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Research Article

Novel Approximations to the Damped Parametric Driven Pendulum Oscillators

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The damped parametric driven nonlinear pendulum equation/oscillator (NPE), also known as the damped disturbed NPE, is examined, along with some associated oscillators for arbitrary angles with the vertical pivot. For analyzing and solving the current pendulum equation, we reduce this equation to the damped Duffing equation (DDE) with variable coefficients. After that, the DDE with variable coefficients is divided into two cases. In the first case, two analytical approximations to the damped undisturbed NPE are obtained. The first approximation is determined using the ansatz method while the second one is derived using He's frequency formulation. In the second case, i.e., the damped disturbed NPE, three analytical approximations in terms of the trigonometric and Jacobi elliptic functions are derived and discussed using the ansatz method. The semianalytical solutions of the two mentioned cases are graphically compared with the 4^{th}-order Runge–Kutta (RK4) approximations. In addition, the maximum error for all the derived approximations is estimated as compared with the RK4 approximation. The proposed approaches as well as the obtained solutions may greatly help in understanding the mysteries of various nonlinear phenomena that arise in different scientific fields such as fluid mechanics, plasma physics, engineering, and electronic chips.

1. Introduction

The complex pendulum is a paradigm for investigating oscillations and numerous other physical problems and nonlinear dynamical phenomena [1, 2]. Several models for the nonlinear pendulum oscillators (NPOs) have been utilized for investigating numerous physical and engineering problems, e.g., the oscillations in chips and electrical circuits, Bose–Einstein condensates, image processing, the movement of satellites, oscillations in different plasma models, and many other problems in several fields [3–5]. Moreover, many evolution equations to the pendulum oscillators have been utilized as a physical model to study several natural problems related to different oscillations, bifurcations, and chaos [6–8]. For instance, He et al. [9] modified the structure of the pendulum oscillations on a dynamical system by using a device with electromagnetic harvesting to control the kinematics of a spring-pendulum system. Based on Lagrange's equations, the authors derived the governing kinematics equations of the NPOs and solved them analytically using the multiple-scale method (MSM). In that investigation, the authors explained various behaviors, which control the mentioned model, such as the stability of fixed points, amplitudes, phases, and the (in)stability regions. Besides that, they motioned that this model is utilized to control sensors in building transportation and industrial sectors. The auto-parametric system of three degrees of freedom consisting of a damped spring pendulum was demonstrated in the study of [10]. The analytical solution to the motion equations up to the third approximation was obtained using the MSM. Furthermore, the stability and instability zones were analyzed and investigated. In the study of [11], the authors investigated the periodic property of a rotating pendulum model. The governing equation for this model was solved analytically using He's homotopy perturbation method (He's HPM). The accuracy of the obtained results was verified by comparing the obtained solution with the numerical one based on the 4^{th}-order Runge–Kutta (RK4) method and with He's frequency formulation (He's FF). HPM and its family succeeded more than other methods in analyzing pendulum equations without both linear and negative-linear terms [12] and many other NPOs [13–15]. He's FF has developed rapidly since its inception; where many researchers have developed this method to give good results for many complicated nonlinear problems without any restrictions [16–21]. The hybrid forced Rayleigh–Van der Pol–Duffing oscillator with higher-order nonlinearity has been solved using the Poincaré–Lindstedt approach. The researchers discovered that the approximate solution and the RK4 numerical solution are in good agreement. The authors found that there is a high agreement between the analytical and numerical approximations [22]. Also, the homotopy analysis method (HAM) has been employed for analyzing the damped Duffing oscillator (DDO) [23]. The Laplace transform, modified differential transform method (MDTM), and Padé approximants have been applied for analyzing and investigating the approximations to many NPOs such as the forced DDO and forced damped van der Pol oscillator [24]. Both damped and undamped Helmholtz–Duffing (H-D) oscillators have been studied and analyzed using the ansatz method with the moving boundary technique to obtain high-accurate approximations [25]. The authors [25] made a comparison between both approximate analytic and numeric solutions to prove the accuracy of the analytic approximations, and it was found that the obtained results were in agreement with each other to a large extent. Moreover, the (un)damped quadratic nonlinear Helmholtz oscillator (HO) have been investigated using ansatz method and the exact solution for the undamped oscillator as well as the approximate solution to the damped oscillator have been derived in terms of the Weiersrtrass elliptic functions [26]. There have been few attempts to solve and analyze damped NPOs while taking friction forces into account [27]. The analytical approximations to some damped NPOs have been derived in terms of the Jacobi elliptic functions [27]. There are many other physical forces, such as periodic and perturbed forces in addition to friction force, that can affect the behavior of pendulum oscillators. For instance, the damped parametric nonlinear pendulum equation/oscillator (NPE/NPO) or known as the damped disturbed NPE/NPO has been discussed and solved numerically using implicit discrete mappings [28, 29]. Also, the parametric NPE with both frictional and excited forces has been solved analytically using the ansatz method and He's FF as well as solved numerically via the Galerkin method [30]. The comparison with the RK4 approximation revealed that the derived analytical and numerical solutions were extremely accurate. Also, utilizing a variety of analytical and numerical techniques, the parametric NPE as well as certain related oscillators have been solved [31]. The authors used the ansatz method for deriving high-accurate approximations to the unforced damped parametric NPE in terms of angular Mathieu functions. Also, the authors applied the ansatz method to find some approximations to the damped (un)forced parametric NPE in terms of trigonometric functions. Moreover, He's FF was applied to obtain some approximations for both undamped and damped parametric NPE. Furthermore, HPM was carried out for analyzing both forced undamped and forced damped parametric NPE. Also, the authors applied the Krylov–Bogoliúbov Mitropolsky method (KBMM) for analyzing forced damped parametric NPE. Finally, the authors made a comparison between all obtained approximations with numerical approximations using both RK4 method and the hybrid Padé-finite difference method. In current work, the parametric (un)disturbed NPE and many other related oscillators will be investigated in detail and some innovative approximations using several effective techniques will be derived using the ansatz method. The present study is divided into two main parts: in the first part, some semianalytical solutions (analytical approximations) to the damped undisturbed NPE will be derived using the ansatz method and He's FF [17, 18, 32]. For the second part, the damped disturbed NPE will be solved analytically via different approaches based on the ansatz method. Furthermore, the comparison between all obtained analytical approximations for the two studied cases and the RK4 numerical approximation will be investigated graphically and numerically.

