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Research Article

Mann Hybrid Deepest-Descent Extragradient Method with Line-Search Process for Hierarchical Variational Inequalities for Countable Nonexpansive Mappings

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In real Hilbert spaces, let the CFPP indicate a common fixed-point problem of asymptotically nonexpansive operator and countably many nonexpansive operators, and suppose that the HVI and VIP represent a hierarchical variational inequality and a variational inequality problem, respectively. We put forward Mann hybrid deepest-descent extragradient approach for solving the HVI with the CFPP and VIP constraints. The proposed algorithms are on the basis of Mann's iterative technique, viscosity approximation method, subgradient extragradient rule with linear-search process, and hybrid deepest-descent rule. Under suitable restrictions, it is shown that the sequences constructed by the algorithms converge strongly to a solution of the HVI with the CFPP and VIP constraints.

1. Introduction

Suppose that P_C is the nearest point projection from H onto C, where H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, and C is a convex closed set with $\varnothing \neq C \subset H$. Let $\mathrm{Fix}(T)$ be the fixed-point set of an operator $T: C \longrightarrow H$ and $\mathbf{R}$ be the real-number set. We use the notations $\rightharpoonup$ and $\longrightarrow$ to denote the weak convergence and strong one in H, respectively. A self-mapping T on C is said to be of asymptotical nonexpansivity iff $\exists\{\theta_k\} \subset [0,+\infty]$ s.t. $\lim_{k\longrightarrow\infty}\theta_k = 0$ and

$$(1+\theta_k)\|u-v\| \geq \left\|T^k u - T^k v\right\|, \forall u, v \in C, k \geq 1. \quad (1)$$

In case $\theta_k = 0$ for each k, T is said to be of non-expansivity. Given an operator $A: H \longrightarrow H$. We consider problem of seeking $x^* \in C$ such that $\langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C$, which is called the classical variational inequality one (VIP). We denote by $\mathrm{VI}(C, A)$ the solution set of the VIP. In particular, if the VIP is defined over C which is the solution set of another problem, then the VIP is called the hierarchical variational inequality (for short, HVI) over the solution set C. It is well known that the extragradient approach is one of the most effective methods for settling the VIP, which is proposed in Korpelevich [1], that is, for any starting $p_0 \in C$, $\{p_k\}$ is fabricated below:

$$\begin{cases} q_k = P_C(p_k - \mu A p_k), \\ p_{k+1} = P_C(p_k - \mu A q_k), \forall k \geq 0, \end{cases} \quad (2)$$

where $\mu \in (0, 1/L)$ and L is the Lipschitzian coefficient of A. In case VI (C, A) is nonempty, $\{p_k\}$ converges weakly to an element in $\mathrm{VI}(C, A)$. At present, the vast literature on Korpelevich's extragradient technique reveals that numerous scholars have given wide attention to it and improved it in different manners (refer to [1–28]).

In 2018, Thong and Van Hieu [20] first invented the inertial-type subgradient extragradient rule, i.e., for any starting $p_0, p_1 \in H$, $\{p_k\}$ is fabricated below:

$$\begin{cases} w_k = p_k + \alpha_k (p_k - p_{k-1}), \\ v_k = P_C (w_k - \mu A w_k), \\ C_k = \{v \in H\colon \langle w_k - \mu A w_k - v_k, v_k - v\rangle \ge 0\}, \\ p_{k+1} = P_{C_k}(w_k - \mu A v_k), \forall k \ge 1, \end{cases} \tag{3}$$

where $\mu \in (0, 1/L)$ and L is the Lipschitzian coefficient of A. Via mild assumptions, they showed that $\{p_k\}$ converges weakly to a point in VI(C, A). Besides, due to the importance and applicability of inertial technique, some new inertial iterative algorithms were recently introduced and analyzed (see [29–32] for more details). Recently, the hybrid inertial-type subgradient extragradient rule with linear-search process in [23] was proposed for settling the VIP with the operator A satisfying both pseudomonotonicity and Lipschitz continuity and the common fixed-point problem (CFPP) of finite nonexpansive operators $\{T_l\}_{l=1}^N$ and asymptotically nonexpansive operator T in H. Let $f\colon H \longrightarrow H$ be a contractive map with coefficient $\delta \in [0, 1]$, and $F\colon H \longrightarrow H$ be an operator satisfying both η-strong monotonicity and κ-Lipschitz continuity, such that $\delta < \tau\colon = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $0 < \rho < 2\eta/\kappa^2$. Suppose that $\{\alpha_k\} \subset [0, 1]$ and $\{\beta_k\}, \{\gamma_k\} \subset (0, 1)$ s.t. $\beta_k + \gamma_k < 1, \forall k$. Besides, one writes $T_k\colon = T_{k \bmod \overline{N}}$ for each $k \ge 1$ with the mod function taking values in $\{1, 2, \ldots, \overline{N}\}$, i.e., in case $k = j\overline{N} + q$ for some $j \ge 0$ and $0 \le q < \overline{N}$, one has that $T_k = T_{\overline{N}}$ for $q = 0$ and $T_k = T_q$ for $0 < q < \overline{N}$.

Under suitable assumptions, they proved the strong convergence of the sequence $\{x_k\}$ to a point in $\Omega = \mathrm{VI}(C, A) \cap (\cap_{l=0}^N \mathrm{Fix}(T_l))$ where $T_0\colon = T$. On the other hand, Reich et al. [25] put forth the modified projection-type rule for handling the VIP with the operator A satisfying both pseudomonotonicity and uniform continuity. Let $\{\alpha_k\} \subset (0, 1)$ and suppose that $f\colon C \longrightarrow C$ is a contractive map with coefficient $\delta \in [0, 1]$.

Under mild assumptions, they proved strong convergence of the sequence $\{x_k\}$ to an element of VI(C, A).

In real Hilbert spaces, let the CFPP stand for a common fixed-point problem of asymptotically nonexpansive operator and countably many nonexpansive operators. Let the HVI indicate a hierarchical variational inequality. We put forward Mann hybrid deepest-descent extragradient approach for solving the HVI with the CFPP and VIP constraints. The proposed algorithms are on the basis of Mann's iterative technique, viscosity approximation method, subgradient extragradient rule with linear-search process, and hybrid deepest-descent rule. Under suitable restrictions, it is shown that the sequences constructed by the algorithms converge strongly to a solution of the HVI with the CFPP and VIP constraints.

The structure of the article is arranged as follows. Basic notions and tools are given in Section 2. The convergence analysis of the suggested algorithms is discussed in Section 3. Section 4 provides an illustrated instance to demonstrate the applicability and implementability of our suggested algorithms. It is worth pointing out that the theorems in this article enhance and develop those associated results with [21, 23, 25] because our algorithms involve solving the VIP with the operator satisfying both pseudomonotonicity and uniform continuity and the CFPP of asymptotically nonexpansive operator and countably many nonexpansive operators.

2. Basic Concepts and Tools

Assume $\emptyset \ne C \subset H$, where C is convex and closed in a real Hilbert space H. For given sequence $\{u_n\} \subset H$, the notations $u_n \rightharpoonup u$ and $u_n \longrightarrow u$ indicate the weak convergence and strong convergence of $\{u_n\}$ to u, respectively. For each $y, z \in C$, a mapping $T\colon C \longrightarrow H$ is said to be

(1) κ-Lipschitzian (or of κ-Lipschitz continuity) iff $\exists \kappa > 0$ s.t. $\|Ty - Tz\| \le \kappa\|y - z\|$

(2) Of monotonicity iff $\langle Ty - Tz, y - z\rangle \ge 0$

(3) Of pseudomonotonicity iff $\langle Ty, z - y\rangle \ge 0 \Rightarrow \langle Tz, z - y\rangle \ge 0$

(4) Of η-strong monotonicity iff $\exists \eta > 0$ s.t. $\langle Ty - Tz, y - z\rangle \ge \alpha\|y - z\|^2$

(5) Of sequentially weak continuity iff for each $\{y_n\}$ in C, one has that $y_n \rightharpoonup y \Rightarrow Ty_n \rightharpoonup Ty$

Note that the class of pseudomonotone operators properly includes the class of monotone operators. Given any y in H, we know that $\exists |$ (nearest point) $z \in C$, written as $z = P_C y$, s.t. $\|y - z\| \le \|y - x\|$ for each x in C. P_C is called a nearest point (or metric) projection from H onto C. According to [33], for each $y, z \in H$, the statements below are valid:

(1) $\langle y - z, P_C y - P_C z\rangle \ge \|P_C y - P_C z\|^2$

(2) $\langle y - P_C y, x - P_C y\rangle \le 0, \forall x \in C$

(3) $\|y - x\|^2 \ge \|y - P_C y\|^2 + \|x - P_C y\|^2, \forall x \in C$

(4) $\|y - z\|^2 = \|y\|^2 - \|z\|^2 - 2\langle y - z, z\rangle$

(5) $\|\lambda y + (1 - \lambda)z\|^2 = \lambda\|y\|^2 + (1 - \lambda)\|z\|^2 - \lambda(1 - \lambda)\|y - z\|^2, \forall \lambda \in [0, 1]$

Definition 1 (see [34]). Let $\{\xi_n\}_{n=1}^\infty \subset [0, 1]$ and suppose that $\{T_n\}_{n=1}^\infty$ is a sequence of nonexpansive operators of C into itself. For each n, the operator $W_n\colon C \longrightarrow C$ is constructed below:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \xi_n T_n U_{n,n+1} + (1 - \xi_n)I, \\ U_{n,n-1} = \xi_{n-1} T_{n-1} U_{n,n} + (1 - \xi_{n-1})I, \\ \cdots \\ U_{n,i} = \xi_i T_i U_{n,i+1} + (1 - \xi_i)I, \\ \cdots \\ U_{n,2} = \xi_2 T_2 U_{n,3} + (1 - \xi_2)I, \\ W_n = U_{n,1} = \xi_1 T_1 U_{n,2} + (1 - \xi_1)I. \end{cases} \tag{4}$$

Such an operator W_n is nonexpansive and is known as the W-mapping constructed by $T_1, \ldots, T_n$ and $\xi_1, \ldots, \xi_{n-1}$.

Initialization: Given any starting $x_1, x_0 \in H$. Let $\mu \in (0,1), \ell \in (0,1), \gamma > 0$.
Iterations: Compute x_{k+1} below:
Step 1. Put $w_k = T_k x_k + \alpha_k (T_k x_k - T_k x_{k-1})$ and calculate $v_k = P_C(w_k - \tau_k A w_k)$, with τ_k being picked to be the largest $\tau \in \{\gamma, \gamma\ell, \gamma\ell^2, \ldots\}$ s.t. $\tau\|Aw_k - Av_k\| \le \mu\|w_k - v_k\|$.
Step 2. Calculate $z_k = P_{C_k}(w_k - \tau_k A v_k)$ with $C_k := \{v \in H: \langle w_k - \tau_k A w_k - v_k, v_k - v\rangle \ge 0\}$.
Step 3. Calculate $x_{k+1} = \beta_k f(x_k) + \gamma_k x_k + ((1-\gamma_k)I - \beta_k \rho F)T^k z_k$.
Again set $k := k+1$ and go to Step 1.

Algorithm 1: Hybrid inertial subgradient extragradient rule (see [23]).

