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Research Article

Well-Posedness of a Hirota–Satsuma System Posed on a Half Line

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This paper studies the initial boundary value problem of the Hirota–Satsuma system posed on the half line. $\begin{cases} u_t - \alpha(u_{xxx} + 6uu_x) = 2\beta vv_x, & x > 0, t > 0, \\ v_t + v_{xxx} + 3uv_x = 0, & x > 0, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x > 0, \\ u(0,t) = f(t), v(0,t) = g(t), & t > 0. \end{cases}$. For $-1/8 < s < 3/2, s \neq 1/2$, we demonstrate that the abovementioned system is locally well-posed in $H^s(\mathbb{R}) \times H^{1+s}(\mathbb{R})$ by utilizing several analytic boundary forcing operators.

1. Introduction

In 1981, Hirota and Satsuma [1] introduced

$$\begin{cases} u_t - \alpha\left(u_{xxx} + 6uu_x\right) = 2\beta vv_x, & x \in \mathbb{R}, t \geq 0, \\ v_t + v_{xxx} + 3uv_x = 0, & x \in \mathbb{R}, t \geq 0, \end{cases} \tag{1}$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$; $u = u(x,t)$, $v = v(x,t)$ are the real functions modeling the interactions of two long waves with different dispersion relations. Nowadays, (1) is called the Hirota–Satsuma system. Additionally, Hirota and Ohta in [2] derived the Hirota–Satsuma system as a reduction of a special hierarchy of coupled bilinear equations.

The initial value problem for the Hirota–Satsuma system on the whole line and periodic domain has been extensively studied. In 2005, Angulo [3] proved that the system is locally well-posed in $H^s_{\text{periodic}}(0,L) \times H^s_{\text{periodic}}(0,L)$, for $s \geq 0$, when $\alpha = -1$, and globally well-posed in $H^s_{\text{periodic}}(0,L) \times H^s_{\text{periodic}}(0,L)$ for $s \geq 1$, when $\alpha \neq -1, 0$. Furthermore, in 2007, Panthee and Silva [4] verified that the system is locally well-posed in $H^s_{\text{periodic}}(0,L) \times H^{1+s}_{\text{periodic}}(0,L)$, for $s \geq -1/2$, when $\alpha = -1$, and globally well-posed in $H^s_{\text{periodic}}(0,L) \times H^{1+s}_{\text{periodic}}(0,L)$, for $s \geq -14/3$, when $\alpha = -1$. Moreover, in 1994, Feng [5] demonstrated that the system is locally well-posed$H^s(\mathbb{R}) \times H^s(\mathbb{R})$, for $s > 2$, and in 2022, Zhao and Lv [6] confirmed that the system is locally well-posed in $H^s(\mathbb{R}) \times H^{1+s}(\mathbb{R})$, for $s \geq -1/8$, when $\alpha = -1, \beta = 1$.

Without losing generalization, for the remainder of this paper, we assume that $\alpha = -1$ and $\beta = 1$.

This study will investigate the initial boundary value problem for the Hirota–Satsuma system posed on the half line as follows:

$$\begin{cases} u_t + u_{xxx} + 6uu_x = 2vv_x, & x > 0, t > 0, \\ v_t + v_{xxx} + 3uv_x = 0, & x > 0, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x > 0, \\ u(0,t) = f(t), v(0,t) = g(t), & t > 0. \end{cases} \tag{2}$$

The main result regarding the IBVP (2) can be stated as follows:

Theorem 1. *Let $-1/8 < s < 3/2, s \neq 1/2$. Then, there exists*

$$T = T\left(\|u_0\|_{H^s(\mathbb{R}^+)}, \|v_0\|_{H^{1+s}(\mathbb{R}^+)}, \|f\|_{H^{(s+1)/3}(\mathbb{R}^+)}, \|g\|_{H^{((s+1)/3}(\mathbb{R}^+)}\right), \tag{3}$$

and a local solution $(u(\cdot,t), v(\cdot,t))$ of IBVP (2), in the distributional sense, such that

$$(u(\cdot,t), v(\cdot,t)) \in \mathscr{C}\left([0,T]; H^s\left(\mathbb{R}^+\right) \times H^{1+s}\left(\mathbb{R}^+\right)\right). \tag{4}$$

Moreover, the mapping $(u_0, v_0, f, g) \mapsto (u(\cdot,t), v(\cdot,t))$ is locally Lipschitz-continuous from $H^s(\mathbb{R}^+) \times H^{1+s}$

$(\mathbb{R}^+)\times H^{(s+1)/3}(\mathbb{R}^+)\times H^{(s+1)/3}(\mathbb{R}^+)$ to $\mathscr{C}([0,T];H^s(\mathbb{R}^+)\times H^{1+s}(\mathbb{R}^+))$.

