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Research Article

(Generalized) Incidence and Laplacian-Like Energies

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In this study, for graph Γ with r connected components (also for connected nonbipartite and connected bipartite graphs) and a real number $\varepsilon(\neq 0, 1)$, we found generalized and improved bounds for the sum of ε-th powers of Laplacian and signless Laplacian eigenvalues of Γ. Consequently, we also generalized and improved results on incidence energy (IE) and Laplacian energy-like invariant (LEL).

1. Introduction

Let Γ denote a finite, simple, and undirected graph of order n. The edge and vertex sets of Γ are denoted by $E(\Gamma) = \{e_1, e_2, \ldots, e_m\}$ and $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$, respectively. If the vertex v_i is neighbour to v_j, then write $v_i \sim v_j$. The degree of the vertex $v_i \in V(\Gamma)$, symbolized by d_i, is the number of vertices adjacent to v_i.

The adjacency matrix and the degree matrix of graph Γ are denoted by $A(\Gamma)$ and $D(\Gamma)$, respectively. Let $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma) = 0$ be the eigenvalues of the Laplacian matrix $L(\Gamma)$ of Γ where $L(\Gamma) = D(\Gamma) - A(\Gamma)$ [1, 2]. Let$q_1(\Gamma) \geq q_2(\Gamma) \geq \cdots \geq q_n(\Gamma)$ be the eigenvalues of the signless Laplacian matrix $Q(\Gamma)$ of Γ where $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ [3]. Since the matrices $A(\Gamma)$, $L(\Gamma)$, and $Q(\Gamma)$ are real and symmetric matrices, thus they have real eigenvalues. So, we can write their eigenvalues such that $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$, $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma)$, and $q_1(\Gamma) \geq q_2(\Gamma) \geq \cdots \geq q_n(\Gamma)$, respectively. $L(\Gamma)$ and $Q(\Gamma)$ are semidefinite matrices, according to the Geršgorin disc theorem. From here, all eigenvalues of Laplacian and signless Laplacian matrices of Γ are non-negative integers. In [3], it has been found that $\mu_i(\Gamma) > 0\, (i = 1, 2, \ldots, n-1)$ for a connected nonbipartite graph Γ. Additionally, Γ is a bipartite graph if and only if $q_n = 0$.

The link between the eigenvalues of a graph and the molecular orbital energy levels of $\pi-$ electrons in conjugated hydrocarbons is the most crucial chemical application of graph theory. The total $\pi-$ electron energy in conjugated hydrocarbons is calculated by the sum of absolute values of the eigenvalues corresponding to the molecular graph Γ which has a maximum of four degree generally for the Hüchkel molecular orbital approximation. The energy of Γ given by Gutman in [4] is as follows:

$$E(\Gamma) = \sum_{i=1}^{n} |\lambda_i(\Gamma)|. \tag{1}$$

Nowadays, there is a lot of study on graph energy, as can be seen from the recent papers [5].

The square roots of the eigenvalues of the matrix MM^T are known as the singular values of some $n \times m$ matrix M and its transpose M^T. Recently, in [2], Nikiforov introduced and explored the notion of graph energy. He defined the energy $E(\Gamma)$ of a graph to be the sum of singular values of any matrix M. Clearly, $E(\Gamma) = E(A(\Gamma))$.

Assume that $I(\Gamma)$ represents the vertex-edge incidence matrix of the graph Γ. Then, for Γ having vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, the $(i, j)-$ entry of $I(\Gamma)$ is 0 if v_i is not incident with e_j and 1 if v_i is incident with e_j. Jooyandeh et al. [6]

introduced the notion of incidence energy of a graph. Accordingly, the incidence energy IE of Γ is the sum of the singular values of the incidence matrix of Γ. The following expression is given by Gutman et al. [7]:

$$\mathrm{IE} = \mathrm{IE}(\Gamma) = \sum_{i=1}^{n} \sqrt{q_i(\Gamma)}. \tag{2}$$

Some basic information on IE may be seen in [6, 7].

As abovementioned, one can compute the incidence energy of a graph Γ by calculating the eigenvalues of signless Laplacian matrix of Γ. However, the problem is much more complicated for some classes of graphs due to the computational complexity of finding eigenvalues of signless Laplacian matrix. Thus, to compute the invariant for some classes of graphs, it is crucial to find their lower and upper bounds. Zhou [8] found the upper bounds on the incidence energy in terms of the first Zagreb index. Different lower and upper bounds on IE have been studied by various researchers.