Initialization: Given any starting $x_1 \in C$. Let $\lambda \in (0, 1/\mu), \ell \in (0,1), \mu > 0$.
Iterations: Compute x_{k+1} below:
Step 1. Calculate $v_k = P_C(x_k - \lambda A x_k)$ and $r_\lambda(x_k) := x_k - v_k$. In case $r_\lambda(x_k) = 0$, one stops; x_k lies in VI(C, A). In case $r_\lambda(x_k) \ne 0$, one goes to Step 2.
Step 2. Calculate $w_k = x_k - \tau_k r_\lambda(x_k)$, with $\tau_k := \ell^{j_k}$ and j_k is the smallest nonnegative integer j satisfying $\langle Ax_k - A(x_k - \ell^j r_\lambda(x_k)), r_\lambda(x_k)\rangle \le \mu/2\|r_\lambda(x_k)\|^2$.
Step 3. Calculate $x_{k+1} = \alpha_k f(x_k) + (1-\alpha_k)P_{C_k}(x_k)$, with $C_k := \{v \in C: \hbar_k(v) \le 0\}$ and $\hbar_k(v) = \langle Aw_k, v - x_k\rangle + \tau_k/2\lambda\|r_\lambda(x_k)\|^2$.
Again set $k := k+1$ and go to Step 1.

Algorithm 2: Modifed projection-type rule (see [25]).

Proposition 1 (see [34]). *Let* $[\{\xi_n\}_{n=1}^\infty \subset 0, 1]$ *and suppose that* $\{T_n\}_{n=1}^\infty$ *is a sequence of nonexpansive operators of C into itself such that* $\cap_{n=1}^\infty Fix(T_n) \ne \emptyset$. *Then,*

(a) W_n *is of nonexpansivity and* $Fix(W_n) = \cap_{n=1}^\infty Fix(T_i), \forall n$
(b) $\lim_{n\longrightarrow\infty} U_{n,i} u$ *exists for all* $u \in C$ *and* $i \ge 1$
(c) The mapping W defined by $Wu := \lim_{n\longrightarrow\infty} W_n u = \lim_{n\longrightarrow\infty} U_{n,1} u, \forall u \in C$ *is nonexpansive operator such that* $Fix(W) = \cap_{n=1}^\infty Fix(T_n)$, *and W is known as the W-operator constructed by* $T_1, T_2, \ldots$ *and* $\xi_1, \xi_2, \ldots$

Proposition 2 (see [35]). *Let* $\{\xi_n\}_{n=1}^\infty \subset (0, \varsigma)$ *for certain* $\varsigma \in (0,1)$ *and suppose* $\{T_n\}_{n=1}^\infty$ *is a sequence of nonexpansive operators of C into itself such that* $\cap_{n=1}^\infty Fix(T_n) \ne \emptyset$. *Then,* $\lim_{n\longrightarrow\infty} \sup_{u\in D}\|W_n u - Wu\| = 0, \forall$ *(bounded)* $D \subset C$.

In what follows, one always assumes that $\{\xi_n\}_{n=1}^\infty \subset (0, \varsigma)$ *for certain* $\varsigma \in (0,1)$. *Using the subdifferential inequality of* $\|\cdot\|^2/2$, *we have the relation below:*

$$\|y+z\|^2 \le \|y\|^2 + 2\langle z, y+z\rangle, \forall y, z \in H. \quad (5)$$

Later, we will exploit the lemmas below to derive our main theorems.

Lemma 1 (see [26]). *Let* H_1 *and* H_2 *be two real Hilbert spaces. Suppose that* $A: H_1 \longrightarrow H_2$ *is uniformly continuous on bounded subsets of* H_1 *and M is a bounded subset of* H_1. *Then,* $A(M)$ *is bounded.*

Lemma 2 (see [36]). *Let h be a real-valued function on H and define* $K := \{x \in C: h(x) \le 0\}$. *If K is nonempty and h is Lipschitz continuous on C with modulus* $\theta > 0$, *then* $dist(x, K) \ge \theta^{-1}\max\{h(x), 0\}, \forall x \in C$, *where* $dist(x, K)$ *denotes the distance of x to K.*

Lemma 3. *Suppose that* $A: C \longrightarrow H$ *is of both pseudomonotonicity and continuity. Given a point* $z^* \in C$. *Then,* $\langle Az^*, y - z^*\rangle \ge 0 \forall y \in C \Leftrightarrow \langle Ay, y - z^*\rangle \ge 0 \forall y \in C$.

Proof. It is easy to check that the conclusion is valid. □

Lemma 4 (see [8]). *Suppose that* $\{\overline{a}_n\} \subset [0, \infty]$ *such that* $\overline{a}_{n+1} \le (1-\zeta_n)\overline{a}_n + \zeta_n \overline{b}_n, \forall n \ge 1$, *with* $\{\zeta_n\}$ *and* $\{\overline{b}_n\}$ *both being real sequences satisfying the conditions: (i)* $\{\zeta_n\} \subset [0,1]$ *and* $\sum_{n=1}^\infty \zeta_n = \infty$, *and (ii)* $\limsup_{n\longrightarrow\infty} \overline{b}_n \le 0$ *or* $\sum_{n=1}^\infty |\zeta_n \overline{b}_n| < \infty$. *Then,* $\lim_{n\longrightarrow\infty} \overline{a}_n = 0$.

Lemma 5 (see [37]). *Suppose that* $\emptyset \ne C \subset X$ *where C is convex and closed in a Banach space X admitting a weakly continuous duality mapping. Let the operator* $T: C \longrightarrow C$ *be of asymptotical nonexpansivity such that* $Fix(T) \ne \emptyset$. *Then,* $I - T$ *is of demiclosedness at zero, that is, for each* $\{u_n\} \subset C$ *with* $u_n \rightharpoonup u \in C$, *the relation holds:* $(I-T)u_n \longrightarrow 0 \Rightarrow (I-T)u = 0$, *with I being the identity mapping of X.*

Lemma 6 (see [38]). *Suppose that* $\{\Gamma_m\}$ *is a real sequence which does not decrease at infinity in the sense that* $\exists\{\Gamma_{m_j}\} \subset \{\Gamma_m\}$ *s.t.* $\Gamma_{m_j} < \Gamma_{m_j+1}, \forall j \ge 1$. *Let* $\{\eta(m)\}_{m \ge m_0}$ *be formulated by* $\eta(m) = \max\{j \le m: \Gamma_j < \Gamma_{j+1}\}$, *with* $m_0 \ge 1$ *s.t.* $\{m \le m_0: \Gamma_m < \Gamma_{m+1}\} \ne \emptyset$. *Then, the statements below are valid:*

(i) $\eta(m_0) \le \eta(m_0+1) \le \cdots$ *and* $\eta(m) \longrightarrow \infty$
(ii) $\Gamma_{\eta(m)} \le \Gamma_{\eta(m)+1}$ *and* $\Gamma_m \le \Gamma_{\eta(m)+1}, \forall m \ge m_0$

Lemma 7 (see [7, Lemma 8]). *Suppose that* λ *lies in* $(0,1]$, T *is a nonexpansive self-mapping on C, and* $T^\lambda: C \longrightarrow H$ *is the mapping formulated by* $T^\lambda x := (I - \lambda\rho F)Tx, \forall x \in C$, *with* $F: C \longrightarrow H$ *being of both* κ*-Lipschitz continuity and* η*-strong* *monotonicity. Then,* T^{λ} *is a contractive map for* $\rho \in (0, 2\eta/\kappa^2)$, *i.e.,* $\|T^{\lambda}y - T^{\lambda}z\| \le (1-\lambda\tau)\|y-z\|, \forall y,z \in C$, *with* $\tau = 1 - \sqrt{1-\rho(2\eta - \rho\kappa^2)} \in (0,1)$.

3. Algorithms and Convergence Analysis

Let $\varnothing \ne C \subset H$, with the feasible set C being convex and closed in a real Hilbert space H.

Condition 1. The following conditions are valid.

(C1) $\{T_n\}_{n=1}^{\infty}$ is a sequence of nonexpansive operators of C into itself and $T\colon C \longrightarrow C$ is asymptotical nonexpansivity operator with $\{\theta_n\}$.

(C2) W_n is the W-mapping constructed by $T_1, \ldots, T_n$ and $\xi_1, \ldots, \xi_n$, with $\{\xi_n\}_{n=1}^{\infty} \subset [0, \varsigma]$ for certain $\varsigma \in (0,1)$.

(C3) $A\colon H \longrightarrow H$ is of both pseudomonotonicity and uniform continuity on C, s.t. $\|Az\| \le \liminf_{n\longrightarrow\infty}\|Au_n\|$ for each $\{u_n\} \subset C$ with $u_n \rightharpoonup z$.

(C4) $f\colon C \longrightarrow H$ is a contractive map with coefficient $\delta \in [0,1]$, and $F\colon C \longrightarrow H$ is of both η-strong monotonicity and κ-Lipschitz continuity s.t. $\tau\colon = 1 - \sqrt{1-\rho(2\eta - \rho\kappa^2)} > \delta$ with $0 < \rho < 2\eta/\kappa^2$.

(C5) $\Omega = \mathrm{VI}(C,A) \cap (\cap_{n=1}^{\infty}\mathrm{Fix}(T_n)) \ne \varnothing$ where $T_0\colon = T$.

(C6) $\{\gamma_n\}, \{\beta_n\} \subset (0,1)$ and $\{\sigma_n\} \subset [0,1]$ s.t.

(i) $1 > \gamma_n + \beta_n$ and $\sum_{n=1}^{\infty}\beta_n = \infty$.

(ii) $\beta_n \longrightarrow 0$ and $\theta_n/\beta_n \longrightarrow 0$ as $n \longrightarrow \infty$.

(iii) $1 > \limsup_{n\longrightarrow\infty}\sigma_n \ge \liminf_{n\longrightarrow\infty}\sigma_n > 0$.

(iv) $1 > \limsup_{n\longrightarrow\infty}\gamma_n \ge \liminf_{n\longrightarrow\infty}\gamma_n > 0$.

Lemma 8. *The Armijo-type search process (Algorithm 1) is formulated well, and the relation is valid:* $\|r_{\lambda}(w_n)\|^2 \le \lambda\langle Aw_n, r_{\lambda}(w_n)\rangle$.

Proof. Using the uniformly continuity of A on C, from $l \in (0,1)$, one has $\lim_{j\longrightarrow\infty}\langle Aw_n - A(w_n - l^j r_{\lambda}(w_n)), r_{\lambda}(w_n)\rangle = 0$. In case $r_{\lambda}(w_n) = 0$, it is clear that $j_n = 0$. In case $r_{\lambda}(w_n) \ne 0$, there exists $j_n \ge 0$ meeting (Algorithm 1). Since P_C is firmly nonexpansive, we obtain that $\langle u - P_C v, u - v\rangle \ge \|u - P_C v\|^2, \forall u \in C, v \in H$. Setting $u = w_n$ and $v = w_n - \lambda Aw_n$, one gets $\lambda\langle Aw_n, w_n - P_C(w_n - \lambda Aw_n)\rangle \ge \|w_n - P_C(w_n - \lambda Aw_n)\|^2$, which attains the desired result. □

Lemma 9. *Let* $p \in \Omega$ *and* h_n *be formulated as in (Algorithm 1). Then,* $h_n(p) \le 0$ *and* $h_n(w_n) = \tau_n/2\lambda\|r_{\lambda}(w_n)\|^2$. *Particularly, in case* $r_{\lambda}(w_n) \ne 0$, *one has* $h_n(w_n) > 0$.