2. Presenting the Solution

2.1. Solution to the Initial Value Problem. We define the linear unitary group

$$e^{-t\partial_x^3}:\mathcal{S}'(\mathbb{R})\longrightarrow\mathcal{S}'(\mathbb{R}),\tag{5}$$

which associated to the linear KdV equation as

$$e^{-t\partial_x^3}\phi(x)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{i\xi x}e^{it\xi^3}\widehat{\phi(x)}d\xi,\tag{6}$$

so that

$$\begin{cases}\left(\partial_t+\partial_x^3\right)e^{-t\partial_x^3}\phi(x,t)=0, & (x,t)\in\mathbb{R}\times\mathbb{R},\\ e^{-t\partial_x^3}(x,0)=\phi(x), & x\in\mathbb{R}.\end{cases}\tag{7}$$

Thus, if we set

$$w(x,t)=e^{-t\partial_x^3}\phi(x),\tag{8}$$

then w solves the linear problem

$$\begin{cases}w_t(x,t)+w_{xxx}(x,t)=0, & (x,t)\in\mathbb{R}^*\times\mathbb{R},\\ u(x,0)=\phi(x), & x\in\mathbb{R}.\end{cases}\tag{9}$$

2.2. Solution to the Boundary Value Problem. First, we introduce the Duhamel boundary forcing operator suggested by Colliander and Kenig [7]. Its definition is

$$\begin{aligned}\mathscr{V}f(x,t)&=3\int_0^t e^{-(t-t')\partial_x^3}\delta_0(x)\mathscr{I}_{-2/3}f(t')dt'\\ &=3\int_0^t A\left(\frac{x}{(t-t')^{1/3}}\right)\frac{\mathscr{I}_{-2/3}f(t')}{(t-t')^{1/3}}dt',\end{aligned}\tag{10}$$

which is defined for all $f\in C_0^\infty(\mathbb{R}^+)$ and A signifies the Airy function

$$A(x)=\frac{1}{2\pi}\int_{\xi}e^{ix\xi}e^{i\xi^3}d\xi.\tag{11}$$

From this definition, we observe that

$$\begin{cases}\left(\partial_t+\partial_x^3\right)\mathscr{V}f(x,t)=3\delta_0(x)\mathscr{I}_{-2/3}f(t), & (x,t)\in\mathbb{R}\times\mathbb{R},\\ \mathscr{V}f(x,0)=0, & x\in\mathbb{R}.\end{cases}\tag{12}$$

Next, we introduce the generalization of operator $\mathscr{V}$ used in Holmer [8].

For $f\in C_0^\infty(\mathbb{R}^+)$ and $\lambda\in\mathbb{C}$ with $\mathrm{Re}\lambda>-3$, we have

$$\mathscr{V}_+^\lambda f(x,t)=\left[\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}*\mathscr{V}\left(\mathscr{I}_{-\lambda/3}f\right)(\cdot,t)\right](x),\tag{13}$$

where $x_-^{\lambda-1}/\Gamma(\lambda)=e^{i\pi\lambda}(-x)_+^{\lambda-1}/\Gamma(\lambda)$. Then, by using (12), we obtain

$$\left(\partial_t+\partial_x^3\right)\mathscr{V}_+^\lambda f(x,t)=3\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}\mathscr{I}_{-2/3-\lambda/3}f(t).\tag{14}$$

For any $-1/8<s<3/2$, $s\neq 1/2$, we can address IBVP (2) by replacing $\mathscr{V}$ in (12) with $\mathscr{V}_+^\lambda$ for a suitable $\lambda=\lambda(s)$.

The proofs of the following Lemmas exhibited in this section are presented in [8].

Lemma 1 (Spatial continuity and decay properties for $\mathscr{V}_+^\lambda f(x,t)$). If $f\in C_0^\infty(\mathbb{R}^+)$ and fix $t\geq 0$. Then, we have

$$\mathscr{V}_+^0 f=\mathscr{V}f.\tag{15}$$

For $\lambda>-2$, $\mathscr{V}_+^\lambda f(x,t)$ is continuous in x for all $x\in\mathbb{R}$. For $-2\leq\lambda\leq 1$ and $0\leq t\leq 1$, $\mathscr{V}_+^\lambda f(x,t)$ satisfies the following decay bounds:

$$\begin{aligned}\left|\mathscr{V}_+^\lambda f(x,t)\right|&\leq c_{m,\lambda,f}\langle x\rangle^{-m}, x\geq 0, m\geq 0,\\ \left|\mathscr{V}_+^\lambda f(x,t)\right|&\leq c_{\lambda,f}\langle x\rangle^{\lambda-1}, x\leq 0.\end{aligned}\tag{16}$$

Lemma 2 (Values of $\mathscr{V}_+^\lambda f(x,t)$ at $x=0$). For $f\in C_0^\infty(\mathbb{R}^+)$ and $Re\lambda>-2$, we have

$$\mathscr{V}_+^\lambda f(0,t)=e^{i\pi\lambda}f(t).\tag{17}$$

Now, by adopting [8], we obtain the solution to the boundary value problem by using the $\mathscr{V}_+^\lambda$ class. Specifically, let $-1<\lambda<1$, $h\in C_0^\infty(\mathbb{R}^+)$, if we set

$$w(x,t)=\mathscr{V}_+^\lambda h(x,t),\tag{18}$$

from Lemmas 1 and 2,

$$\begin{aligned}w(0,t)&=f(t)\\ &=e^{i\pi\lambda_1}h(t).\end{aligned}\tag{19}$$

Moreover, if $f(t)$ is given and we set

$$h(t)=e^{-i\pi\lambda}f(t),\tag{20}$$

then w solves the linear problem

$$\begin{cases}w_t(x,t)+w_{xxx}(x,t)=0, & (x,t)\in\mathbb{R}^*\times\mathbb{R},\\ w(x,0)=0, & x\in\mathbb{R},\\ w(0,t)=f(t), & t>0.\end{cases}\tag{21}$$