In [9], associated to the Laplacian eigenvalues, authors introduced the invariant called the Laplacian energy-like invariant (or Laplacian-like energy) which is defined as follows:

$$\mathrm{LEL} = \mathrm{LEL}(\Gamma) = \sum_{i=1}^{n-1} \sqrt{\mu_i}. \tag{3}$$

Firstly, it was examined in [9] that LEL and Laplacian energy have similar characteristics. It has also been shown that it resembles to graph energy much more closely. For detailed information, see [10].

For a graph Γ of order n and a real number ε not equal to 0 and 1 in [8], the sum of the εth powers of the nonzero Laplacian eigenvalues is defined as follows:

$$\sigma_\varepsilon = \sigma_\varepsilon(\Gamma) = \sum_{i=1}^{n-1} \mu_i^\varepsilon. \tag{4}$$

If ε is 0 and 1, then the cases are trivial as $\sigma_0 = n-1$ and $\sigma_1 = 2m$, where m denotes the cardinality of the edge set of Γ. It is clear that $\sigma_{1/2}$ is equal to LEL. We should note that $n\sigma_{-1}$ is also equal to the Kirchhoff index of Γ (for more detail (one can see [11, 12]). Many studies on σ_ε have recently been published in the literature. For details, see [13, 14].

Similar to the definitions of IE, LEL, and σ_ε, Akbari et al. [15] defined the sum of the εth powers of the signless Laplacian eigenvalues of Γ as follows:

$$s_\varepsilon = s_\varepsilon(\Gamma) = \sum_{i=1}^{n} q_i^\varepsilon, \tag{5}$$

and they also gave some connections between σ_ε and s_ε. If ε is 0 and 1, then the cases are trivial as $s_0 = n$ and $s_1 = 2m$. Note that $s_{1/2}$ is equal to the incidence energy IE. We observed that Laplacian eigenvalues and signless Laplacian eigenvalues of bipartite graphs are equal [1, 3, 16]. Therefore, for bipartite graphs, σ_ε and s_ε are equal, and hence, LEL is equal to IE [17]. Recently, different properties, as well as different lower and upper bounds of s_ε have been established in [15, 17, 18].

Lemma 1 (see [19]). *Let $a_1, a_2, \ldots, a_n$ be nonnegative numbers. Then,*

$$\begin{aligned} n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}\right] &\le n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2 \\ &\le n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}\right]. \end{aligned} \tag{6}$$

The equality among them holds if and only if $a_1 = a_2 = \cdots = a_n$.

We aim to obtain some strong bounds using the efficient inequality technique in Lemma 1 for main results. Also, we give some generalizations for s_ε, σ_ε, indicence energy IE, and the Laplacian energy-like invariant LEL of graphs (with r connected components, connected nonbipartite, and connected bipartite).

The following main lemmas are required for our main results.

Let $t = t(\Gamma)$ denote the number of spanning trees of a graph Γ. Let $\Gamma_1 \times \Gamma_2$ be the Cartesian product of the graphs Γ_1 and Γ_2. We define the following number for a graph Γ.

$$t_1 = t_1(\Gamma) = \frac{2t(\Gamma \times K_2)}{t(\Gamma)}. \tag{7}$$

Lemma 2 (see [20]). *If Γ is a connected bipartite graph with n vertices, then $\prod_{i=1}^{n-1}\mu_i = \prod_{i=1}^{n-1} q_i = nt(\Gamma)$. If Γ is a connected nonbipartite graph with n vertices, then $\prod_{i=1}^{n} q_i = t_1$.*

Lemma 3 (see [21]). *Let Γ be a connected graph with $n \ge 3$ vertices and maximum degree Δ. Then, $\mu_2 = \cdots = \mu_{n-1}$ if and only if $\Gamma \cong K_n$ or $\Gamma \cong K_{1,n-1}$ or $\Gamma \cong K_{\Delta,\Delta}$.*

Lemma 4 (see [21]). *Let Γ be a connected graph of order n. Then, $\mu_1 = \cdots = \mu_{n-1}$ if and only if $\Gamma \cong K_n$.*

Lemma 5 (see [3]). *The spectra of $L(\Gamma)$ and $Q(\Gamma)$ coincide if and only if the graph Γ is bipartite.*

2. Main Results

After above preliminary informations, we are ready to give our main results.

It is well known that if a graph Γ has r connected components, the spectrum of Γ is the union of the spectra of Γ_i, $1 \le i \le r$ (and multiplicities are added). The same also holds for the Laplacian and the signless Laplacian spectrum.