Proof. It is clear that $h_n(w_n) = \tau_n/2\lambda\|r_{\lambda}(w_n)\|^2$. In what follows, we claim $h_n(p) \le 0$. Indeed, in terms of Lemma 3, one gets $\langle At_n, t_n - p\rangle \ge 0$. Hence, one has $h_n(p) = \langle At_n, t_n - w_n\rangle + \langle At_n, p - t_n\rangle + \tau_n/2\lambda \quad \|r_{\lambda}(w_n)\|^2 \le -\tau_n\langle At_n, r_{\lambda}(w_n)\rangle + \tau_n/2\lambda\|r_{\lambda}(w_n)\|^2$. Using (Algorithm 1) and Lemma 8, one gets $\langle At_n, r_{\lambda}(w_n)\rangle \ge \langle Aw_n, r_{\lambda}(w_n)\rangle - \mu/2\|r_{\lambda}(w_n)\|^2 \ge (1/\lambda - \mu/2)\|r_{\lambda}(w_n)\|^2$, which hence arrives at $h_n(p) \le -\tau_n/2(1/\lambda - \mu)\|r_{\lambda}(w_n)\|^2$. Therefore, the claim is valid. □

Lemma 10. *Suppose that the sequences* $\{z_n\}, \{y_n\}, \{x_n\}, \{w_n\}$ *fabricated in Algorithm 3, are of boundedness. Assume that* $x_{n+1} - x_n \longrightarrow 0, x_n - w_n \longrightarrow 0, y_n - w_n \longrightarrow 0, z_n - w_n \longrightarrow 0$ *and* $T^{n+1}x_n - T^n x_n \longrightarrow 0$. *Then,* $\omega_w(\{x_n\}) \subset \Omega$, *with* $\omega_w(\{x_n\}) = \{z \in H\colon x_{n_k} \rightharpoonup z \text{ for certain } \{x_{n_k}\} \subset \{x_n\}\}$.

Proof. Take a fixed $z \in \omega_w(\{x_n\})$ arbitrarily. Then, $\exists\{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \rightharpoonup z \in H$. Owing to $x_n - w_n \longrightarrow 0$, one knows that $\exists\{w_{n_k}\} \subset \{w_n\}$ s.t. $w_{n_k} \rightharpoonup z \in H$. In what follows, we claim $z \in \Omega$. In fact, observe that $x_n - w_n = \sigma_n(x_n - W_n x_n), \forall n$. Thus, $\|x_n - w_n\| = \sigma_n \|x_n - W_n x_n\|$. Using the assumptions $\liminf_{n\longrightarrow\infty}\sigma_n > 0$ and $x_n - w_n \longrightarrow 0$, we have

$$\lim_{n\longrightarrow\infty}\|x_n - W_n x_n\| = 0. \tag{6}$$

Putting $v_n\colon = \beta_n f(x_n) + \gamma_n x_n + ((I - \gamma_n)I - \beta_n\rho F)T^n z_n$, by Algorithm 3 we obtain that $x_{n+1} = P_C v_n$ and $v_n - T^n z_n = \beta_n f(x_n) + \gamma_n(x_n - T^n z_n) - \beta_n\rho FT^n z_n$, which immediately yields

$$\begin{aligned}\|x_n - T^n z_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n z_n\| \le \|x_n - x_{n+1}\| + \|v_n - T^n z_n\| \\ &- \le \|x_n - x_{n+1}\| + \beta_n\|f(x_n)\| + \gamma_n\|x_n - T^n z_n\| + \beta_n\|\rho FT^n z_n\|.\end{aligned} \tag{7}$$

Hence, one gets $\|x_{n+1} - x_n\| + \beta_n(\|f(x_n)\| + \|\rho FT^n z_n\|) \ge (1-\gamma_n)\|T^n z_n - x_n\|$. Since $x_{n+1} - x_n \longrightarrow 0, \beta_n \longrightarrow 0$, $\liminf_{n\longrightarrow\infty}(1-\gamma_n) > 0$ and $\{x_n\}, \{z_n\}$ are of boundedness, one gets $\lim_{n\longrightarrow\infty}\|T^n z_n - x_n\| = 0$. We claim that $\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$. Indeed, since T is of asymptotical nonexpansivity, we obtain

Initial Step: Given any starting $x_1 \in C$. Let $\mu > 0, \lambda \in (0, 1/\mu), l \in (0, 1)$.
Iterations: Given the current iterate x_n, calculate x_{n+1} below:
Step 1. Compute $w_n = (1-\sigma_n)x_n + \sigma_n W_n x_n$, $y_n = P_C(w_n - \lambda A w_n)$ and $r_\lambda(w_n) := w_n - y_n$.
Step 2. Compute $t_n = w_n - \tau_n r_\lambda(w_n)$, where $\tau_n := l^{j_n}$ and the integer j_n is the smallest nonnegative one j s.t $\mu/2\|r_\lambda(w_n)\|^2 \geq \langle A(w_n - l^j r_\lambda(w_n)) - Aw_n, y_n - w_n\rangle$
Step 3. Calculate
$x_{n+1} = P_C[\beta_n f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \beta_n \rho F)T^n z_n]$
with $z_n = P_{C_n}(w_n)$, $C_n := \{u \in C: h_n(u) \leq 0\}$ and $h_n(u) = \langle At_n, u - w_n\rangle + \tau_n/2\lambda\|r_\lambda(w_n)\|^2$.
Set $n := n+1$ and go to Step 1.

Algorithm 3: The 1st Mann hybrid deepest-descent extragradient rule.

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - T^n z_n\| + \|T^n z_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + \|T^{n+1} x_n - T^{n+1} z_n\| \\ &\quad + \|T^{n+1} z_n - Tx_n\| \\ &\leq (2+\theta_1)\|x_n - T^n z_n\| + (2+\theta_n+\theta_{n+1})\|z_n - x_n\| + \|T^n x_n - T^{n+1} x_n\| \\ &\leq (2+\theta_1)\|x_n - T^n z_n\| + (2+\theta_n+\theta_{n+1})\left(\|z_n - w_n\| + \|w_n - x_n\|\right) + \|T^n x_n - T^{n+1} x_n\|.\end{aligned} \tag{8}$$

Noticing $x_n - w_n \longrightarrow 0$, $z_n - w_n \longrightarrow 0$, and $x_n - T^n z_n \longrightarrow 0$, we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0. \tag{9}$$

Also, we claim $\lim_{n\longrightarrow\infty}\|Wx_n - x_n\| = 0$. In fact, it is clear that $\|x_n - w_n\| = \sigma_n\|x_n - W_n x_n\|$. Since $\liminf_{n\longrightarrow\infty}\sigma_n > 0$ and $x_n - w_n \longrightarrow 0$, one has $\lim_{n\longrightarrow\infty}\|W_n x_n - x_n\| = 0$. Note that $\|Wx_n - x_n\| \leq \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\| \leq \sup_{u\in D}\|Wu - W_n u\| + \|W_n x_n - x_n\|$, where $D = \{x_n: n \geq 1\}$. Using Proposition 2, we obtain

$$\lim_{n\longrightarrow\infty}\|Wx_n - x_n\| = 0. \tag{10}$$

Now, we claim $z \in \mathrm{VI}(C, A)$. Indeed, because C is of both convexity and closedness, using $\{w_n\} \subset C$ and $w_{n_k} \rightharpoonup z$, one has that z lies in C. In case $Az = 0$, it is easily known that $z \in \mathrm{VI}(C, A)$ due to $\langle Az, x - z\rangle \geq 0, \forall x \in C$. In case $Az \neq 0$, combining $w_n - y_n \longrightarrow 0$ and $w_{n_k} \rightharpoonup z$ yields $y_{n_k} \rightharpoonup z$ as $k \longrightarrow \infty$. By the condition imposed on A, one gets $0 < \|Az\| \leq \liminf_{k\longrightarrow\infty}\|Ay_{n_k}\|$. So, we may presume $\|Ay_{n_k}\| \neq 0, \forall k$. In addition, using $y_n = P_C(w_n - \lambda Aw_n)$, one has $\langle w_n - \lambda Aw_n - y_n, x - y_n\rangle \leq 0, \forall x \in C$, and hence

$$\begin{aligned}\langle Aw_n, x - w_n\rangle &\geq \langle Aw_n, y_n - w_n\rangle \\ &\quad + \frac{1}{\lambda}\langle w_n - y_n, x - y_n\rangle, \forall x \in C.\end{aligned} \tag{11}$$

Since A is uniformly continuous on C, $\{Aw_n\}$ is of boundedness (owing to Lemma 1). Note that $\{y_n\}$ is bounded as well. Thus, using (11), one gets $\liminf_{k\longrightarrow\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \geq 0, \forall x \in C$. It is clear that $\langle Ay_n, x - y_n\rangle = \langle Ay_n - Aw_n, x - w_n\rangle + \langle Aw_n, x - w_n\rangle + \langle Ay_n, w_n - y_n\rangle$. Thus, using $y_n - w_n \longrightarrow 0$, one gets $Aw_n - Ay_n \longrightarrow 0$ and hence attains $\liminf_{k\longrightarrow\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0, \forall x \in C$.

In what follows, one picks $\{\varsigma_k\} \subset (0, 1)$ s.t. $\varsigma_j \downarrow 0$ $(j \longrightarrow \infty)$. For any $j \geq 1$, one writes by m_j the smallest natural number s.t.

$$\langle Ay_{n_k}, x - y_{n_k}\rangle + \varsigma_j \geq 0, \forall k \geq m_j. \tag{12}$$

Noticing the fact that $\{\varsigma_j\}$ is decreasing, we can readily see that $\{m_j\}$ is increasing. From $Ay_{m_j} \neq 0, \forall j$ (owing to $\{Ay_{m_j}\} \subset \{Ay_{n_j}\}$), one puts $\nu_{m_j} = Ay_{m_j}/\|Ay_{m_j}\|^2$, and one has $\langle Ay_{m_j}, \nu_{m_j}\rangle = 1, \forall j$. Thus, using (12), one gets $\langle Ay_{m_j}, x + \varsigma_j\nu_{m_j} - y_{m_j}\rangle \geq 0, \forall j$. Also, since A is pseudomonotone, one has $\langle A(x + \varsigma_j\nu_{m_j}), x + \varsigma_j\nu_{m_j} - y_{m_j}\rangle \geq 0, \forall j$, which hence yields

$$\begin{aligned}\langle Ax, x - y_{m_j}\rangle &\geq \langle Ax - A\left(x + \varsigma_j\nu_{m_j}\right), x + \varsigma_j\nu_{m_j} - y_{m_j}\rangle \\ &\quad - \varsigma_j\langle Ax, \nu_{m_j}\rangle.\end{aligned} \tag{13}$$