2.3. Solution to the Initial-Boundary Value Problem. From the Lemmas presented and considering

$$w(x,t)=e^{-t\partial_x^3}\phi(x)+\mathscr{V}_+^\lambda h(x,t),\tag{22}$$

where

$$h(t)=e^{-i\pi\lambda}\left(f-e^{-t\partial_x^3}\phi|_{x=0}\right)(x,t),\tag{23}$$

then w solves the linear problem

$$\begin{cases} w_t(x,t) + w_{xxx}(x,t) = 0, & (x,t) \in \mathbb{R}^* \times \mathbb{R}, \\ w(x,0) = \phi(x), & x \in \mathbb{R}, \\ w(0,t) = f(t), & t > 0. \end{cases} \quad (24)$$

2.4. Nonhomogeneous Versions. We introduce the Duhamel nonhomogeneous solution operator $\mathscr{K}$ associated with the KdV equation as

$$\mathscr{K}w(x,t) = \int_0^t e^{-(t-t')\partial_x^3} w(x,t')dt', \quad (25)$$

and it follows that

$$\begin{cases} \left(\partial_t + \partial_x^3\right)\mathscr{K}w(x,t) = w(x,t), & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ \mathscr{K}w(x,0) = 0, & x \in \mathbb{R}. \end{cases} \quad (26)$$

2.5. Solution to the Hirota–Satsuma System. Using (9), (21), (24), and (26), we obtain the solution of IBVP (2) as

$$\begin{aligned} u(x,t) &= e^{-t\partial_x^3} u_0(x) + \mathscr{V}_+^{\lambda 1} h_1(x,t) + \mathscr{K}\left[-6(uu_x) + 2(vv_x)\right](x,t), \\ v(x,t) &= e^{-t\partial_x^3} v_0(x) + \mathscr{V}_+^{\lambda 2} h_2(x,t) + \mathscr{K}\left[-3(uv_x)\right](x,t), \end{aligned} \quad (27)$$

where

$$\begin{aligned} h_1(t) &= e^{-i\pi\lambda 1}\left(f - e^{-t\partial_x^3}\phi|_{x=0} - \mathscr{K}\left[-6(uu_x) + 2(vv_x)\right](0,t)\right)(x,t), \\ h_2(t) &= e^{-i\pi\lambda 2}\left(g - e^{-t\partial_x^3}\phi|_{x=0} - \mathscr{K}\left[-3(uv_x)\right](0,t)\right)(x,t). \end{aligned} \quad (28)$$

3. Main Estimates

In this section, we introduce the space trace estimates, the time trace estimates, and the Bourgain regularities.

Let $s \in \mathbb{R}$ and $b < 1/2$; we introduce the classical Bourgain spaces $X^{s,b}$ associated with $\partial_t + \partial_x^3$ as the completion of the Schwartz space $S(R^2)$ under the norm

$$\|u\|_{X^{s,b}} = \left(\int\int \langle\xi\rangle^{2s} \langle\tau - \xi^3\rangle^{2b} |\widehat{u}(\xi,\tau)|^2 d\xi d\tau\right)^{1/2}, \quad (29)$$

where $\langle\xi\rangle = 1 + |\xi|$.

To obtain our results, we define the following auxiliary modified Bourgain spaces. Let $Y^{s,b}$ and Z^{α} be the completion of $S'(\mathbb{R}^2)$ with respect to the norms

$$\|u\|_{Y^{s,b}} = \left(\int\int \langle\tau\rangle^{2s/3} \langle\tau - \xi^3\rangle^{2b} |\widehat{u}(\xi,\tau)|^2 d\xi d\tau\right)^{1/2}, \quad (30)$$

and

$$\|u\|_{Z^{\alpha}} = \left(\int\int \langle\tau\rangle^{2\alpha} |\widehat{u}(\xi,\tau)|^2 d\xi d\tau\right)^{1/2}. \quad (31)$$

The following Lemma is an important estimate presented [9].

Lemma 3. *Let $-1/2 < b' < b \leq 0$ or $0 \leq b' < b < 1/2, w \in X^{s,b}$, and $s \in \mathbb{R}$. Then, we have*

$$\left\|\psi_T w\right\|_{X^{s,b'}(\phi)} \leq cT^{b-b'} \|w\|_{X^{s,b}(\phi)}. \quad (32)$$

3.1. Space Trace Estimates

Lemma 4. *Let $s \in \mathbb{R}$.*

(a) If $\phi \in H^s(\mathbb{R})$, then we have

$$\left\|e^{-t\partial_x^3}\phi(x)\right\|_{\mathscr{C}(\mathbb{R}_t;H^s(\mathbb{R}_x))} \leq c\|\phi\|_{H^s(\mathbb{R})}. \quad (33)$$