Firstly, we give lower and upper bounds on s_ε and σ_ε for a graph with r connected components.

Theorem 6. *Let Γ be a graph of order n with r connected components such that p of them are connected bipartite. Then,*

$$\begin{aligned}\sqrt{\sigma_{2\varepsilon} + (n-r)(n-r-1)R_{n-r}^{2\varepsilon/(n-r)}} \le \sigma_\varepsilon \le \sqrt{\sigma_{2\varepsilon}(n-r-1) + (n-r)R_{n-r}^{2\varepsilon/(n-r)}},\\ \sqrt{s_{2\varepsilon} + (n-p)(n-p-1)\Delta_{n-p}^{2\varepsilon/(n-p)}} \le s_\varepsilon \le \sqrt{s_{2\varepsilon}(n-p-1) + (n-p)\Delta_{n-p}^{2\varepsilon/(n-p)}},\end{aligned} \tag{8}$$

where $R_{n-r} = \prod_{i=1}^{n-r}\mu_i$ and $\Delta_{n-p} = \prod_{i=1}^{n-p} q_i$. Equalities occur in both bounds if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-r}$ and $q_1 = q_2 = \cdots = q_{n-p}$, respectively.

Proof. Note that 0 is an eigenvalue of Laplacian matrix with multiplicity r. Taking $a_i = \mu_i^{2\varepsilon}$, replacing n by $n-r$ in Lemma 1, we obtain the following equation:

$$W \le (n-r)\sum_{i=1}^{n-r}\mu_i^{2\varepsilon} - \left(\sum_{i=1}^{n-r}\mu_i^{\varepsilon}\right)^2 \le (n-r)W, \tag{9}$$

where

$$W = (n-r)\left[\frac{1}{n-r}\sum_{i=1}^{n-r}\mu_i^{2\varepsilon} - \left(\prod_{i=1}^{n-r}\mu_i^{2\varepsilon}\right)^{1/(n-r)}\right]. \tag{10}$$

Since $\sum_{i=1}^{n-r}\mu_i^{\varepsilon} = \sigma_\varepsilon$, we have the following equation:

$$W \le (n-r)\sigma_{2\varepsilon} - \sigma_\varepsilon^2 \le (n-r)W. \tag{11}$$

Observe that

$$\begin{aligned} W &= (n-r)\left[\frac{1}{n-r}\sum_{i=1}^{n-r}\mu_i^{2\varepsilon} - \left(\prod_{i=1}^{n-r}\mu_i^{2\varepsilon}\right)^{1/(n-r)}\right]\\ &= (n-r)\left[\frac{1}{n-r}\sigma_{2\varepsilon} - R_{n-r}^{2\varepsilon/(n-r)}\right]\\ &= \sigma_{2\varepsilon} - (n-r)R_{n-r}^{2\varepsilon/(n-r)}.\end{aligned} \tag{12}$$

Hence, we get the result.

From Lemma 1, the equalities hold if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-r}$.

It is known that 0 is an eigenvalues of signless Laplacian matrix with multiplicity p. For s_ε, the proof is similar, replacing n by $n-p$ and taking $a_i = q_i^{2\varepsilon}$ in Lemma 1.

As a special case, if we take $\varepsilon = 1/2$, we get the bounds for the LEL and IE given as follows: □

Corollary 7. *Let Γ be a graph of order n with r connected components such that p of them are connected bipartite. Then,*

$$\begin{aligned}\sqrt{2m + (n-r)(n-r-1)R_{n-r}^{1/(n-r)}} \le \mathrm{LEL} \le \sqrt{2m(n-r-1) + (n-r)R_{n-r}^{1/(n-r)}},\\ \sqrt{2m + (n-p)(n-p-1)\Delta_{n-p}^{1/(n-p)}} \le IE \le \sqrt{2m(n-p-1) + (n-p)\Delta_{n-p}^{1/(n-p)}},\end{aligned} \tag{13}$$

where $R_{n-r} = \prod_{i=1}^{n-r}\mu_i$ and $\Delta_{n-p} = \prod_{i=1}^{n-p} q_i$. Equalities hold in both bounds if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-r}$ and $q_1 = q_2 = \cdots = q_{n-p}$, respectively.

Note that, if we take $r = 1$ and $p = 0$ in Theorem 6, we reach the following result.