Let us show $\lim_{j\longrightarrow\infty}\varsigma_j\nu_{m_j} = 0$. In fact, using $w_{n_j} \rightharpoonup z \in C$ and $y_n - w_n \longrightarrow 0$, one obtains $y_{n_j} \rightharpoonup z$. Noticing $\{y_{m_j}\} \subset \{y_{n_j}\}$ and $\varsigma_j \downarrow 0$, one deduces that $0 \leq \limsup_{j\longrightarrow\infty}\|\varsigma_j\nu_{m_j}\| = \limsup_{j\longrightarrow\infty}\varsigma_j/\|Ay_{m_j}\| \leq \limsup_{j\longrightarrow\infty}\varsigma_j/\liminf_{j\longrightarrow\infty}\|Ay_{n_j}\| = 0$. Hence, we get $\varsigma_j\nu_{m_j} \longrightarrow 0$ as $j \longrightarrow \infty$. So, it follows that the right-hand side of (13) tends to zero by the uniform continuity of A and the boundedness of $\{y_{m_j}\}, \{\nu_{m_j}\}$, and $\lim_{j\longrightarrow\infty}\varsigma_j\nu_{m_j} = 0$. Therefore, $\langle Ax, x - z\rangle = \liminf_{j\longrightarrow\infty}\langle Ax, x - y_{m_j}\rangle \geq 0, \forall x \in C$. Using Lemma 3, one has $z \in \mathrm{VI}(C, A)$. Last, we claim $z \in \Omega$. In fact, using $x_n - w_n \longrightarrow 0$ and $w_{n_j} \rightharpoonup z$, one gets $x_{n_j} \rightharpoonup z$. Note that (9) guarantees $x_{n_j} - Tx_{n_j} \longrightarrow 0$. By Lemma 5, one knows that $I - T$ is of demiclosedness at zero. Thus, using $x_{n_j} \rightharpoonup z$, one gets $(I - T)z = 0$, that is, $z \in \mathrm{Fix}(T)$. Besides, we claim $z \in \mathrm{Fix}(W) = \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$. Actually, noticing $x_{n_j} \rightharpoonup z$ and $x_{n_j} - Wx_{n_j} \longrightarrow 0$ (due to (10)), from Proposition 1 and

Lemma 5, we obtain that $I - W$ is demiclosed at zero. This hence yields $z \in \mathrm{Fix}(W) = \cap_{n=1}^{\infty}\mathrm{Fix}(T_i)$. Consequently, $z \in \mathrm{VI}(C, A) \cap (\cap_{n=1}^{\infty}\mathrm{Fix}(T_i)) = \Omega$. □

Lemma 11. *Let $\{w_n\}$ be the sequence fabricated in Algorithm 3. Then,*

$$\tau_n \| r_\lambda (w_n) \|^2 \longrightarrow 0 \Rightarrow y_n - w_n \longrightarrow 0. \tag{14}$$

Proof. Assume $\limsup_{n\longrightarrow\infty}\|y_n - w_n\| = \alpha > 0$. Picking $\{n_k\} \subset \{n\}$, one has $\lim_{k\longrightarrow\infty}\|w_{n_k} - y_{n_k}\| = \alpha > 0$. Note that $\lim_{k\longrightarrow\infty}\tau_{n_k}\|r_\lambda(w_{n_k})\|^2 = 0$. Consider two cases. If $\liminf_{k\longrightarrow\infty}\tau_{n_k} > 0$, one may presume that $\exists d > 0$ s.t. $\tau_{n_k} \ge d > 0, \forall k$. So, one knows that $\|w_{n_k} - y_{n_k}\|^2 = 1/\tau_{n_k}\tau_{n_k} \|w_{n_k} - y_{n_k}\|^2 \le 1/d \cdot \tau_{n_k}\|w_{n_k} - y_{n_k}\|^2 = 1/d \cdot \tau_{n_k}\|r_\lambda(w_{n_k})\|^2$, which immediately leads to $0 < a^2 = \lim_{k\longrightarrow\infty} \|w_{n_k} - y_{n_k}\|^2 \le \lim_{k\longrightarrow\infty}\{1/d \cdot \tau_{n_k} \|r_\lambda(w_{n_k})\|^2\} = 0$. This reaches a contradiction.

If $\liminf_{k\longrightarrow\infty}\tau_{n_k} = 0$, there exists a subsequence of $\{\tau_{n_k}\}$, still denoted by $\{\tau_{n_k}\}$, s.t. $\lim_{k\longrightarrow\infty}\tau_{n_k} = 0$. We now put $q_{n_k} := 1/l\tau_{n_k} y_{n_k} + (1 - 1/l\tau_{n_k})w_{n_k} = w_{n_k} - 1/l\tau_{n_k}(w_{n_k} - y_{n_k})$. Then, from $\lim_{k\longrightarrow\infty}\tau_{n_k}\|r_\lambda(w_{n_k})\|^2 = 0$, we deduce that $\lim_{k\longrightarrow\infty} \|q_{n_k} - w_{n_k}\|^2 = \lim_{k\longrightarrow\infty} 1/l^2\tau_{n_k} \cdot \tau_{n_k}\|y_{n_k} - w_{n_k}\|^2 = 0$. Using (6), one obtains $\langle Aw_{n_k} - Aq_{n_k}, w_{n_k} - y_{n_k}\rangle > \mu/2\|w_{n_k} - y_{n_k}\|^2$. Since A is uniformly continuous on bounded subsets of C, this ensures that $\lim_{k\longrightarrow\infty}\|Aw_{n_k} - Aq_{n_k}\| = 0$, which hence yields $\lim_{k\longrightarrow\infty}\|y_{n_k} - w_{n_k}\| = 0$. This reaches a contradiction. Therefore, $y_n - w_n \longrightarrow 0$ as $n \longrightarrow \infty$. □

Theorem 1. *Let $\{x_n\}$ be the sequence fabricated in Algorithm 3. Assume $T^{n+1}x_n - T^n x_n \longrightarrow 0$. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$, which is only a solution to the VIP: $\langle (f - \rho F)x^*, x - x^*\rangle \le 0, \forall x \in \Omega$.*

Proof. Thanks to $1 > \limsup_{n\longrightarrow\infty}\sigma_n \ge \liminf_{n\longrightarrow\infty}\sigma_n > 0$ and $\theta_n/\beta_n \longrightarrow 0$, one may presume that $(0,1) \supset [a, b] \supset \{\sigma_n\}$ and $\beta_n(\tau - \delta)/2 \ge \theta_n, \forall n$. It is easy to check that $P_\Omega(I - \rho F + f)\colon C \longrightarrow C$ is a contractive map. Thus, $\exists | x^* \in C$ s.t. $x^* = P_\Omega(I - \rho F + f)x^*$. Hence, $\exists | x^* \in \Omega$ satisfying the VIP:

$$\langle (f - \rho F)x^*, x - x^*\rangle \le 0, \forall x \in \Omega. \tag{15}$$

Next in the rest of the proof, we divide it into a few steps. □

Step 1. One claims that $\{x_n\}$ is of boundedness. Indeed, choose any $p \in \Omega = \mathrm{VI}(C, A) \cap (\cap_{n=1}^{\infty}\mathrm{Fix}(T_i))$. Then, $Tp = p$ and $W_n p = p, \forall n$. Let us show the relation below:

$$-\mathrm{dist}^2(w_n, C_n) + \|w_n - p\|^2 \ge \|z_n - p\|^2, \forall p \in \Omega. \tag{16}$$

In fact, one has $\|z_n - p\|^2 = \|P_{C_n}w_n - p\|^2 \| \le w_n - p\|^2 - \mathrm{dist}^2(w_n, C_n)$, which hence yields

$$\|z_n - p\| \le \|w_n - p\|, \forall n \ge 1. \tag{17}$$

Using the formulation of w_n, one gets $\|w_n - p\| \le (1 - \sigma_n)\|x_n - p\| + \sigma_n\|W_n x_n - p\| \le \|x_n - p\|$, which together with (17) yields

$$\|z_n - p\| \le \|w_n - p\| \le \|x_n - p\|, \forall n \ge 1. \tag{18}$$

Noticing $1 > \gamma_n + \beta_n$, from (18) and Lemma 7, we obtain

$$\begin{aligned} \|x_{n+1} - p\| &\le \|\beta_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \beta_n \rho F)T^n z_n - p\| \\ &= \|\beta_n(f(x_n) - f(p)) + \gamma_n(x_n - p) + (1 - \gamma_n) \\ &\quad \times [(I - \beta_n/1 - \gamma_n\rho F)T^n z_n - (I - \beta_n/1 - \gamma_n\rho F)p] + \beta_n(f - \rho F)p\| \\ &\le \beta_n\delta\|x_n - p\| + \gamma_n\|x_n - p\| + (1 - \gamma_n) \\ &\quad \times (1 - \beta_n/1 - \gamma_n\tau)(1 + \theta_n)\|z_n - p\| + \beta_n\|(f - \rho F)p\| \\ &\le [\beta_n\delta + \gamma_n + (1 - \gamma_n - \beta_n\tau) + \theta_n]\|x_n - p\| + \beta_n\|(f - \rho F)p\| \\ &\le \max\{\|x_n - p\|, 2\|(f - \rho F)p\|/\tau - \delta\}. \end{aligned} \tag{19}$$

Using the induction, we get $\|x_n - p\| \le \max\{\|x_1 - p\|, 2\|(f - \rho F)p\|/\tau - \delta\}, \forall n \ge 1$. Hence, $\{x_n\}$ is of boundedness. Therefore, the sequences $\{w_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}, \{At_n\}, \{W_n x_n\}, \{T^n z_n\}$ are of boundedness as well.

Step 2. One claims that $\|v_n - x_{n+1}\|^2 \le -\|x_{n+1} - p\|^2 + \|x_n - p\|^2 + \beta_n M_1$ for certain $M_1 > 0$. In fact, using Lemma 7 and the convexity of $g(s) = s^2, \forall s \in \mathbf{R}$, we obtain that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|v_n-p\|^2-\|v_n-x_{n+1}\|^2 = \|\beta_n(f(x_n)-f(p))+\gamma_n(x_n-p)+(1-\gamma_n)\\&\quad\times\left[\left(I-\frac{\beta_n}{1-\gamma_n}\rho F\right)T^n z_n-\left(I-\frac{\beta_n}{1-\gamma_n}\rho F\right)p\right]+\beta_n(f-\rho F)p\|^2-\|v_n-x_{n+1}\|^2\\&\le\{\beta_n\|f(x_n)-f(p)\|+\gamma_n\|x_n-p\|+(1-\gamma_n)\\&\quad\times\left\|\left(I-\frac{\beta_n}{1-\gamma_n}\rho F\right)T^n z_n-\left(I-\frac{\beta_n}{1-\gamma_n}\rho F\right)p\right\|\Bigg\}^2+2\beta_n\langle(f-\rho F)p,v_n-p\rangle-\|v_n-x_{n+1}\|^2\\&\le\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+[(1-\gamma_n-\beta_n\tau)+\theta_n]\|z_n-p\|^2\\&\quad+2\beta_n\langle(f-\rho F)p,v_n-p\rangle-\|v_n-x_{n+1}\|^2,\end{aligned}\tag{20}$$

where $v_n:=\beta_n f(x_n)+\gamma_n x_n+((1-\gamma_n)I-\beta_n\rho F)T^n z_n$ and $x_{n+1}=P_C(v_n)$. Furthermore, according to Algorithm 3, we obtain

$$\begin{aligned}\|z_n-p\|^2\|w_n-p\|^2-\|w_n-z_n\|^2&=(1-\sigma_n)\|x_n-p\|^2\\&\quad+\sigma_2\|W_nx_n-p\|^2-\sigma_n(1-\sigma_n)\|W_nx_n-x_n\|^2-\|w_n-z_n\|^2.\end{aligned}\tag{21}$$

Substituting (21) into (20), one gets

$$\begin{aligned}\|x_{n+1}-p\|^2&\le\beta_n\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+[(1-\gamma_n-\beta_n\tau)+\theta_n]\Big\{\|x_n-p\|^2\Big]\\&\quad-\sigma_n(1-\sigma_n)\|W_nx_n-x_n\|^2-\|w_n-z_n\|^2\Big\}+2\beta_n\langle(f-\rho F)p,v_n-p\rangle-\|v_n-x_{n+1}\|^2\\&\le[1-\beta_n(\tau-\delta)/2]\|x_n-p\|^2-[(1-\gamma_n-\beta_n\tau)+\theta_n]\Big\{\sigma_n(1-\sigma_n)\|W_nx_n-x_n\|^2\\&\quad+\|w_n-z_n\|^2\Big\}+2\beta_n\langle(f-\rho F)p,v_n-p\rangle-\|v_n-x_{n+1}\|^2\\&\le\|x_n-p\|^2-[(1-\gamma_n-\beta_n\tau)+\theta_n]\{\sigma_n(1-\sigma_n)\|W_nx_n-x_n\|^2\\&\quad+\|w_n-z_n\|^2\Big\}+\beta_nM_1-\|v_n-x_{n+1}\|^2.\end{aligned}\tag{22}$$

where $\sup_{n\ge1}2\|(f-\rho F)p\|\|v_n-p\|\le M_1$ for some $M_1>0$. This immediately attains the claim.