(b) If $s - 5/2 < \lambda < s + 1/2, \lambda < 1/2$ and $supp(f) \subset [0,1]$, then we have

$$\left\|\mathscr{V}_+^{\lambda} f(x,t)\right\|_{\mathscr{C}(\mathbb{R}_t;H^s(\mathbb{R}_x))} \leq c\|f\|_{H_0^{(s+1)/3}(\mathbb{R}^+)}. \quad (34)$$

(c) If $1/2 < d < 1$, then we have

$$\|\psi(t)\mathscr{K}w(x,t)\|_{\mathscr{C}(\mathbb{R}_t;H^s(\mathbb{R}_x))} \leq c\|w\|_{X^{s,d-1}}. \quad (35)$$

Proof. See reference [8]. □

3.2. Time Trace Estimates

Lemma 5. *Let $s \in \mathbb{R}$.*

(a) For all $\phi \in H^s(\mathbb{R})$, we have

$$\left\|\psi(t)e^{-t\partial_x^3}\phi(x)\right\|_{\mathscr{C}(\mathbb{R}_x;H^{(s+1)/3}(\mathbb{R}_t))} \leq c\|\phi\|_{H^s(\mathbb{R})}. \quad (36)$$

(b) For all $-2 < \lambda < 1$, we have

$$\left\|\psi(t)\mathcal{V}_{+}^{\lambda} f(x,t)\right\|_{\mathscr{C}\left(\mathbb{R}_x;H_0^{(s+1)/3}\left(\mathbb{R}_t^+\right)\right)} \leq c\|f\|_{H_0^{(s+1)/3}(\mathbb{R}^+)}. \quad (37)$$

(c) *For all* $1/2 < d < 1$, *we have*

$$\|\psi(t)\mathscr{K}w(x,t)\|_{\mathscr{C}\left(\mathbb{R}_x;H^{(s+1)/3}(\mathbb{R}_t)\right)} \lesssim \begin{cases} \|w\|_{X^{s,d-1}}, & -1 \leq s \leq \dfrac{1}{2}, \\ \|w\|_{X^{s,d-1}} + \|w\|_{Y^{s,d-1}}, & s \in \mathbb{R}. \end{cases} \quad (38)$$

Proof. See reference [8]. □

3.3. Bourgain Regularities

Lemma 6. *Let* $s \in \mathbb{R}$.

(a) *For all* $0 < b < 1$, *if* $\phi \in H^s(\mathbb{R})$, *we have*

$$\left\|\psi(t)e^{-t\partial_x^3}\phi(x)\right\|_{X^{s,b}\cap Z^{\alpha}} \leq c\|\phi\|_{H^s(\mathbb{R})}. \quad (39)$$

(b) *For all* $s-1 \leq \lambda < s+1/2, \lambda < 1/2, \alpha \leq (s-\lambda+2)/3$, *and* $0 \leq b < 1/2$, *we have*

$$\left\|\psi(t)\mathcal{V}_{+}^{\lambda} f(x,t)\right\|_{X^{s,b}\cap Z^{\alpha}} \leq c\|f\|_{H_0^{(s+1)/3}(R^+)}. \quad (40)$$

(c) *For all* $1/2 < b < 1$ *and* $\alpha > 1/2$, *we have*

$$\|\psi(t)\mathscr{K}w(x,t)\|_{X^{s,b}\cap Z^{\alpha}} \leq \|w\|_{X^{s,b-1}}. \quad (41)$$

Proof. See references [8, 10]. □

3.4. Bilinear Estimates. The following Lemma was proposed by Zhao and its proof can be found in [6].

Lemma 7. *If* $s \geq -1/8, 1/2 < b < 9/16$, *then* $\forall b' > 1/2$, *we have*

$$\left\|(u_1)_x u_2\right\|_{X^{1+s,b-1}} \leq C\left\|u_1\right\|_{X^{1+s,b'}}\left\|u_2\right\|_{X^{s,b'}}, \quad (42)$$

and

$$\left\|(u_1u_2)_x\right\|_{X^{s,b-1}} \leq C\left\|u_1\right\|_{X^{s,b'}}\left\|u_2\right\|_{X^{s,b'}}. \quad (43)$$

4. Proof of the Main Theorem

Before proving the main theorem, some helpful properties of the Sobolev spaces are outlined in the results that follow. The proof of them can be found in [7].

Lemma 8. *Let* $-1/2 < s < 1/2$. *If* $f \in H^s(\mathbb{R})$, *then there is a constant* c_s *such that*

$$\left\|\chi_{(0,+\infty)} f\right\|_{H^s(\mathbb{R})} \leq c_s\|f\|_{H^s(\mathbb{R})}. \quad (44)$$

Lemma 9. *Let* $1/2 < s < 3/2$. *If* $f \in H^s(\mathbb{R}^+)$ *with* $f(0)=0$; *then, we have* $H_0^s(\mathbb{R}^+) = \{f \in H^s(\mathbb{R}^+); f(0)=0\}$ *and there is a constant* c_s *such that*

$$\left\|\chi_{(0,+\infty)} f\right\|_{H_0^s(\mathbb{R}^+)} \leq c_s\|f\|_{H^s(\mathbb{R}^+)}. \quad (45)$$

