Research Article

(Generalized) Incidence and Laplacian-Like Energies

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In this study, for graph $\Gamma$ with $r$ connected components (also for connected nonbipartite and connected bipartite graphs) and a real number $\epsilon \neq 0, 1$, we found generalized and improved bounds for the sum of $\epsilon$-th powers of Laplacian and signless Laplacian eigenvalues of $\Gamma$. Consequently, we also generalized and improved results on incidence energy (IE) and Laplacian energy-like invariant (LEL).

1. Introduction

Let $\Gamma$ denote a finite, simple, and undirected graph of order $n$. The edge and vertex sets of $\Gamma$ are denoted by $E(\Gamma) = \{e_1, e_2, \ldots, e_m\}$ and $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$, respectively. If the vertex $v_i$ is neighbour to $v_j$, then write $v_i \sim v_j$. The degree of the vertex $v_i \in V(\Gamma)$, symbolized by $d_i$, is the number of vertices adjacent to $v_i$.

The adjacency matrix and the degree matrix of graph $\Gamma$ are denoted by $A(\Gamma)$ and $D(\Gamma)$, respectively. Let $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma) = 0$ be the eigenvalues of the Laplacian matrix $L(\Gamma)$ of $\Gamma$ where $L(\Gamma) = D(\Gamma) - A(\Gamma)$ [1, 2]. Let $q_1(\Gamma) \geq q_2(\Gamma) \geq \cdots \geq q_m(\Gamma)$ be the eigenvalues of the signless Laplacian matrix $Q(\Gamma)$ of $\Gamma$ where $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ [3]. Since the matrices $A(\Gamma)$, $L(\Gamma)$, and $Q(\Gamma)$ are real and symmetric matrices, thus they have real eigenvalues. So, we can write their eigenvalues such that $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$, $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma)$, and $q_1(\Gamma) \geq q_2(\Gamma) \geq \cdots \geq q_m(\Gamma)$, respectively. $L(\Gamma)$ and $Q(\Gamma)$ are semidefinite matrices, according to the Gersgorin disc theorem. From here, all eigenvalues of Laplacian and signless Laplacian matrices of $\Gamma$ are non-negative integers. In [3], it has been found that $\mu_i(\Gamma) > 0 (i = 1, 2, \ldots, n - 1)$ for a connected nonbipartite graph $\Gamma$. Additionally, $\Gamma$ is a bipartite graph if and only if $q_m = 0$.

The link between the eigenvalues of a graph and the molecular orbital energy levels of $\pi$-electrons in conjugated hydrocarbons is the most crucial chemical application of graph theory. The total $\pi$-electron energy in conjugated hydrocarbons is calculated by the sum of absolute values of the eigenvalues corresponding to the molecular graph $\Gamma$ which has a maximum of four degree generally for the Hückel molecular orbital approximation. The energy of $\Gamma$ given by Gutman in [4] is as follows:

$$E(\Gamma) = \sum_{i=1}^{n} |\lambda_i(\Gamma)|.$$ (1)

Nowadays, there is a lot of study on graph energy, as can be seen from the recent papers [5].

The square roots of the eigenvalues of the matrix $MM^T$ are known as the singular values of some $n \times m$ matrix $M$ and its transpose $M^T$. Recently, in [2], Nikiforov introduced and explored the notion of graph energy. He defined the energy $E(\Gamma)$ of a graph to be the sum of singular values of any matrix $M$. Clearly, $E(\Gamma) = E(A(\Gamma))$.

Assume that $I(\Gamma)$ represents the vertex-edge incidence matrix of the graph $\Gamma$. Then, for $\Gamma$ having vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, the $(i, j)$-entry of $I(\Gamma)$ is 0 if $v_i$ is not incident with $e_j$ and 1 if $v_i$ is incident with $e_j$. Jooyandeh et al. [6]
introduced the notion of incidence energy of a graph. Accordingly, the incidence energy IE of Γ is the sum of the singular values of the incidence matrix of Γ. The following expression is given by Gutman et al. [7]:

\[ IE = IE(\Gamma) = \sum_{i=1}^{n} \sqrt{q_i(\Gamma)}. \]  

(2)

Some basic information on IE may be seen in [6, 7].

As abovementioned, one can compute the incidence energy of a graph Γ by calculating the eigenvalues of the incidence matrix of Γ. However, the problem is much more complicated for some classes of graphs due to the computational complexity of finding eigenvalues of the incidence matrix. Thus, to compute the invariant for some classes of graphs, it is crucial to find their lower and upper bounds. Zhou [8] found the upper bounds on the incidence energy in terms of the first Zagreb index. Different lower and upper bounds on IE have been studied by various researchers.

In [9], associated to the Laplacian eigenvalues, authors introduced the invariant called the Laplacian energy-like invariant (or Laplacian-like energy) which is defined as follows:

\[ LEL = LEL(\Gamma) = \sum_{i=1}^{n-1} \mu_i. \]  

(3)

Firstly, it was examined in [9] that LEL and Laplacian energy have similar characteristics. It has also been shown that it resembles to graph energy much more closely. For detailed information, see [10].