Step 3. One claims that $[(1-\gamma_n-\beta_n\tau)+\theta_n][\tau_n/2\lambda L\|r_\lambda(w_n)\|^2]^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_nM_1$. In fact, let us show that $\exists L>0$ s.t.

$$\|z_n-p\|^2\le\|w_n-p\|^2-\left[\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2\right]^2.\tag{23}$$

Since the sequence $\{At_n\}$ is bounded, there exists $L>0$ such that $\|At_n\|\le L,\forall n\ge1$. This ensures that $|h_n(u)-h_n(v)|=|\langle At_n,u-v\rangle|\le L\|u-v\|\forall u,v\in C_n$, which hence implies that $h_n(\cdot)$ is L-Lipschitz continuous on C_n. By Lemmas 2 and 9, we obtain

$$\operatorname{dist}(w_n, C_n) \geq \frac{1}{L} h_n(w_n) = \frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2. \tag{24}$$

Combining (16) and (24), we get $\|z_n - p\|^2 \leq \|w_n - p\|^2 - [\tau_n/2\lambda L\|r_\lambda(w_n)\|^2]^2$.

From (18), (20), and (23), it follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n\|\delta x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + [(1 - \gamma_n - \beta_n\tau) + \theta_n]\|z_n - p\|^2 \\
&\quad + 2\beta_n\langle (f - \rho F)p, v_n - p\rangle \\
&\leq \beta_n\delta\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + [(1 - \gamma_n - \beta_n\tau) + \theta_n]\left\{\|w_n - p\|^2 - \left[\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2\right]^2\right\} \\
&\quad + 2\beta_n\langle (f - \rho F)p, v_n - p\rangle \\
&\leq \|x_n - p\|^2 - [(1 - \gamma_n - \beta_n\tau) + \theta_n]\left[\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2\right]^2 + \beta_n M_1.
\end{aligned} \tag{25}$$

This immediately yields the claim.

Step 4. We show that

$$\|x_{n+1} - p\|^2 \leq [1 - \beta_n(\tau - \delta)]\|x_n - p\|^2 + \beta_n(\tau - \delta)\left[\frac{2\langle (f - \rho F)p, v_n - p\rangle}{\tau - \delta} + \frac{\theta_n}{\beta_n}\cdot\frac{M}{\tau - \delta}\right], \tag{26}$$

for some $M > 0$. In fact, from (20), one obtains

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq \|v_n - p\|^2 &\leq \beta_n\delta\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + (1 - \gamma_n - \beta_n\tau + \theta_n) \\
&\quad \times\|z_n - p\|^2 + 2\beta_n\langle (f - \rho F)p, v_n - p\rangle \\
&\leq [1 - \beta_n(\tau - \delta)]\|x_n - p\|^2 + \beta_n(\tau - \delta)\left[2\frac{\langle (f - \rho F)p, v_n - p\rangle}{\tau - \delta} + \frac{\theta_n}{\beta_n}\cdot\frac{M}{\tau - \delta}\right],
\end{aligned} \tag{27}$$

where $\sup_{n\geq 1}\|z_n - p\|^2 \leq M$ for some $M > 0$.

Step 5. One claims that $x_n \longrightarrow x^* \in \Omega$, which is only a solution of the VIP (15).

In fact, setting $p = x^*$ in (26), one obtains

$$\beta_n(\tau - \delta)\left[\frac{2\langle (f - \rho F)x^*, v_n - x^*\rangle}{\tau - \delta} + \frac{\theta_n}{\beta_n}\cdot\frac{M}{\tau - \delta}\right] + (1 - \beta_n(\tau - \delta))\|x_n - x^*\|^2 \geq \|x_{n+1} - x^*\|^2. \tag{28}$$

Setting $\Gamma_n = \|x_n - x^*\|^2$, we demonstrate the convergence of $\{\Gamma_n\}$ to zero via the two cases below.

Case 1. Presume that $\exists n_0 \geq 1$ s.t. $\{\Gamma_n\}$ is nonincreasing. Then, $\lim_{n\longrightarrow\infty}\Gamma_n = \hat{h} < +\infty$ and $\Gamma_n - \Gamma_{n+1} \longrightarrow 0\,(n\longrightarrow\infty)$. Setting $p = x^*$, from Step 2 and $\{\sigma_n\} \subset [a, b] \subset (0, 1)$, we obtain

$$\begin{aligned}
&[(1 - \gamma_n - \beta_n\tau) + \theta_n]\left\{(1 - b)a\|x_n - W_n x_n\|^2 + \|z_n - w_n\|^2\right\} + \|v_n - x_{n+1}\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n M_1 = \Gamma_n - \Gamma_{n+1} + \beta_n M_1.
\end{aligned} \tag{29}$$

Because $\liminf_{n\longrightarrow\infty}(1 - \gamma_n) > 0$, $\theta_n \longrightarrow 0, \beta_n \longrightarrow 0$, and $\Gamma_{n+1} - \Gamma_n \longrightarrow 0$, one has

Initial Step: Given any starting $x_1 \in C$. Let $\mu > 0, \lambda \in (0, 1/\mu), l \in (0, 1)$.
Iterations: Given the current iterate x_n, calculate x_{n+1} below:
Step 1. Compute $w_n = (1 - \sigma_n)x_n + \sigma_n W_n x_n$, $y_n = P_C(w_n - \lambda A w_n)$ and $r_\lambda(w_n) := w_n - y_n$.
Step 2. Compute
$t_n = w_n - \tau_n r_\lambda(w_n)$, with $\tau_n := l^{j_n}$ and integer j_n being the smallest nonnegative j s.t. $\mu/2\|r_\lambda(w_n)\|^2 \geq \langle A(w_n - l^j r_\lambda(w_n)) - Aw_n, y_n - w_n\rangle$
Step 3. Calculate
$x_{n+1} = P_C[\beta_n f(x_n) + \gamma_n w_n + ((1-\gamma_n)I - \beta_n \rho F)T^n z_n]$
with $z_n = P_{C_n}(w_n)$, $C_n := \{u \in C: h_n(u) \leq 0\}$ and $h_n(u) = \langle At_n, u - w_n\rangle + \tau_n/2\lambda\|r_\lambda(w_n)\|^2$.
Put $n := n+1$ and return to Step 1.
It is worthy to mention that Lemmas 8–11 remain true for Algorithm 4.

Algorithm 4: The 2nd Mann hybrid deepest-descent extragradient rule.

$$\lim_{n\longrightarrow\infty}\|x_n - W_n x_n\| = \lim_{n\longrightarrow\infty}\|z_n - w_n\| = \lim_{n\longrightarrow\infty}\|v_n - x_{n+1}\| = 0. \tag{30}$$

Also, noticing $\beta_n(f(x_n) - \rho F T^n z_n) + (1-\gamma_n)\ (T^n z_n - x^*) + \gamma_n(x_n - x^*) = v_n - x^*$, one gets

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|x_{n+1} - v_n + v_n - x^*\|^2\\ &= \|\gamma_n(x_n - x^*) + (1-\gamma_n)(T^n z_n - x^*) + \beta_n(f(x_n) - \rho F T^n z_n) + x_{n+1} - v_n\|^2\\ &\|x_n - x^*\|^2 + (1-\gamma_n)[1 + (2+\theta_n)\theta_n]\|x_n - x^*\|^2 - (1-\gamma_n)\gamma_n\|T^n z_n - x_n\|^2\\ &\quad + 2\{\beta_n(\|f(x_n)\| + \|\rho F T^n z_n\|) + \|x_{n+1} - v_n\|\}\|x_{n+1} - x^*\|\\ &\|x_n - x^*\|^2 + (2+\theta_n)\theta_n\|x_n - x^*\|^2 - (1-\gamma_n)\gamma_n\|T^n z_n - x_n\|^2\\ &\quad + 2\{\beta_n(\|f(x_n)\| + \|\rho F T^n z_n\|) + \|x_{n+1} - v_n\|\}\|x_{n+1} - x^*\|,\end{aligned} \tag{31}$$

which immediately arrives at

$$\begin{aligned}&\gamma_n(1-\gamma_n)\|x_n - T^n z_n\|^2\\ &\leq \Gamma_n - \Gamma_{n+1} + \theta_n(2+\theta_n)\Gamma_n + 2\{\beta_n(\|f(x_n)\| + \|\rho F T^n z_n\|) + \|x_{n+1} - v_n\|\}\Gamma_{n+1.}^{1/2}\end{aligned} \tag{32}$$

Because $1 > \liminf_{n\longrightarrow\infty}\gamma_n \geq \liminf_{n\longrightarrow\infty}\gamma_n > 0$, $\theta_n \longrightarrow 0, \beta_n \longrightarrow 0, \Gamma_{n+1} - \Gamma_n \longrightarrow 0$, and $v_n - x_{n+1} \longrightarrow 0$, from the boundedness of $\{T^n z_n\}, \{f(x_n)\}$, one obtains $\lim_{n\longrightarrow\infty}\|T^n z_n - x_n\| = 0$. So, we know from Algorithm 3 and (30) that $\|x_n - w_n\| = \sigma_n\|x_n - W_n x_n\| \leq \|x_n - W_n x_n\| \longrightarrow 0$ $(n\longrightarrow\infty)$, and

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|x_{n+1} - v_n\| + \|v_n - x_n\|\\ &= \|x_{n+1} - v_n\| + \|\beta_n f(x_n) + (1-\gamma_n)(T^n z_n - x_n) - \beta_n \rho F T^n z_n\|\\ &\leq \|x_{n+1} - v_n\| + \|T^n z_n - x_n\| + \beta_n(\|f(x_n)\| + \|\rho F T^n z_n\|) \longrightarrow 0\ (n\longrightarrow\infty).\end{aligned} \tag{33}$$

Setting $p = x^*$, in terms of Step 3, one has $[(1-\gamma_n-\beta_n\tau)+\theta_n][\tau_n/2\lambda L\|r_\lambda(w_n)\|^2]^2 \le \Gamma_n-\Gamma_{n+1}+\beta_n M_1$. Since $0<\liminf_{n\longrightarrow\infty}(1-\gamma_n)$, $\theta_n\longrightarrow 0, \beta_n\longrightarrow 0$, and $\Gamma_{n+1}-\Gamma_n\longrightarrow 0$, one gets $\lim_{n\longrightarrow\infty}[\tau_n/2\lambda L\|r_\lambda(w_n)\|^2]^2=0$. Hence, by Lemma 11, we deduce that

$$\lim_{n\longrightarrow\infty}\left\|y_n-w_n\right\|=0. \tag{34}$$

Since $\{x_n\}$ is bounded, we deduce that $\exists\{x_{n_k}\}\subset\{x_n\}$ s.t.