Proof of Theorem 1. Let $\widetilde{u}_0, \widetilde{v}_0, \widetilde{f}$, and $\widetilde{g}$ be the extensions of u_0, v_0, f, and g such that $\|\widetilde{u}_0\|_{H^s(\mathbb{R})} \leq c\|u_0\|_{H^s(\mathbb{R}+)}$, $\|\widetilde{v}_0\|_{H^{1+s}(\mathbb{R})} \leq c\|v_0\|_{H^{1+s}(\mathbb{R}^+)}$, $\|\widetilde{f}\|_{H^{(s+1)/3}(\mathbb{R})} \leq c\|f\|_{H^{(s+1)/3}(\mathbb{R}^+)}$, and $\|\widetilde{g}\|_{H^{(s+1)/3}(\mathbb{R})} \leq c\|g\|_{H^{(s+1)/3}(\mathbb{R}^+)}$. Let $b < 9/16$ such that Lemma 7 is valid. □

Step 1. Solution mapping and spaces.

Take $-1 < \lambda_1, \lambda_2 < 1$, $h_1, h_2 \in C_0^{\infty}(\mathbb{R}^+)$. By Section 2.5 we define the operator $\Lambda = (\Lambda_1, \Lambda_2)$, which is given by

$$\Lambda_1(u,v) = \psi(t)e^{-t\partial_x^3}\widetilde{u}_0(x) + \psi(t)\mathscr{K}\left(-3\psi_T\left(u^2\right)_x + \psi_T\left(v^2\right)_x\right)(x,t) + \psi(t)e^{-i\pi\lambda_1}\mathcal{V}_{+}^{\lambda_1}h_1(x,t), \quad (46)$$

and

$$\Lambda_2(u,v) = \psi(t)e^{-t\partial_x^3}\widetilde{v}_0(x) + \psi(t)\mathscr{K}\left(-3\psi_T\left(uv_x\right)\right)(x,t) + \psi(t)e^{-i\pi\lambda_2}\mathcal{V}_{+}^{\lambda_2}h_2(x,t), \quad (47)$$

where h_1 and h_2 are given by

$$h_1(t)=\left(\psi(t)\tilde{f}(t)-\psi(t)e^{-t\partial_x^3}\tilde{u}_0|_{x=0}-\psi(t)\mathscr{K}\left(-3\psi_T\left(u^2\right)_x+\psi_T\left(v^2\right)_x\right)(0,t)\right)|_{(0,+\infty)}, \tag{48}$$

and

$$h_2(t)=\left(\psi(t)\tilde{g}(t)-\psi(t)e^{-t\partial_x^3}\tilde{v}_0|_{x=0}-\psi(t)\mathscr{K}\left(-3\psi_T(uv_x)\right)(0,t)\right)|_{(0,+\infty)}. \tag{49}$$

We consider Λ in the Banach space $W=W_1\times W_2$, where

$$\begin{aligned} W_1&=\mathscr{C}\left(\mathbb{R}_t;H^s(\mathbb{R}_x)\right)\cap\mathscr{C}\left(\mathbb{R}_x;H^{(s+1)/3}(\mathbb{R}_t)\right)\cap X^{s,b}\cap Z^{\alpha},\\ W_2&=\mathscr{C}\left(\mathbb{R}_t;H^{1+s}(\mathbb{R}_x)\right)\cap\mathscr{C}\left(\mathbb{R}_x;H^{(s+1)/3}(\mathbb{R}_t)\right)\cap X^{1+s,b}\cap Z^{\alpha}, \end{aligned} \tag{50}$$

with norm

$$\begin{aligned} \|(u,v)\|_W=\|u\|_{W_1}+\|v\|_{W_2}:=&\|u\|_{\mathscr{C}(\mathbb{R}_t;H^s(\mathbb{R}_x))}+\|u\|_{\mathscr{C}(\mathbb{R}_x;H^{(s+1)/3}(\mathbb{R}_t))}+\|u\|_{X^{s,b}}+\|u\|_{Z^{\alpha}}\\ &+\|v\|_{\mathscr{C}(\mathbb{R}_t;H^{1+s}(\mathbb{R}_x))}+\|v\|_{\mathscr{C}(\mathbb{R}_x;H^{(s+1)/3}(\mathbb{R}_t))}+\|v\|_{X^{1+s,b}}+\|v\|_{Z^{\alpha}}. \end{aligned} \tag{51}$$

Step 2. Estimates of boundary values.

For this purpose, by using Lemmas 4–6 (b), we show that $h_1\in H_0^{(s+1)/3}(\mathbb{R}^+)$ and $h_2\in H_0^{(s+1)/3}(\mathbb{R}^+)$.