The rest of the present work is organized as follows. In Section 2, a glimpse about the equation of motion of the parametric pendulum equation and its solution is recovered. In Section 3, two-different analytical approximations (sometimes are called semianalytical solutions) to the damped undisturbed NPO are obtained using a suitable ansatz method and He's FF. In Section 4, some semianalytical solutions to the damped parametric driven NPO are discussed in detail. In this section, three-different formulas for the semianalytical solutions are derived. The results and discussion are introduced in Section 5. We summarized and introduced the most important results in Section 6.

2. The Damped Disturbed NPE and Its Solution

In this section, we construct the differential equation that describes the behavior of the parametric driven NPE on the pivot vertical. A simple mathematical parametric pendulum system is modeled by a point mass m in kg unit, hanging at the end of a massless wire with length l in m unit and fixed to a supporting point "O," swinging to and from in a vertical plane. It is assumed that the end of the massless wire is moving harmonically with a small harmonic disturbance [28]: $y(t) = \pm \varepsilon \cos(\gamma t)$ (the motion of the vibrating base), where ε is a dimensionless small parameter and γ represents the frequency of harmonic motion. By analyzing the pendulum motion, we obtain the following equation:

$$\begin{cases} x(t) = l\sin\varphi, \\ y(t) = -l\cos\varphi + \varepsilon\cos(\gamma t), \end{cases} \tag{1}$$

where φ denotes the angular displacement about downward vertical.