For a graph Γ of order n and a real number ε not equal to 0 and 1 in [8], the sum of the εth powers of the nonzero Laplacian eigenvalues is defined as follows:

\[ \sigma_\varepsilon = \sigma_\varepsilon(\Gamma) = \sum_{i=1}^{n-1} \mu_i^\varepsilon. \]  

(4)

If ε is 0 and 1, then the cases are trivial as \( \sigma_0 = n - 1 \) and \( \sigma_1 = 2m \), where m denotes the cardinality of the edge set of Γ. It is clear that \( \sigma_{1/2} \) is equal to LEL. We should note that \( n\sigma_{-1} \) is also equal to the Kirchhoff index of Γ (for more detail (one can see [11, 12]). Many studies on \( \sigma_\varepsilon \) have recently been published in the literature. For details, see [13, 14].

Similar to the definitions of IE, LEL, and \( \sigma_\varepsilon \), Akbari et al. [15] defined the sum of the εth powers of the signless Laplacian eigenvalues of Γ as follows:

\[ s_\varepsilon = s_\varepsilon(\Gamma) = \sum_{i=1}^{n} q_i^\varepsilon, \]  

(5)

and they also gave some connections between \( \sigma_\varepsilon \) and \( s_\varepsilon \). If ε is 0 and 1, then the cases are trivial as \( s_0 = n \) and \( s_1 = 2m \). Note that \( s_{1/2} \) is equal to the incidence energy IE. We observed that Laplacian eigenvalues and signless Laplacian eigenvalues of bipartite graphs are equal [1, 3, 16]. Therefore, for bipartite graphs, \( \sigma_\varepsilon \) and \( s_\varepsilon \) are equal, and hence, LEL is equal to IE [17]. Recently, different properties, as well as different lower and upper bounds of \( s_\varepsilon \) have been established in [15, 17, 18].

**Lemma 1** (see [19]). Let \( a_1, a_2, \ldots, a_n \) be nonnegative numbers. Then,

\[ n\left[ 1 - \frac{1}{n} \sum_{i=1}^{n} a_i - \left( \prod_{i=1}^{n} a_i \right)^{1/n} \right] \leq n - 1 \sum_{i=1}^{n} a_i - \left( \prod_{i=1}^{n} a_i \right)^{2} \]

\[ \leq n(n - 1) \left[ 1 - \frac{1}{n} \sum_{i=1}^{n} a_i - \left( \prod_{i=1}^{n} a_i \right)^{1/n} \right]. \]  

(6)

\[ t_1 = t_1(\Gamma) = \frac{2t(\Gamma \times K_2)}{t(\Gamma)}. \]  

(7)

**Lemma 2** (see [20]). If Γ is a connected bipartite graph with n vertices, then \( \prod_{i=1}^{n} a_i = \prod_{i=1}^{n} q_i = n t(\Gamma) \). If Γ is a connected nonbipartite graph with n vertices, then \( \prod_{i=1}^{n} \frac{1}{q_i} = t_1 \).

**Lemma 3** (see [21]). Let Γ be a connected graph with \( n \geq 3 \) vertices and maximum degree Δ. Then, \( \mu_2 = \cdots = \mu_{n-1} \) if and only if Γ \( \cong K_n \) or Γ \( \cong K_{1,n-1} \) or Γ \( \cong K_{n,\Delta} \).

**Lemma 4** (see [21]). Let Γ be a connected graph of order n. Then, \( \mu_1 = \cdots = \mu_{n-1} \) if and only if Γ \( \cong K_n \).
Lemma 5 (see [3]). The spectra of $L(\Gamma)$ and $Q(\Gamma)$ coincide if and only if the graph $\Gamma$ is bipartite.

2. Main Results

After above preliminary informations, we are ready to give our main results.

It is well known that if a graph $\Gamma$ has $r$ connected components, the spectrum of $\Gamma$ is the union of the spectra of $\Gamma_i$, $1 \leq i \leq r$ (and multiplicities are added). The same also holds for the Laplacian and the signless Laplacian spectrum.

Firstly, we give lower and upper bounds on $s_\epsilon$ and $\sigma_\epsilon$ for a graph with $r$ connected components.

**Theorem 6.** Let $\Gamma$ be a graph of order $n$ with $r$ connected components such that $p$ of them are connected bipartite. Then,

$$W = (n - r)\left[ \frac{1}{n - r} \sum_{i=1}^{n-r} \mu_i^2 - \left( \frac{1}{n-r} \sum_{i=1}^{n-r} \mu_i \right)^2 \right];$$

where $R_{n-r} = \prod_{i=1}^{n-r} \mu_i$ and $\Delta_{n-p} = \prod_{i=1}^{n-p} q_i$. Equalities occur in both bounds if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-r}$ and $q_1 = q_2 = \cdots = q_{n-p}$, respectively.