By hypothesizing $f\in H^{(s+1)/3}(\mathbb{R}^+)$, Lemmas 8 and 5 (a) imply

$$\left\|\left(\psi(t)e^{-t\partial_x^3}\tilde{u}_0|_{x=0}\right)|_{(0,+\infty)}\right\|_{H^{(s+1)/3}(\mathbb{R}^+)}\leq\left\|\left(\psi(t)e^{-t\partial_x^3}\tilde{u}_0|_{x=0}\right)\right\|_{H^{(s+1)/3}(\mathbb{R})}\leq c\|\tilde{u}_0\|_{H^s(\mathbb{R})}. \tag{52}$$

Now, Lemmas 3, 5 (c), 7, and 8 imply

$$\begin{aligned} &\left\|\psi(t)\mathscr{K}\left[-3\psi_T\left(u^2\right)_x+\psi_T\left(v^2\right)_x\right](0,t)|_{(0,+\infty)}\right\|_{H^{(s+1)/3}(\mathbb{R}^+)}\\ &\leq\left\|\psi(t)\mathscr{K}\left[\psi_T\left(-3\left(u^2\right)_x+\left(v^2\right)_x\right)\right](0,t)\right\|_{H^{(s+1)/3}(\mathbb{R})}\\ &\leq c\left\|\psi_T\left(-3\left(u^2\right)_x+\left(v^2\right)_x\right)\right\|_{X^{s,b-1}}\leq cT^{\epsilon}\left\|\left(-3\left(u^2\right)_x+\left(v^2\right)_x\right)\right\|_{X^{s,b'-1}}\\ &\leq cT^{\epsilon}\left[\|u\|_{X^{s,b}}^2+\|v\|_{X^{s,b}}^2\right]\leq cT^{\epsilon}\left[\|u\|_{X^{s,b}}^2+\|v\|_{X^{1+s,b}}^2\right], \end{aligned} \tag{53}$$

where $\epsilon=b'-b$ is adequately small.

If $-1/8<s<1/2$, then $7/24<(s+1)/3<1/2$, and Lemma 8 shows that $H^{(s+1)/3}(\mathbb{R}^+)=H_0^{(s+1)/3}(\mathbb{R}^+)$. If $1/2<s<3/2$, then $1/2<(s+1)/3<5/6$ due to the compatibility condition, and Lemma 9 shows that $H^{(s+1)/3}(\mathbb{R}^+)=H_0^{(s+1)/3}(\mathbb{R}^+)$. Thus, equations (52) and (53) reveal that $h_1\in H_0^{(s+1)/3}(\mathbb{R}^+)$.

By combining the abovementioned, we obtain

$$\|h_1\|_{H_0^{(s+1)/3}(\mathbb{R}^+)}\leq c\|f\|_{H^{(s+1)/3}(\mathbb{R}^+)}+c\|u_0\|_{H^s(\mathbb{R}^+)}+cT^{\epsilon}\left[\|u\|_{X^{s,b}}^2+\|v\|_{X^{1+s,b}}^2\right]. \tag{54}$$

Similarly, from Lemmas 3, 5 (a), 5 (c), 7, and 8, we have

$$\begin{aligned}
&\left\|\left(\psi(t)\widetilde{g}-\psi(t)e^{-t\partial_x^3}\widetilde{v}_0|_{x=0}-\psi(t)\psi(t)\mathscr{K}\left[-3\psi_T\left(uv_x\right)\right](0,t)\right)|_{(0,+\infty)}\right\|_{H^{(s+1)/3}(\mathbb{R}^+)}\\
&\leq\left\|\left(\psi(t)\widetilde{g}-\psi(t)e^{-t\partial_x^3}\widetilde{v}_0|_{x=0}-\psi(t)\mathscr{K}\left[-3\psi_T\left(uv_x\right)\right](0,t)\right)\right\|_{H^{(s+1)/3}(\mathbb{R})},\\
&\leq c\left(\|g\|_{H^{(s+1)/3}(\mathbb{R}^+)}+\left\|v_0\right\|_{H^{1+s}(\mathbb{R}^+)}+\left\|\psi_T\left(uv_x\right)\right\|_{X^{s,b-1}}\right),\\
&\leq c\left(\|g\|_{H^{(s+1)/3}(\mathbb{R}^+)}+\left\|v_0\right\|_{H^{1+s}(\mathbb{R}^+)}+T^{\epsilon}\left\|\left(uv_x\right)\right\|_{X^{s,b'-1}}\right),\\
&\leq c\left(\|g\|_{H^{(s+1)/3}(\mathbb{R}^+)}+\left\|v_0\right\|_{H^{1+s}(\mathbb{R}^+)}+T^{\epsilon}\left[\|u\|_{X^{s,b}}\|v\|_{X^{1+s,b}}\right]\right).
\end{aligned}\tag{55}$$

It follows that

$$\psi(t)\left(\widetilde{g}-e^{-t\partial_x^3}\widetilde{v}_0|_{x=0}-\mathscr{K}\left[-3\psi_T\left(uv_x\right)\right](0,t)\right)|_{(0,+\infty)}\in H^{(s+1)/3}\left(\mathbb{R}^+\right).\tag{56}$$

Similarly, as h_1, we obtain

$$\psi(t)\left(\widetilde{g}-e^{-t\partial_x^3}\widetilde{v}_0|_{x=0}-\mathscr{K}\left[-3\psi_T\left(uv_x\right)\right](0,t)\right)|_{(0,+\infty)}\in H_0^{(s+1)/3}\left(\mathbb{R}^+\right).\tag{57}$$

Indeed, the abovementioned equations show that $h_2\in H_0^{(s+1)/3}(\mathbb{R}^+)$.