$$\limsup_{n\longrightarrow\infty}\langle(f-\rho F)x^*, x_n-x^*\rangle=\lim_{k\longrightarrow\infty}\langle(f-\rho F)x^*, x_{n_k}-x^*\rangle. \tag{35}$$

According to the reflexivity of H and boundedness of $\{x_n\}$, one may presume that $x_{n_k}\rightharpoonup\tilde{x}$. Hence, by (35), one gets

$$\limsup_{n\longrightarrow\infty}\langle(f-\rho F)x^*, x_n-x^*\rangle=\langle(f-\rho F)x^*, \tilde{x}-x^*\rangle. \tag{36}$$

So, we know from $x_{n_k}\rightharpoonup\tilde{x}$ and $x_n-w_n\longrightarrow 0$ that $w_{n_k}\rightharpoonup\tilde{x}$. Because $x_{n+1}-x_n\longrightarrow 0, x_n-w_n\longrightarrow 0, y_n-w_n\longrightarrow 0, z_n-w_n\longrightarrow 0$, and $T^{n+1}x_n-T^n x_n\longrightarrow 0$, from Lemma 10, we infer that $\tilde{x}$ lies in Ω. Consequently, using (15) and (36), we obtain

$$\limsup_{n\longrightarrow\infty}\langle(f-\rho F)x^*, x_n-x^*\rangle=\langle(f-\rho F)x^*, \tilde{x}-x^*\rangle\le 0. \tag{37}$$

This along with (30)–(33) arrives at

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\langle(f-\rho F)x^*, v_n-x^*\rangle\\&=\limsup_{n\longrightarrow\infty}\left[\langle(f-\rho F)x^*, v_n-x_{n+1}+x_{n+1}-x_n\rangle+\langle(f-\rho F)x^*, x_n-x^*\rangle\right]\le 0.\end{aligned} \tag{38}$$

Note that $\beta_n(\tau-\delta)\in[0,1]\forall n, \sum_{n=1}^{\infty}\beta_n(\tau-\delta)=\infty$, and $\limsup_{n\longrightarrow\infty}[2\langle(f-\rho F)x^*, v_n-x^*\rangle/\tau-\delta+\theta_n/\beta_n\cdot M/\tau-\delta]\le 0$. So, using Lemma 4 to (28), we have $\lim_{n\longrightarrow\infty}\|x_n-x^*\|^2=0$.

Case 2. Suppose that $\exists\{\Gamma_{n_k}\}\subset\{\Gamma_n\}$ s.t. $\Gamma_{n_k}<\Gamma_{n_k+1}, \forall k\in\mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. Let $\eta\colon\mathcal{N}\longrightarrow\mathcal{N}$ be defined as $\eta(n)\colon=\max\{j\le n\colon \Gamma_j<\Gamma_{j+1}\}$. Using Lemma 6, one obtains that $\Gamma_n\le\Gamma_{\eta(n)+1}$ and $\Gamma_{\eta(n)}\le\Gamma_{\eta(n)+1}$. Putting $p=x^*$, from Step 2, we have

$$\begin{aligned}&\left[\left(1-\gamma_{\eta(n)}-\beta_{\eta(n)}\tau\right)+\theta_{\eta(n)}\right]\left\{(1-b)a\left\|x_{\eta(n)}-W_{\eta(n)}x_{\eta(n)}\right\|^2\right.\\&\quad\left.+\left\|z_{\eta(n)}-w_{\eta(n)}\right\|^2\right\}+\left\|v_{\eta(n)}-x_{\eta(n)+1}\right\|^2\\&\le\left\|x_{\eta(n)}-x^*\right\|^2-\left\|x_{\eta(n)+1}-x^*\right\|^2+\beta_{\eta(n)}M_1\le\Gamma_{\eta(n)}-\Gamma_{\eta(n)+1}+\beta_{\eta(n)}M_1.\end{aligned} \tag{39}$$

This hence leads to $\lim_{n\longrightarrow\infty}\|x_{\eta(n)}-W_{\eta(n)}x_{\eta(n)}\|=\lim_{n\longrightarrow\infty}\|z_{\eta(n)}-w_{\eta(n)}\|=\lim_{n\longrightarrow\infty}\|v_{\eta(n)}-x_{\eta(n)+1}\|=0$. Setting $p=x^*$, in terms of Step 3, one gets

$$\left[\left(1-\gamma_{\eta(n)}-\beta_{\eta(n)}\tau\right)+\theta_{\eta(n)}\right]\left[\frac{\tau_{\eta(n)}}{2\lambda L}\left\|r_\lambda\left(w_{\eta(n)}\right)\right\|^2\right]^2\le\Gamma_{\eta(n)}-\Gamma_{\eta(n)+1}+\beta_{\eta(n)}M_1, \tag{40}$$

which hence leads to $\lim_{n\longrightarrow\infty}[\tau_{\eta(n)}/2\lambda L\|r_\lambda(w_{\eta(n)})\|^2]^2=0$. Using the same inferences as in the proof of Case 1, one obtains that $\lim_{n\longrightarrow\infty}\|y_{\eta(n)}-w_{\eta(n)}\|=\lim_{n\longrightarrow\infty}\|x_{\eta(n)}-w_{\eta(n)}\|=\lim_{n\longrightarrow\infty}\|x_{\eta(n)}-x_{\eta(n)+1}\|=0$, and

$$0\ge\limsup_{n\longrightarrow\infty}\langle(f-\rho F)x^*, v_{\eta(n)}-x^*\rangle. \tag{41}$$

Furthermore, using (28), one has

$$\beta_{\eta(n)}(\tau-\delta)\Gamma_{\eta(n)} \le \Gamma_{\eta(n)} - \Gamma_{\eta(n)+1} + \beta_{\eta(n)}(\tau-\delta)\left[\frac{2\langle (f-\rho F)x^*, v_{\eta(n)} - x^*\rangle}{\tau-\delta} + \frac{\theta_{\eta(n)}}{\beta_{\eta(n)}}\cdot\frac{M}{\tau-\delta}\right]$$
$$- \le \beta_{\eta(n)}(\tau-\delta)\left[\frac{2\langle (f-\rho F)x^*, v_{\eta(n)} - x^*\rangle}{\tau-\delta} + \frac{\theta_{\eta(n)}}{\beta_{\eta(n)}}\cdot\frac{M}{\tau-\delta}\right], \tag{42}$$

which hence arrives at

$$\limsup_{n\longrightarrow\infty}\Gamma_{\eta(n)} \le \limsup_{n\longrightarrow\infty}\left[\frac{2\langle (f-\rho F)x^*, v_{\eta(n)} - x^*\rangle}{\tau-\delta} + \frac{\theta_{\eta(n)}}{\beta_{\eta(n)}}\cdot\frac{M}{\tau-\delta}\right] \le 0. \tag{43}$$

Thus, $\lim_{n\longrightarrow\infty}\Gamma_{\eta(n)} = 0$. Also, observe that

$$\begin{aligned}&\left\|x_{\eta(n)+1} - x^*\right\|^2\\ &= \left\|x_{\eta(n)+1} - x_{\eta(n)}\right\|^2 + 2\langle x_{\eta(n)+1} - x_{\eta(n)}, x_{\eta(n)} - x^*\rangle + \left\|x_{\eta(n)} - x^*\right\|^2\\ &\le \left\|x_{\eta(n)+1} - x_{\eta(n)}\right\|^2 + 2\left\|x_{\eta(n)+1} - x_{\eta(n)}\right\| + \left\|x_{\eta(n)} - x^*\right\| + \left\|x_{\eta(n)} - x^*\right\|^2.\end{aligned} \tag{44}$$

Owing to $\Gamma_n \le \Gamma_{\eta(n)+1}$, we get

$$\begin{aligned}\left\|x_n - x^*\right\|^2 &\le \left\|x_{\eta(n)+1} - x^*\right\|^2\\ &\le \left\|x_{\eta(n)+1} - x_{\eta(n)}\right\|^2 + 2\left\|x_{\eta(n)+1} - x_{\eta(n)}\right\|\left\|x_{\eta(n)} - x^*\right\| + \left\|x_{\eta(n)} - x^*\right\|^2 \longrightarrow 0\ (n\longrightarrow\infty).\end{aligned} \tag{45}$$

Consequently, $\lim_{n\longrightarrow\infty}\|x_n - x^*\|^2 = 0$.

Theorem 2. *Suppose that T is a nonexpansive self-mapping on C and $\{x_n\}$ is fabricated by the modification of Algorithm 3, i.e., for any starting $x_1 \in C$,*

$$\begin{cases} w_n = (1-\sigma_n)x_n + \sigma_n W_n x_n,\\ y_n = P_C(w_n - \lambda A w_n),\\ t_n = (1-\tau_n)w_n + \tau_n y_n,\\ x_{n+1} = P_C\left[\beta_n f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \beta_n\rho F)TP_{C_n}(w_n)\right], \forall n \ge 1,\end{cases} \tag{46}$$

where for any n, C_n and τ_n are picked as in Algorithm 3. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$, which is only a solution to the VIP: $\langle (f-\rho F)x^*, x - x^*\rangle \le 0, \forall x \in \Omega$.

Proof. We write $z_n := P_{C_n}(w_n)$, and divide the proof into a few steps.

Step 1. One claims that $\{x_n\}$ is of boundedness. In fact, via the similar inferences to those in Step 1 of the proof of Theorem 1, one attains the claim.

Step 2. One claims that

$$\begin{aligned}(1-\gamma_n-\beta_n\tau)\left\{(1-\sigma_n)\sigma_n\|x_n-W_nx_n\|^2+\|z_n-w_n\|^2\right\}+\|v_n-x_{n+1}\|^2,\\ \le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\beta_n\langle(f-\rho F)p,v_n-p\rangle,\end{aligned}\tag{47}$$

where $v_n=\beta_nf(x_n)+\gamma_nx_n+((1-\gamma_n)I-\beta_n\rho F)Tz_n$. In fact, via the similar inferences to those in Step 2 of the proof of Theorem 1, one attains the claim.

Step 3. One claims that

$$(1-\gamma_n-\beta_n\tau)\left[\frac{\tau_n}{2\lambda L}\|r_\lambda(w_n)\|^2\right]^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\beta_n\langle(f-\rho F)p,v_n-p\rangle,\tag{48}$$

In fact, via the similar inferences to those in Step 3 of the proof of Theorem 1, one attains the claim.

Step 4. One claims that

$$\|x_{n+1}-p\|^2\le[1-\beta_n(\tau-\delta)]\|x_n-p\|^2+\beta_n(\tau-\delta)\cdot\frac{2\langle(f-\rho F)p,v_n-p\rangle}{\tau-\delta}.\tag{49}$$

In fact, via the similar inferences to those in Step 4 of the proof of Theorem 1, one attains the claim.

Step 5. One claims that $x_n\longrightarrow x^*\in\Omega$, which is only a solution of the VIP (15). In fact, via the similar inferences to those in Step 3 of the proof of Theorem 1, one attains the claim.