Accordingly, the velocities in (x, y)-directions read

$$\begin{cases} \dot{x}(t) = \dot{\varphi} l\cos\varphi, \\ \dot{y}(t) = \pm\dot{\varphi} l\sin\varphi - \varepsilon\gamma\sin(\gamma t). \end{cases} \tag{2}$$

Using the Lagrangian method, the equation of motion could be obtained taking damping force of the medium into account [28]:

$$\mathbb{R}_1 \equiv \ddot{\varphi} + 2\beta\dot{\varphi} + \phi(t)\sin\varphi = 0, \tag{3}$$

where $\phi(t) = \omega_0^2 \pm Q_0\cos(\gamma t)$, $\omega_0 = \sqrt{g/l}$ denotes the eigenfrequency of the system, $\beta = \mu/(2ml)$ indicates the coefficient of the damping term, $Q_0 = \varepsilon\omega_2$ is the excitation amplitude, and $\omega_2 = \gamma^2/l$. Here, g represents the gravitational acceleration in unit m/s^2, $g = 9.8m/s^2$. More details about the deriving (3) can be found in the studies of [28, 29].

As a particular case, in the absence of the dissipative (friction) forces, i.e., for $\beta = 0$, the parametric pendulum equation reduces to the undamped disturbed nonlinear pendulum equation:

$$\ddot{\varphi} + \phi(t)\sin\varphi = 0. \tag{4}$$

For $\varepsilon = 0$ and undamping $\beta = 0$, the parametric pendulum equation reduces to the simple pendulum equation:

$$\ddot{\varphi} + \omega_0^2\sin\varphi = 0. \tag{5}$$

In the following sections, we proceed to solve and analyze all the mentioned cases in detail.

3. Analytical Approximations to the Damped Undisturbed NPE

It is supposed that $\varepsilon = 0$, then (3) reduces into the damped simple pendulum equation:

$$\begin{cases} \mathbb{R}_2 \equiv \ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2\sin\varphi = 0, \\ \varphi(0) = \varphi_0 \text{ and } \dot{\varphi}(0) = \dot{\varphi}_0. \end{cases} \tag{6}$$

For the Chebyshev approximation $\sin\varphi \approx (332/333)\varphi - (13/85)\varphi^3$, the initial value problem (i.v.p.) (6) can be rewritten in the following damped Duffing i.v.p. [33]:

$$\begin{cases} \Bbbk \equiv \ddot{\varphi} + 2\beta\dot{\varphi} + p\varphi + q\varphi^3 = 0, \\ \varphi(0) = \varphi_0 \text{ and } \dot{\varphi}(0) = \dot{\varphi}_0. \end{cases} \tag{7}$$

where $p = (332/333)\omega_0^2$ and $q = -(13/85)\omega_0^2$.

3.1. First Formula: Trigonometric Solution. Now, we seek a solution in the following ansatz:

$$\varphi = \exp(-\beta t)[A\sin(f(t)) + B\cos(f(t))], \tag{8}$$

where the coefficients A and B are undetermined coefficients which can be obtained the initial conditions (ICs) $\varphi(0) = \varphi_0$ and $\varphi'(0) = \dot{\varphi}_0$ and $f \equiv f(t)$, $f(0) = 0$ is a function to be determined later. Applying the mentioned ICs, the values of A and B are obtained as

$$\begin{cases} A = \pm\sqrt{\pm\dfrac{4\beta^2 + \sqrt{\left(-4\beta^2 + 4p + 3q\varphi_0^2\right)^2 + 48q\left(\beta\varphi_0 + \dot{\varphi}_0\right)^2} - 4p - 3q\varphi_0^2}{6q}}, \\ B = \varphi_0. \end{cases} \tag{9}$$

The substitution of (8) into (7) gives us

$$\Bbbk = S_1\sin(f) + S_2\sin(3f) + S_3\cos(f) + S_4\cos(3f), \tag{10}$$

with

$$
\begin{aligned}
S_1 &= -\frac{e^{-3\beta t}}{4}\begin{pmatrix} -3A^3q - 3AB^2q + 4A(\dot{f})^2e^{2\beta t} & e^{2\beta t} \\ -4Ape^{2\beta t} + 4A\beta^2e^{2\beta t} + 4B\ddot{f} & \end{pmatrix},\\
S_2 &= -\frac{e^{-3\beta t}}{4}\left(A^3q - 3AB^2q\right),\\
S_3 &= -\frac{e^{-3\beta t}}{4}\left(-3A^2Bq - 4A\ddot{f}e^{2\beta t} - 3B^3q + 4B(\dot{f})^2e^{2\beta t} - 4Bpe^{2\beta t} + 4\beta^2Be^{2\beta t}\right),\\
S_4 &= -\frac{e^{-3\beta t}}{4}\left(3A^2Bq - B^3q\right).
\end{aligned} \tag{11}
$$