Thus, we obtain

$$\left\|h_2\right\|_{H_0^{(s+1)/3}(\mathbb{R}^+)}\leq c\|g\|_{H^{(s+1)/3}(\mathbb{R}^+)}+c\left\|v_0\right\|_{H^{1+s}(\mathbb{R}^+)}+cT^{\epsilon}\left[\|u\|_{X^{s,b}}\|v\|_{X^{1+s,b}}\right].\tag{58}$$

Step 3. Contraction of the solution mapping.

From Lemmas 3, 4, 7, and equation (54), we have

$$\left\|\Lambda_1(u,v)\right\|_{\mathscr{C}\left(\mathbb{R}_t;H^s\left(\mathbb{R}_x\right)\right)}\leq c\left(\left\|u_0\right\|_{H^s(\mathbb{R}^+)}+\|f\|_{H^{(s+1)/3}(\mathbb{R}^+)}+T^{\epsilon}\|u\|_{X^{s,b}}^2+T^{\epsilon}\|v\|_{X^{1+s,b}}^2\right).\tag{59}$$

From Lemmas 3, 5, 7, and equation (54), we have

$$\left\|\Lambda_1(u,v)\right\|_{\mathscr{C}\left(\mathbb{R}_x;H^{(s+1)/3}\left(\mathbb{R}_t\right)\right)}\leq c\left(\left\|u_0\right\|_{H^s(\mathbb{R}^+)}+\|f\|_{H^{(s+1)/3}(\mathbb{R}^+)}+T^{\epsilon}\|u\|_{X^{s,b}}^2+T^{\epsilon}\|v\|_{X^{1+s,b}}^2\right).\tag{60}$$

From Lemmas 3, 6, 7, and equation (54), we have

$$\left\|\Lambda_1(u,v)\right\|_{X^{s,b}\cap Z^{\alpha}}\leq c\left(\left\|u_0\right\|_{H^s(\mathbb{R}^+)}+\|f\|_{H^{(s+1)/3}(\mathbb{R}^+)}+T^{\epsilon}\|u\|_{X^{s,b}}^2+T^{\epsilon}\|v\|_{X^{1+s,b}}^2\right),\tag{61}$$

by combining the abovementioned, we obtain

$$\|\Lambda_1(u,v)\|_{W_1} \leq c\Big(\|u_0\|_{H^s(\mathbb{R}^+)} + \|f\|_{H^{(s+1)/3}(\mathbb{R}^+)} + T^{\epsilon}\|u\|^2_{X^{s,b}} + T^{\epsilon}\|v\|^2_{X^{1+s,b}}\Big). \tag{62}$$

From Lemmas 3–7, and equation (58), we have

$$\|\Lambda_2(u,v)\|_{W_2} \leq c\Big(\|v_0\|_{H^{1+s}(\mathbb{R}^+)} + \|g\|_{H^{(s+1)/3}(\mathbb{R}^+)} + T^{\epsilon}\big[\|u\|_{X^{s,b}}\|v\|_{X^{1+s,b}}\big]\Big). \tag{63}$$

Inspired from [11], we have

$$\begin{aligned}
\Lambda_1(u_1,v_1) - \Lambda_1(u_2,v_2) &= \psi(t)\mathscr{K}\big[-3\psi_T\big((u_1+u_2)(u_1-u_2)\big)_x + \psi_T\big((v_1+v_2)(v_1-v_2)\big)_x\big](x,t) + \psi(t)e^{-i\pi\lambda_1}\mathscr{V}_+^{\lambda_1}h_1(x,t),\\
\Lambda_2(u_1,v_1) - \Lambda_2(u_2,v_2) &= \psi(t)\mathscr{K}\big[-3\psi_T\big(u_1(v_1)_x - u_2(v_2)_x\big)\big](x,t) + \psi(t)e^{-i\pi\lambda_2}\mathscr{V}_+^{\lambda_2}h_2(x,t),\\
\Lambda(u_1,v_1) - \Lambda(u_2,v_2) &= \begin{pmatrix}\Lambda_1(u_1,v_1) - \Lambda_1(u_2,v_2)\\ \Lambda_2(u_1,v_1) - \Lambda_2(u_2,v_2)\end{pmatrix},
\end{aligned} \tag{64}$$

where h_1 and h_2 are given by

$$h_1(t) = -\Big(\psi(t)\mathscr{K}\big[-3\psi_T\big((u_1+u_2)(u_1-u_2)\big)_x + \psi_T\big((v_1+v_2)(v_1-v_2)\big)_x\big](0,t)|_{(0,+\infty)}\Big), \tag{65}$$

and

$$h_2(t) = -\Big(\psi(t)\mathscr{K}\big[-3\psi_T\big(u_1(v_1)_x - u_2(v_2)_x\big)\big](0,t)|_{(0,+\infty)}\Big). \tag{66}$$

Similarly, as abovementioned, we have

$$\begin{aligned}
&\|\Lambda(u_1,v_1) - \Lambda(u_2,v_2)\|_W \leq cT^{\epsilon}\\
&\Big\{\big(\|u_1\|_{X^{s,b}} + \|u_2\|_{X^{s,b}}\big)\|u_1-u_2\|_{X^{s,b}} + \|v_1-v_2\|_{X^{1+s,b}}\big(\|v_1\|_{X^{1+s,b}} + \|v_2\|_{X^{1+s,b}}\big) + \|v_1\|_{X^{1+s,b}}\|u_1-u_2\|_{X^{s,b}} + \|u_2\|_{X^{s,b}}\|v_1-v_2\|_{X^{1+s,b}}\Big\}.
\end{aligned} \tag{67}$$