Next, we introduce modified hybrid deepest-descent extragradient approach. □

Theorem 3. *Let $\{x_n\}$ be the sequence fabricated in Algorithm 4. Assume $T^{n+1}x_n-T^nx_n\longrightarrow 0$. Then, $\{x_n\}$ converges strongly to $x^*\in\Omega$, which is only a solution to the VIP: $\langle(f-\rho F)x^*,x-x^*\rangle\le 0,\forall x\in\Omega$.*

Proof. Because $1>\limsup_{n\longrightarrow\infty}\sigma_n\ge\liminf_{n\longrightarrow\infty}\sigma_n>0$ and $\theta_n/\beta_n\longrightarrow 0$, one may presume that $(0,1)\supset[a,b]\supset\{\sigma_n\}$ and $\beta_n(\tau-\delta)/2\ge\theta_n,\forall n$. It is easy to verify that $P_\Omega(I-\rho F+f)\colon C\longrightarrow C$ is a contractive map. Hence, $\exists\,|\,x^*\in C$ s.t. $x^*=P_\Omega(I-\rho F+f)x^*$. Thus, $\exists|x^*\in\Omega$ satisfying the VIP (15).

In what follows, in the rest of the proof, we divide it into some steps.

Step 1. One claims that $\{x_n\}$ is of boundedness. In fact, using the similar inferences to those in Step 1 of the proof of Theorem 1, we obtain that relations (16)–(30) hold. Thus, using (30) and $1>\gamma_n+\beta_n$, one deduces that

$$\begin{aligned}\|x_{n+1}-p\|&\le\|\beta_n(f(x_n)-f(p))+\gamma_n(w_n-p)+(1-\gamma_n)\\&\quad\times[(I-\beta_n/1-\gamma_n\rho F)T^nz_n-(I-\beta_n/1-\gamma_n\rho F)p]+\beta_n(f-\rho F)p\|\\&\le[\beta_n\delta+\gamma_n+(1-\gamma_n-\beta_n\tau)+\theta_n]\|x_n-p\|+\beta_n\|(f-\rho F)p\|\\&\le\max\left\{\|x_n-p\|,\frac{2\|(f-\rho F)p\|}{\tau-\delta}\right\}.\end{aligned}\tag{50}$$

Using the induction, one obtains $\|x_n-p\|\le\max\{\|x_1-p\|,2\|(f-\rho F)p\|/\tau-\delta\},\forall n$.

Hence, $\{x_n\}$ is bounded, and so are the sequences $\{w_n\},\{y_n\},\{z_n\},\{f(x_n)\},\{At_n\},\{W_nx_n\},\{T^nz_n\}$.

Step 2. One claims that

$$\begin{aligned}&[(1-\gamma_n-\beta_n\tau)+\theta_n]\{(1-\sigma_n)\sigma_n\|x_n-W_nx_n\|^2+\|z_n-w_n\|^2\}+\|v_n-x_{n+1}\|^2\\&-\leq-\|x_{n+1}-p\|^2+\|x_n-p\|^2+\beta_nM_1.\end{aligned}\tag{51}$$

for certain $M_1>0$, where $v_n=\beta_nf(x_n)+\gamma_nw_n+((1-\gamma_n)I-\beta_n\rho F)T^nz_n$ and $x_{n+1}=P_Cv_n$. To prove this, we first note that

$$\begin{aligned}\|x_{n+1}-p\|^2&\leq\beta_n(f(x_n)-f(p))+\gamma_n(w_n-p)+(1-\gamma_n)\\&\quad\times[(I-\beta_n/1-\gamma_n\rho F)T^nz_n-(I-\beta_n/1-\gamma_n\rho F)p]+\beta_n(f-\rho F)p\|^2-\|v_n-x_{n+1}\|^2\\&\leq\beta_n\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+[(1-\gamma_n-\beta_n\tau)+\theta_n]\|z_n-p\|^2\\&\quad+2\beta_n\langle(f-\rho F)p,v_n-p\rangle-\|v_n-x_{n+1}\|^2.\end{aligned}\tag{52}$$

Furthermore, via the similar reasoning to that in (21), we get

$$\|z_n-p\|^2\leq\|x_n-p\|^2-(1-\sigma_n)\sigma_n\|x_n-W_nx_n\|^2-\|z_n-w_n\|^2.\tag{53}$$

Substituting (38) into (37), one gets

$$\begin{aligned}\|x_{n+1}-p\|^2&\leq\beta_n\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+[(1-\gamma_n-\beta_n\tau)+\theta_n]\Big\{\|x_n-p\|^2\\&\quad-(1-\sigma_n)\sigma_n\|x_n-W_nx_n\|^2-\|z_n-w_n\|^2\Big\}+2\beta_n\langle(f-\rho F)p,v_n-p\rangle-\|v_n-x_{n+1}\|^2\\&\leq\|x_n-p\|^2-[(1-\gamma_n-\beta_n\tau)+\theta_n]\Big\{(1-\sigma_n)\sigma_n\|x_n-W_nx_n\|^2+\|z_n-w_n\|^2\Big\}+\beta_nM_1\\&\quad-\|v_n-x_{n+1}\|^2.\end{aligned}\tag{54}$$

with $\sup_{n\geq1}2\|(\rho F-f)p\|\|v_n-p\|\leq M_1$ for certain $M_1>0$. This immediately arrives at the claim.

Step 3. One claims that

$$[(1-\gamma_n-\beta_n\tau)+\theta_n]\Big[\tau_n/2\lambda L\|r_\lambda(w_n)\|^2\Big]^2\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_nM_1.\tag{55}$$

In fact, via the similar inferences to those in (27), one obtains that for certain $L>0$,

$$\|z_n-p\|^2\leq\|w_n-p\|^2-\Big[\tau_n/2\lambda L\|r_\lambda(w_n)\|^2\Big]^2.\tag{56}$$

From (39), (30), and (43) we know that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+[(1-\gamma_n-\beta_n\tau)+\theta_n]\|z_n-p\|^2+\beta_nM_1\\ &\le \beta_n\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+[(1-\gamma_n-\beta_n\tau)+\theta_n]\Big\{\|w_n-p\|^2\\ &\quad -\Big[\tau_n/2\lambda L\,\|r_\lambda(w_n)\|^2\Big]^2\Big\}+\beta_nM_1\\ &\le \|x_n-p\|^2-[(1-\gamma_n-\beta_n\tau)+\theta_n]\Big[\tau_n/2\lambda L\|r_\lambda(w_n)\|^2\Big]^2+\beta_nM_1.\end{aligned} \tag{57}$$

which hence yields the claim.

Step 4. One claims that $\|x_{n+1}-p\|^2\le[1-\beta_n(\tau-\delta)]\|x_n-p\|^2+\beta_n(\tau-\delta)[2\langle(f-\rho F)p,v_n-p\rangle/\tau-\delta+\theta_n/\beta_n\cdot M/\tau-\delta]$ for some $M>0$. In fact, using Lemma 7 and (30), one has

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|v_n-p\|^2=\beta_n(f(x_n)-f(p))+\gamma_n(w_n-p)+(1-\gamma_n)\\ &-\times[(I-\beta_n/1-\gamma_n\rho F)T^n z_n-(I-\beta_n/1-\gamma_n\rho F)p]+\beta_n(f-\rho F)p^2\\ &-\le\beta_n\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+(1-\gamma_n-\beta_n\tau+\theta_n)\\ &-\times\|z_n-p\|^2+2\beta_n\langle(f-\rho F)p,v_n-p\rangle\\ &-\le[1-\beta_n(\tau-\delta)]\|x_n-p\|^2+\beta_n(\tau-\delta)[2\langle(f-\rho F)p,v_n-p\rangle/\tau-\delta+\theta_n/\beta_n\cdot M/\tau-\delta],\end{aligned} \tag{58}$$

where $\sup_{n\ge1}\|x_n-p\|^2\le M$ for certain $M>0$.

Step 5. One claims that $x_n\longrightarrow x^*\in\Omega$ which is only a solution of the VIP (15). In fact, setting $p=x^*$, in terms of Step 4, one deduces that

$$\|x_{n+1}-x^*\|^2\le[1-\beta_n(\tau-\delta)]\|x_n-x^*\|^2+\beta_n(\tau-\delta)[2\langle(f-\rho F)x^*,v_n-x^*\rangle/\tau-\delta+\theta_n/\beta_n\cdot M/\tau-\delta]. \tag{59}$$

Setting $\Gamma_n=\|x_n-x^*\|^2$, one demonstrates the convergence of $\{\Gamma_n\}$ to zero via the two cases below. □

Case 3. Presume that $\exists n_0\ge1$ s.t. $\{\Gamma_n\}$ is nonincreasing. Then, $\lim_{n\longrightarrow\infty}\Gamma_n=\hat{h}<+\infty$ and $\Gamma_n-\Gamma_{n+1}\longrightarrow0$ $(n\longrightarrow\infty)$. Setting $p=x^*$, from Step 2 and $\{\sigma_n\}\subset[a,b]\subset(0,1)$, we obtain

$$\begin{aligned}&[(1-\gamma_n-\beta_n\tau)+\theta_n]\Big\{(1-b)a\|x_n-W_nx_n\|^2+\|z_n-w_n\|^2\Big\}+\|v_n-x_{n+1}\|^2\\ &-\le[(1-\gamma_n-\beta_n\tau)+\theta_n]\Big\{(1-\sigma_n)\sigma_n\|x_n-W_nx_n\|^2+\|z_n-w_n\|^2\Big\}+\|v_n-x_{n+1}\|^2\\ &-\le\Gamma_n-\Gamma_{n+1}+\beta_nM_1,\end{aligned} \tag{60}$$

which hence yields

$$\lim_{n\longrightarrow\infty}\|x_n-W_nx_n\|=\lim_{n\longrightarrow\infty}\|z_n-w_n\|=\lim_{n\longrightarrow\infty}\|v_n-x_{n+1}\|=0. \tag{61}$$

Putting $p = x^*$, from Step 3, we obtain $[(1-\gamma_n-\beta_n\tau)+\theta_n][\tau_n/2\lambda L\ \|r_\lambda(w_n)\|^2]^2 \le \Gamma_n - \Gamma_{n+1} + \beta_n M_1$, which immediately leads to $\lim_{n\longrightarrow\infty}[\tau_n/2\lambda L\|r_\lambda(w_n)\|^2]^2 = 0$. Using the similar reasoning to that in Case 1 of the proof of Theorem 1, one deduces that $\lim_{n\longrightarrow\infty}\|w_n - y_n\| = \lim_{n\longrightarrow\infty}\|w_n - x_n\| = \lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0$, and $\limsup_{n\longrightarrow\infty}\langle (f-\rho F)x^*, v_n - x^*\rangle \le 0$. Accordingly, using Lemma 4 to (59), we attain $\lim_{n\longrightarrow\infty}\|x_n - x^*\|^2 = 0$.

Case 4. Presume that $\exists\{\Gamma_{n_j}\} \subset \{\Gamma_n\}$ s.t. $\Gamma_{n_j} < \Gamma_{n_j+1}, \forall j \in \mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. Let $\eta\colon \mathcal{N} \longrightarrow \mathcal{N}$ be defined as $\eta(n)\colon = \max\{j \le n\colon \Gamma_j < \Gamma_{j+1}\}$. Using Lemma 6, one gets $\Gamma_{\eta(n)} \le \Gamma_{\eta(n)+1}$ and $\Gamma_n \le \Gamma_{\eta(n)+1}$. In the rest of the proof, applying the similar reasoning to that in Case 2 of the proof of Theorem 1, one obtains the claim.