The ODE for the function f could be obtained by eliminating $\ddot{f}$ from the system

$$S_1 = 0\ \&S_3 = 0. \tag{12}$$

The ODE to be solved is

$$\begin{cases} \dot{f} = \sqrt{(3/4)q\left(A^2+B^2\right)e^{-2\beta t} - \beta^2 + p}, \\ f(0) = 0. \end{cases} \tag{13}$$

The solution to the ODE (13) reads

$$f = \frac{1}{2\beta}\left[\sqrt{\Gamma} - \sqrt{\Gamma e^{-2t\beta}} + 2\sqrt{p-\beta^2}\left(\sinh^{-1}(\Theta) - \sinh^{-1}(\Theta)\right)\right], \tag{14}$$

with

$$\Theta = 2\sqrt{\frac{p-\beta^2}{3\left(A^2+B^2\right)q}}e^{t\beta}\ \text{and}\ \Gamma = 4p - 4\beta^2 + 3\left(A^2+B^2\right)q. \tag{15}$$

For $\beta \longrightarrow 0$, the function f reduces to

$$f = \sqrt{p + (3/4)\left(A^2+B^2\right)t}. \tag{16}$$

Example 1. Let $(\beta, \omega_0) = (0.1, 1)$, then we obtain the following equation:

$$\begin{cases} \ddot{\varphi} + 0.2\dot{\varphi} + \sin\varphi = 0, \\ \varphi(0) = 0\ \text{and}\ \dot{\varphi}(0) = 0.25. \end{cases} \tag{17}$$

According to the above analysis, the approximate analytical solution reads

$$\varphi_{\text{Approx}} = -0.252583e^{-0.1t}\sin\begin{pmatrix} 5.\sqrt{3.94799 - 0.0292721e^{-0.2t}} \\ +(-9.93477i)\sin^{-1}\left(11.6134e^{0.1t}\right) \\ +(21.3316 + 15.6055i) \end{pmatrix}. \tag{18}$$

The comparison between the approximation (18) and the RK4 numerical approximation is carried out as shown in Figure 1. Also, the maximum distance error (MDE) L_∞ to the approximation (18) as compared to the RK4 approximation is estimated:

$$L_\infty = \max_{0\le t\le 30}\left|\varphi_{\text{Approx}} - \varphi_{\text{RK}}\right| = 0.00134826. \tag{19}$$

It is clear from the value of the MDE L_∞ that the analytical approximation (8) is characterized by the high-accuracy as compared to the RK4 numerical approximation, which enhances the effectiveness of this solution.

3.2. Second Approach: He's Frequency Formulation (He's FF). In order to apply He's FF, the following i.v.p.

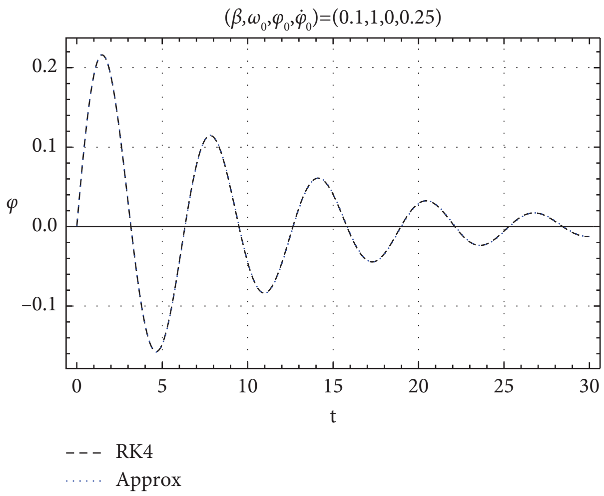

Figure 1: A comparison between the approximation (18) and the RK numerical approximation to the i.v.p. (17).