We set the ball of W as

$$B = \big\{(u,v)\in W; \|u\|_{W_1} \leq M_1, \|v\|_{W_2} \leq M_2\big\}, \tag{68}$$

where $M_1 = 2c\left(\|u_0\|_{H^s(\mathbb{R}^+)} + \|f\|_{H^{(s+1)/3}(\mathbb{R}^+)}\right)$ and $M_2 = 2c\left(\|v_0\|_{H^{1+s}(\mathbb{R}^+)} + \|g\|_{H^{(s+1)/3}(\mathbb{R}^+)}\right)$.

By restricting (u,v) on the ball B, by (62) and (63) and (67), we obtain

$$\begin{aligned}
\|\Lambda_1(u,v)\|_{W_1} &\leq \frac{M_1}{2} + cT^{\epsilon}\big(M_1^2 + M_2^2\big),\\
\|\Lambda_2(u,v)\|_{W_2} &\leq \frac{M_2}{2} + cT^{\epsilon}M_1M_2,\\
\|\Lambda(u_1,v_1) - \Lambda(u_2,v_2)\|_{W_1} &\leq cT^{\epsilon}\Big[M_1\|u_1-u_2\|_{W_1} + M_2\|v_1-v_2\|_{W_2}\Big],\\
\|\Lambda(u_1,v_1) - \Lambda(u_2,v_2)\|_{W_2} &\leq cT^{\epsilon}(M_1+M_2)\Big[\|u_1-u_2\|_{W_1} + \|v_1-v_2\|_{W_2}\Big].
\end{aligned} \tag{69}$$

Then, we take $T = T(M_1, M_2)$ to be suitably small, such that

$$\begin{aligned} \|\Lambda_1(u,v)\|_{W_1} &\le M_1, \\ \|\Lambda_2(u,v)\|_{W_2} &\le M_2, \end{aligned} \tag{70}$$

and

$$\|\Lambda(u_1,v_1) - \Lambda(u_2,v_2)\|_W \le \frac{1}{2}\|(u_1,v_1) - (u_2,v_2)\|_{W}. \tag{71}$$

Thus, Λ defines a contraction in $W \cap B$, and we obtain a fixed point in (u,v) in B. Therefore, the IBVP (2) is solved in the sense of distributions by

$$(u,v) := \left(u|(x,t) \in \mathscr{C}([0,T]; H^s(\mathbb{R}^+)), v|(x,t) \in \mathscr{C}\left([0,T]; H^{1+s}(\mathbb{R}^+)\right)\right). \tag{72}$$

5. Conclusions

According to [4, 6], the asymmetry of the nonlinear component uv_x makes the asymmetrical product space $H^s(\mathbb{R}) \times H^{1+s}(\mathbb{R})$ better suited for the Hirota–Satsuma system than the symmetric product space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. Thus, this paper studies the IBVP of the Hirota–Satsuma system in the asymmetrical product space. Inspired by [8, 11], we propose the following directions for further consideration:

Remark 1. The skill used in this paper is also applicable to the IBVP of the Hirota–Satsuma system posed on the left half line $\mathbb{R}^- = (-\infty, 0)$:

$$\begin{cases} u_t - \alpha(u_{xxx} + 6uu_x) = 2\beta v v_x, & x \in (-\infty,0), t \ge 0, \\ v_t + v_{xxx} + 3uv_x = 0, & x \in (-\infty,0), t \ge 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in (-\infty,0), \\ u(0,t) = f(t), v(0,t) = g_1(t), v_x(0,t) = g_2(t), & t \ge 0. \end{cases} \tag{73}$$

Remark 2. In addition to discussing the left half line, we may consider other boundary conditions on the right half line. More specifically, let A denote the differential operator associated with the KdV equation

$$AX = \left(-\frac{\mathrm{d}^3 u}{\mathrm{d}x^3}, -\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}\right)^T, \tag{74}$$

where $X = (u,v)^T$. Since

$$\langle AX, X\rangle = \langle A^* X, X\rangle = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\overline{u}\Big|_0^\infty + \frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\Big|_0^\infty - \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\overline{v}\Big|_0^\infty + \frac{1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\Big|_0^\infty = 0, \tag{75}$$

with any of the following boundary values:

$$\begin{aligned} &\mathbf{a}. u_{xx}(0,t) = v_{xx}(0,t) = 0, \\ &\mathbf{b}. u(0,t) = v_{xx}(0,t) = 0, \\ &\mathbf{c}. u(0,t) - v(0,t) = 0, \\ &u_{xx}(0,t) + v_{xx}(0,t) = 0, \\ &\mathbf{d}. u(0,t) + v_{xx}(0,t) = 0, \\ &u_{xx}(0,t) - v(0,t) = 0. \end{aligned} \tag{76}$$

For any one of the boundary values a, b, c, and d, operator A generates an operator group by the Phillips–Lumper theorem. We will investigate those questions in the future.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

An earlier version of this article [12] was presented as a preprint on the Research Square website.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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