Corollary 8. *Let Γ be a nonbipartite connected graph of order n. Let t and t_1 be as given in Lemma 2. Then,*

$$\sqrt{\sigma_{2\varepsilon} + (n-1)(n-2)(nt)^{2\varepsilon/(n-1)}} \le \sigma_\varepsilon \le \sqrt{\sigma_{2\varepsilon}(n-2) + (n-1)(nt)^{2\varepsilon/(n-1)}}, \tag{14}$$

and

$$\sqrt{s_{2\varepsilon} + n(n-1)t_1^{2\varepsilon/n}} \le s_\varepsilon \le \sqrt{s_{2\varepsilon}(n-1) + nt_1^{2\varepsilon/n}}. \quad (15)$$

Inequalities (14) and (15) hold in both bounds if and only if $\Gamma \cong K_n$ and $q_1 = q_2 = \cdots = q_n$, respectively.

Taking $\varepsilon = 1/2$ in Corollary 7, we have the following corollary.

Corollary 9. *Let Γ be a nonbipartite connected graph of order n and t and t_1 be as given in Lemma 2. Then,*

$$\sqrt{2m + (n-1)(n-2)(nt)^{1/(n-1)}} \le \text{LEL} \le \sqrt{2m(n-2) + (n-1)(nt)^{1/(n-1)}}, \quad (16)$$

and

$$\sqrt{2m + n(n-1)t_1^{1/n}} \le \text{IE} \le \sqrt{2m(n-1) + nt_1^{1/n}}. \quad (17)$$

Equalities (16) and (17) hold in both bounds if and only if $\Gamma \cong K_n$ and $q_1 = q_2 = \cdots = q_n$, respectively.

Now, we consider the bipartite graph case of the above theorem (Theorem 6). In the next corollary, we actually improved the results which were obtained in [22].

Corollary 10. *Let Γ be a connected bipartite graph with n vertices. Let t be as given in Lemma 2. Then,*

$$\sqrt{s_{2\varepsilon} + (n-1)(n-2)(nt)^{2\varepsilon/(n-1)}} \le s_\varepsilon = \sigma_\varepsilon \le \sqrt{s_{2\varepsilon}(n-2) + (n-1)(nt)^{2\varepsilon/(n-1)}}, \quad (18)$$

and

$$\sqrt{2m + (n-1)(n-2)(nt)^{1/(n-1)}} \le \text{IE} = \text{LEL} \le \sqrt{2m(n-2) + (n-1)(nt)^{1/(n-1)}}. \quad (19)$$

Equalities (18) and (19) hold in both bounds if and only if $\Gamma \cong K_n$, $\Gamma \cong K_{1,n-1}$, or $\Gamma \cong K_{\Delta,\Delta}$, where Δ is the maximum degree.

As it is well known in graph theory, every tree is bipartite. In addition, for a tree T, $m = n-1$ and $t = 1$. From Corollary 10, we have the following.

Corollary 11. *Let T be a tree of order n. Then,*

$$\begin{aligned} &\sqrt{\sigma_{2\varepsilon} + (n-1)(n-2)n^{2\varepsilon/(n-1)}} \le s_\varepsilon(T) = \sigma_\varepsilon(T) \le \sqrt{\sigma_{2\varepsilon}(n-2) + (n-1)n^{2\varepsilon/(n-1)}}, \\ &\sqrt{(n-1)\left[2 + (n-2)n^{1/(n-1)}\right]} \le \text{IE}(T) = \text{LEL}(T) \le \sqrt{(n-1)\left[2(n-2) + n^{1/(n-1)}\right]}. \end{aligned} \quad (20)$$

Equalities hold in both bounds if and only if $T \cong K_{1,n-1}$.

Remark 12. It is pertinent to mention here that in equations (15) and (17), for connected nonbipartite graphs, we recover the same lower bounds as in Theorem 2.6 (i) and Corollary 2.7 (i) in [22] through a different approach. For connected bipartite graphs, it can be seen that lower bounds (18) and (19) are better than lower bounds obtained in Theorem 2.6 (ii) and Corollary 2.7 (ii) in [22], respectively. Moreover, we obtained extra upper bounds for the relevant parameters and generalized them as different forms [22].

3. Accomplishment Remarks

In this paper, we have obtained new results for the graph invariants s_ε and σ_ε of a simple graph Γ with r connected components (connected nonbipartite and connected bipartite), where $\varepsilon(\neq 0, 1)$ is a real number. Also, as a result, we generalized and improved the results on incidence energy (IE) and Laplacian energy-like invariant (LEL).

Data Availability

All data and materials used to obtain the results are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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