$$\begin{cases} \mathbb{R}_2 \equiv \ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2 \sin\varphi = 0, \\ \varphi(0) = \varphi_0 \text{ and } \dot{\varphi}(0) = \dot{\varphi}_0, \end{cases} \tag{20}$$

replaces by the i.v.p.

$$\begin{cases} \ddot{\varphi} + 2\beta\dot{\varphi} + f(\varphi) = 0, \\ \varphi(0) = \varphi_0 \text{ and } \dot{\varphi}(0) = \dot{\varphi}_0, \end{cases} \tag{21}$$

where the function $f(\varphi) \equiv f(x)$ can be obtained from the Chebyshev approximation as

$$f(x) = \omega_0^2 \sin x \approx \omega_0^2\left(x - \frac{2}{13}x^3\right). \tag{22}$$

According to He's FF, we state

$$\varphi = Ae^{-\rho t}\cos\left(w(t) + \arccos\left(\frac{\varphi_0}{A}\right)\right), \tag{23}$$

with

$$\dot{w}(t) = \mu\sqrt{(f(x)/x)} \text{ and } x = \frac{\sqrt{a}}{2}Ae^{-\rho t}, \tag{24}$$

which leads to

$$\dot{w}(t) = \mu\sqrt{\left(1 - (3/26)A^2e^{-2\rho t}\right)\omega_0^2}. \tag{25}$$

Solving the ODE (25) using the initial condition (IC) $w(0) = 0$, we have

$$\begin{aligned} w(t) &= \mu\int_0^t \sqrt{\left(1 - \frac{3}{26}A^2e^{-2\rho t}\right)\omega_0^2}\,d\tau, \\ &= \mu\left(\frac{\sqrt{Y(0)\omega_0^2} - \sqrt{Y(t)\omega_0^2}}{\sqrt{26}\,\rho} - \frac{\sqrt{-Y(0)}\,\omega_0^2\csc^{-1}\left(\sqrt{(3/26)}\,A\right)}{\rho\sqrt{Y(0)\omega\$_0^2}}\right. \\ &\quad \left. - \frac{Ae^{\rho t}\sqrt{3A^2 - 26e^{2\rho t}}\sqrt{Y(t)\omega_0^2}\csc^{-1}\left(\sqrt{(3/26)}\,\mathrm{A}e^{-\rho t}\right)}{\rho\left(3\mathrm{A}^2 - 26e^{2\rho t}\right)}\right), \end{aligned} \tag{26}$$

with

$$Y(t) = \left(26 - 3A^2e^{-2\rho t}\right). \tag{27}$$

The value of the coefficient A can be obtained by using the value of $w(t)$ (given in (26)) and applying the IC $\dot{\varphi}(0) = \dot{\varphi}_0$

$$\Gamma \equiv \left(-3A^2\varphi_0^2 + 3A^4 - 26A^2 + 26B^2\right)\mu^2\omega_0^2 + 26\varphi_0^2\rho^2 + 52\varphi_0\rho\dot{\varphi}_0 + 26\dot{\varphi}_0^2 = 0. \tag{28}$$

Using command Solve [Γ == 0, A] in MATHEMATICA, we finally get the value of A as follows:

$$A = \pm\frac{\sqrt{\pm\left(\sqrt{9\varphi_0^4\mu^2\omega_0^2 - 156\varphi_0^2\mu\omega_0^2 - 312\varphi_0^2\rho^2 - 624\varphi_0\rho\dot{\varphi}_0 + 676\mu^2\omega_0^2 - 312\dot{\varphi}_0^2}\,/\mu\omega_0\right) + 3\varphi_0^2 + 26}}{\sqrt{6}}. \tag{29}$$

Here, ρ and μ are free parameters that are chosen in order to optimize the approximate solution, i.e., minimize the residual error.