Theorem 4. *Suppose that T is a nonexpansive self-mapping on C and $\{x_n\}$ is fabricated by the modification of Algorithm 4, i.e., for each starting $x_1 \in C$,*

$$\begin{cases} w_n = (1-\sigma_n)x_n + \sigma_n W_n x_n, \\ y_n = P_C(w_n - \lambda A w_n), \\ t_n = (1-\tau_n)w_n + \tau_n y_n, \\ x_{n+1} = P_C\left[\beta_n f(x_n) + \gamma_n w_n + ((1-\gamma_n)I - \beta_n\rho F)TP_{C_n}(w_n)\right], \forall n \ge 1, \end{cases} \tag{62}$$

where for any n, C_n and τ_n are picked as in Algorithm 4. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$, which is only a solution to the VIP: $\langle (f-\rho F)x^*, x - x^*\rangle \le 0, \forall x \in \Omega$.

Proof. One writes $z_n\colon = P_{C_n}(w_n)$ and divides the proof of the theorem into a few steps.

Step 1. One claims that $\{x_n\}$ is of boundedness. In fact, via the similar inferences to those in Step 1 of the proof of Theorem 3, one obtains the claim.

Step 2. One claims that

$$\begin{aligned} &(1-\gamma_n-\beta_n\tau)\left\{(1-\sigma_n)\sigma_n\|x_n - W_n x_n\|^2 + \|z_n - w_n\|^2\right\} + \|v_n - x_{n+1}\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\beta_n\langle (f-\rho F)p, v_n - p\rangle, \end{aligned} \tag{63}$$

with $v_n = \beta_n f(x_n) + \gamma_n w_n + ((1-\gamma_n)I - \beta_n\rho F)Tz_n$. In fact, via the similar inferences to those in Step 2 of the proof of Theorem 3, one obtains the claim.

Step 3. One claims that

$$(1-\gamma_n-\beta_n\tau)\left[\tau_n/2\lambda L\|r_\lambda(w_n)\|^2\right]^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\beta_n\langle (f-\rho F)p, v_n - p\rangle. \tag{64}$$

In fact, via the similar inferences to those in Step 3 of the proof of Theorem 3, one obtains the claim.

Step 4. One claims that $\|x_{n+1} - p\|^2 \le [1-\beta_n(\tau-\delta)]\|x_n - p\|^2 + \beta_n(\tau-\delta)\cdot 2\langle (f-\rho F)p, v_n - p\rangle/\tau - \delta$. Indeed, via the similar inferences to those in Step 4 of the proof of Theorem 3, one obtains the claim.

Step 5. One claims that $x_n \longrightarrow x^* \in \Omega$ which is only a solution of the VIP (15). In fact, via the similar inferences to those in Step 5 of the proof of Theorem 3, one obtains the claim. □

4. Applicability and Implementability

In this section, we provide an illustrated instance to demonstrate the applicability and implementability of our proposed algorithms. Put $\rho = 2, \lambda = l = \mu = 1/2, \sigma_n = 1/3, \beta_n = 1/3(n+1)$, and $\gamma_n = 1/3(n+1)$.

We first provide an example of Lipschitz continuous and pseudomonotone mapping A, asymptotically nonexpansive mapping T, and countably many nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ with $\Omega = \mathrm{VI}(C, A) \cap (\cap_{n=1}^{\infty} \mathrm{Fix}(T_i)) \neq \varnothing$ where $T_0: = T$. Put $C = [-3, 3]$ and $H = \mathbf{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. The starting point x_1 is arbitrarily selected in C. Put $f(x) = F(x) = 1/2x, \forall x \in C$ s.t.

$$\delta = 1/2 < \tau = 1 - \sqrt{1 - \rho\left(2\eta - \rho\kappa^2\right)} = 1. \tag{65}$$

Assume that $A: H \longrightarrow H$ and $T, T_n: C \longrightarrow C$ are formulated as $Av: = 1/1 + |\sin v| - 1/1 + |v|$, $Tv: = 3/5 \sin v$, and $T_n v = Sv = \sin v, \forall v \in C, n \geq 1$. We now claim that A is of both pseudomonotonicity and Lipschitz continuity. Indeed, for each $v, u \in H$, one has

$$\begin{aligned} \|Av - Au\| &\leq \left|\frac{\|v\| - \|u\|}{(1 + \|u\|)(1 + \|v\|)}\right| + \left|\frac{\|\sin v\| - \|\sin u\|}{(1 + \|\sin u\|)(1 + \|\sin v\|)}\right| \\ - &\leq \frac{\|\mathrm{v} - \mathrm{u}\|}{(1 + \|\mathrm{u}\|)(1 + \|\mathrm{v}\|)} + \frac{\|\sin \mathrm{v} - \sin \mathrm{u}\|}{(1 + \|\sin \mathrm{u}\|)(1 + \|\sin \mathrm{v}\|)} \leq 2\|\mathrm{v} - \mathrm{u}\|. \end{aligned} \tag{66}$$

Accordingly, A is of Lipschitz continuity. In what follows, one claims that A is of pseudomonotonicity. For any $v, u \in H$, it is clear that $\langle Av, u - v\rangle = (1/1 + |\sin v| - 1/1 + |u|)(u - v) \geq 0 \Rightarrow \langle Au, u - v\rangle = (1/1 + |\sin u| - 1/1 + |u|)\ (u - v) \geq 0$. Moreover, it is easy to check that T is of asymptotical nonexpansivity with $\theta_n = (3/5)^n, \forall n$, s.t. $\|T^n x_n - T^{n+1} x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. In fact, one observes that $\|T^n v - T^n u\| \leq 3/5 \|T^{n-1} v - T^{n-1} u\| \leq \cdots \leq (3/5)^n \quad \|v - u\| \leq (1 + \theta_n)\|v - u\|$, and

$$\left\|T^{n+1} x_n - T^n x_n\right\| \leq (3/5)^{n-1} \left\|T^2 x_n - T x_n\right\| = (3/5)^{n-1} \left\|\frac{3}{5} \sin(T x_n) - 3/5 \sin x_n\right\| \leq 2\left(\frac{3}{5}\right)^n \longrightarrow 0. \tag{67}$$

It is clear that $\mathrm{Fix}(T) = \{0\}$ and $\lim_{n\longrightarrow\infty} \theta_n/\beta_n = \lim (3/5)^n / 1/3(n+1)_{n\longrightarrow\infty} = 0$. Additionally, it is readily known that $T_n = S$ is of nonexpansivity and $\mathrm{Fix}(S) = \{0\}$. Thus, $\Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T) = \{0\} \neq \varnothing$. So, from $W_n = S$ and $(1 - \gamma_n)I - \beta_n \rho F = (1 - n/3(n+1))I - 1/\ \ 3(n+1)2 \cdot 1/2I = 2/3I$, we reduce Algorithm 3 to the following:

$$\begin{cases} w_n = \dfrac{2}{3} x_n + \dfrac{1}{3} S x_n, \\ y_n = P_C\left(w_n - \dfrac{1}{2} A w_n\right), \\ t_n = (1 - \tau_n) w_n + \tau_n y_n, \\ x_{n+1} = \dfrac{1}{3(n+1)} \cdot \dfrac{1}{2} x_n + \dfrac{n}{3(n+1)} x_n + \dfrac{2}{3} T^n P_{C_n}(w_n), \forall n \geq 1, \end{cases} \tag{68}$$

where for any n, C_n and τ_n are picked as in Algorithm 3. Then, by Theorem 1, one deduces that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$.

Particularly, noticing the fact that $Tu: = 3/5 \sin u$ is of nonexpansivity, we also present the modification of Algorithm 3, i.e.,

$$\begin{cases} w_n = \dfrac{2}{3} x_n + \dfrac{1}{3} S x_n, \\ y_n = P_C\left(w_n - \dfrac{1}{2} A w_n\right), \\ t_n = (1 - \tau_n) w_n + \tau_n y_n, \\ x_{n+1} = \dfrac{1}{3(n+1)} \cdot \dfrac{1}{2} x_n + \dfrac{n}{3(n+1)} x_n + \dfrac{2}{3} T P_{C_n}(w_n), \forall n \geq 1, \end{cases} \tag{69}$$

where for any n, C_n and τ_n are picked as above. Then, by Theorem 2, one infers that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$.

5. Concluding Remarks

Compared with the associated theorems of Kraikaew and Saejung [21], Ceng and Shang [23], and Reich et al. [25], our theorems enhance, extend, and develop them in the following ways.

(i) The issue of seeking a point of $\mathrm{VI}(C, A)$ in [21] is developed into our issue of seeking a point of $\mathrm{VI}(C, A) \cap (\cap_{i=1}^{\infty} \mathrm{Fix}(T_i))$ with T_n being of nonexpansivity for any n and $T_0 = T$ being of asymptotical nonexpansivity. The Halpern subgradient extragradient rule for settling the VIP in [21] is developed into our Mann hybrid deepest-descent extragradient approach for handling a HVI with the CFPP and VIP constraints, which is on the basis of Mann's iterative technique, viscosity approximation method, subgradient extragradient rule with linear-search process, and hybrid deepest-descent rule.

(ii) The issue of seeking a point of $\mathrm{VI}(C, A)$ in [25] is developed into our issue of seeking a point of $\mathrm{VI}(C, A) \cap (\cap_{i=1}^{\infty} \mathrm{Fix}(T_i))$ with T_n being of nonexpansivity for any n and $T_0 = T$ being of asymptotical nonexpansivity. The modified projection-type rule with linear-search process for settling the VIP in [25] is developed into Mann hybrid deepest-descent extragradient approach for settling a HVI with the CFPP and VIP constraints, which is on the basis of Mann's iterative technique, viscosity approximation method, subgradient extragradient rule with linear-search process, and hybrid deepest-descent rule.

(iii) The issue of seeking a point of $\mathrm{VI}(C, A) \cap (\cap_{i=1}^{\infty} \mathrm{Fix}(T_i))$ with Lipschitz continuity and sequentially weak continuity mapping A in [23] is developed into our issue of seeking a point of $\mathrm{VI}(C, A) \cap (\cap_{i=1}^{\infty} \mathrm{Fix}(T_i))$ with A being uniform continuity mapping satisfying $\|Az\| \le \liminf_{n \longrightarrow \infty} \|Ax_n\|$ for any $\{x_n\} \subset C$ with $x_n \rightharpoonup z \in C$. The hybrid inertial-type subgradient extragradient rule with linear-search process in [23] is developed into Mann hybrid deepest-descent extragradient approach, e.g., the original inertial-type iteration $''w_n = T_n x_n + \alpha_n (T_n x_n - T_n x_{n-1})''$ is replaced by Mann-type iteration $''w_n = (1 - \sigma_n) x_n + \sigma_n W_n x_n''$, and the original viscosity iteration $''x_{n+1} = \beta_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \beta_n \rho F) T^n z_n''$ is replaced by our hybrid viscosity iteration "$x_{n+1} = P_C[\beta_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \beta_n \rho F) T^n z_n]$." It is worthy to point out that the definition of z_n in the former formula of x_{n+1} is quite different from the definition of z_n in the latter formula of x_{n+1}.

Data Availability

All data generated or analyzed during this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors have read and approved the final manuscript.

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