P^{n_2} \times \cdots \times P^{n_m}\right) &= 7 + \left(2^{k_1+1} - 1\right) + \left(2^{k_2+1} - 1\right) + \cdots + \left(2^{k_m+1} - 1\right) + 1, \\
\mathrm{TC}\left(P^{n_1} \times P^{n_2} \times \cdots \times P^{n_m}\right) &= \left(2^{k_1+1} - 1\right) + \left(2^{k_2+1} - 1\right) + \cdots + \left(2^{k_m+1} - 1\right) + 1.
\end{aligned} \quad (29)$$

Proof. The proof follows by calculating the zero cup-length of product and Theorem 12. □

Example 2. $\mathrm{TC}(P^1 \times P^3 \times P^5) = 12$ and $\mathrm{TC}(P^2 \times P^3 \times P^4 \times P^5 \times P^6) = 28$.

Next example shows that Proposition 26 is not true for any arbitrary product of real projective spaces. Consider the following examples.

Example 3. Note that $\mathrm{TC}(P^{10}) = \mathrm{imd}(P^{10}) + 1 = 16 + 1 = 17$ and $\mathrm{zcl}(P^{10} \times P^{10}) = 30$, and thus

$$30 = \mathrm{zcl}\left(P^{10} \times P^{10}\right) < \mathrm{TC}\left(P^{10} \times P^{10}\right) \leq \mathrm{TC}\left(P^{10}\right) + \mathrm{TC}\left(P^{10}\right) - 1 = 33. \quad (30)$$

Note that $\mathrm{TC}(P^{15}) = \mathrm{imd}(P^{15}) + 1 = 22 + 1 = 23$ and $\mathrm{zcl}(P^{15} \times P^{15}) = 30$, and thus

$$30 = \mathrm{zcl}\left(P^{15} \times P^{15}\right) < \mathrm{TC}\left(P^{15} \times P^{15}\right) \leq \mathrm{TC}\left(P^{15}\right) + \mathrm{TC}\left(P^{15}\right) - 1 = 45. \quad (31)$$

As we see there is a gap between lower and upper bounds.

For $2^k + 1 < n < 2^{k+1}$ in Example 3, to calculate topological complexity of $P^n \times P^n$ from zero cup-length that we have used in our calculation, we will find a gap between lower and upper bounds of topological complexity. Therefore, we cannot use this technique to find topological complexity of arbitrary products.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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