**Proof.** Note that 0 is an eigenvalue of Laplacian matrix with multiplicity $r$. Taking $a_i = \mu_i^2$, replacing $n$ by $n - r$ in Lemma 1, we obtain the following equation:

$$W \leq (n - r) \sum_{i=1}^{n-r} \mu_i^2 - \left( \sum_{i=1}^{n-r} \mu_i \right)^2 \leq (n - r)W,$$

where

$$W = (n - r)\left[ \frac{1}{n - r} \sum_{i=1}^{n-r} \mu_i^2 - \left( \frac{1}{n-r} \sum_{i=1}^{n-r} \mu_i \right)^2 \right].$$

Since $\sum_{i=1}^{n-r} \mu_i^2 = \sigma$e, we have the following equation:

$$W \leq (n - r)\sigma - \sigma^2 \leq (n - r)W.$$ 

Observe that

$$\sqrt{2m + (n - r)(n - r - 1)R_{n-r}^{1/(n-r)}} \leq LEL \leq \sqrt{2m (n - r - 1) + (n - r)R_{n-r}^{1/(n-r)}},$$

$$\sqrt{2m + (n - p)(n - p - 1)\Delta_{n-p}^{1/(n-p)}} \leq IEL \leq \sqrt{2m (n - p - 1) + (n - p)\Delta_{n-p}^{1/(n-p)}},$$

where $R_{n-r} = \prod_{i=1}^{n-r} \mu_i$ and $\Delta_{n-p} = \prod_{i=1}^{n-p} q_i$. Equalities hold in both bounds if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-r}$ and $q_1 = q_2 = \cdots = q_{n-p}$, respectively.

Note that, if we take $r = 1$ and $p = 0$ in Theorem 6, we reach the following result.

**Corollary 7.** Let $\Gamma$ be a graph of order $n$ with $r$ connected components such that $p$ of them are connected bipartite. Then,
and
\[ \sqrt{2s_2 + n(n-1)t_1^{2/3}} \leq s_2 \leq \sqrt{2s_2 + (n-1) + nt_1^{2/3}}. \]  
\hfill (15)

Inequalities (14) and (15) hold in both bounds if and only if \( \Gamma \cong K_n \) and \( q_1 = q_2 = \cdots = q_n \), respectively. Moreover, we have
\[ \sqrt{2m + (n-1)(n-2)(nt)^{1/(n-1)}} \leq \text{LEL} \leq \sqrt{2m(n-2) + (n-1)(nt)^{1/(n-1)}}, \]  
\hfill (16)

and
\[ \sqrt{2m + n(n-1)t_1^{1/3}} \leq \text{IE} \leq \sqrt{2m(n-1) + nt_1^{1/3}}. \]  
\hfill (17)

Equalities (16) and (17) hold in both bounds if and only if \( \Gamma \cong K_n \) and \( q_1 = q_2 = \cdots = q_n \), respectively.

Equalities hold in both bounds if and only if \( \Gamma \cong K_n \), \( \Gamma \cong K_{1,n-1} \), or \( \Gamma \cong K_{\Delta,n} \), where \( \Delta \) is the maximum degree.

Taking \( \varepsilon = 1/2 \) in Corollary 7, we have the following corollary.

**Corollary 9.** Let \( \Gamma \) be a nonbipartite connected graph of order \( n \) and \( t \) and \( t_1 \) be as given in Lemma 2. Then,
\[ \sqrt{\sigma_{2\varepsilon} + (n-1)(n-2)n^{2\varepsilon/(n-1)}} \leq \sigma_{\varepsilon}(T) = \sigma_{\varepsilon}(T) \leq \sqrt{\sigma_{2\varepsilon} + (n-1)n^{2\varepsilon/(n-1)}}, \]
\[ \sqrt{(n-1)[2 + (n-2)n^{1/(n-1)}]} \leq \text{IE}(T) = \text{LEL}(T) \leq \sqrt{(n-1)[2(n-2) + n^{1/(n-1)}]}. \]  
\hfill (20)

Equalities hold in both bounds if and only if \( T \cong K_{1,n-1} \).

**Remark 12.** It is pertinent to mention here that in equations (15) and (17), for connected nonbipartite graphs, we recover the same lower bounds as in Theorem 2.6 (i) and Corollary 2.7 (i) in [22] through a different approach. For connected bipartite graphs, it can be seen that lower bounds (18) and (19) are better than lower bounds obtained in Theorem 2.6 (ii) and Corollary 2.7 (ii) in [22], respectively. Moreover, we obtained extra upper bounds for the relevant parameters and generalized them as different forms [22].

3. **Accomplishment Remarks**

In this paper, we have obtained new results for the graph invariants \( s_2 \) and \( \sigma_{\varepsilon} \) of a simple graph \( \Gamma \) with \( r \) connected components (connected nonbipartite and connected bipartite), where \( \varepsilon (\neq 0, 1) \) is a real number. Also, as a result, we...
generalized and improved the results on incidence energy (IE) and Laplacian energy-like invariant (LEL).

Data Availability
All data and materials used to obtain the results are included within the article.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

References