Example 2. We can apply He's FF (23) on Example 1, which is given in equation (20). The approximation (23) and the RK4 numerical approximation are compared with each other as shown in Figure 2 and the error L_∞ is estimated at the best values of $(\rho,\mu) = (0.104, 0.99)$ as

$$\begin{cases} L_\infty = \max_{0\le t\le 30}\left|\varphi_{\text{Approx}} - \varphi_{\text{RK}}\right| = 0.00136089. \\ L_\infty = \max_{0\le t\le 30}\left|\varphi_{\text{He's}-FF}(t) - \varphi_{\text{RK}}(t)\right| = 0.00531151. \end{cases} \tag{30}$$

It is clear from the errors of both trigonometric solution (8) and He's FF (23) for the same values of parameters that the Trigonometric solution (8) is better than the solution of He's FF (23). However, for higher-order Chebyshev approximation to the function $f(x)$:

$$f(x) = \omega_0^2 \sin x \approx \omega_0^2\left(x - \frac{1}{6}x^3 - \frac{1}{127}x^5\right). \tag{31}$$

We cannot get more accuracy but in this case the third-order Chebyshev approximation is better than the fifth-order Chebyshev approximation.

4. Some Analytical Approximations to the Damped Disturbed NPE

Some different formulas to the analytical approximations to the following i.v.p. will be discussed in detail:

$$\begin{cases} \mathbb{R}_1 \equiv \ddot{\varphi} + 2\beta\dot{\varphi} + \phi(t)\sin\varphi = 0, \\ \varphi(0) = \varphi_0 \text{ and } \dot{\varphi}(0) = \dot{\varphi}_0. \end{cases} \tag{32}$$

In the next section, three different formulas with high accuracy are investigated.

4.1. First Formula: Jacobi Elliptic Form. Taking the approximation (13) into account and with the help of the approximation $\sin\varphi \approx (\lambda_0\varphi - \lambda_1\varphi^3)$, the i.v.p. (32) reduces to the following variable coefficients damped Duffing i.v.p.:

$$\begin{cases} \mathbb{R} \equiv \ddot{\varphi} + 2\beta\dot{\varphi} + \phi(t)\left(\lambda_0\varphi - \lambda_1\varphi^3\right) = 0, \\ \varphi(0) = \varphi_0 \text{ and } \dot{\varphi}(0) = \dot{\varphi}_0, \end{cases} \tag{33}$$

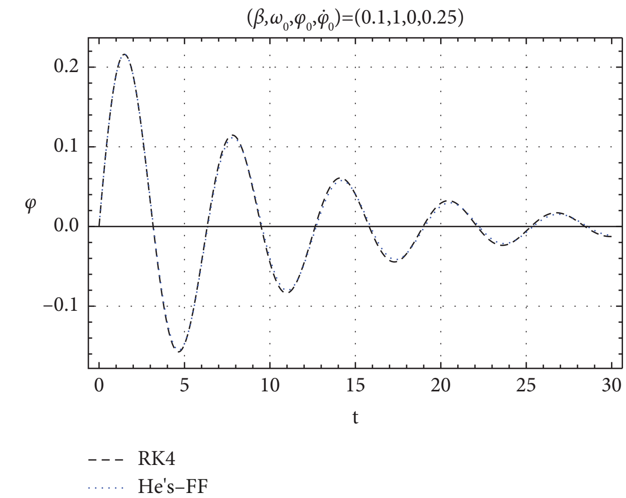

Figure 2: A comparison between the He's FF approximation (23) and the RK numerical approximation to the i.v.p. (6).

where $\lambda_0 = (332/333)$ and $\lambda_1 = (13/85)$. Note that the values of $\sin\varphi \approx (\lambda_0\varphi - \lambda_1\varphi^3)$ are obtained from the Chebyshev approximation.

Now, we seek approximate analytic solution to the i.v.p. (33) in the following ansatz form:

$$\varphi|_{\text{Approx}(1)} = \exp(-\beta t)\theta(f(t)), \tag{34}$$

where $\theta \equiv \theta(t)$ is a solution to the following i.v.p.:

$$\begin{cases} \ddot{\theta} + \kappa\left(\lambda_0\theta - \lambda_1\theta^3\right) = 0, \\ \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \dot{\theta}_0, \end{cases} \tag{35}$$

where $\kappa = (\omega_0^2 - \varepsilon\omega_2)$.

The solution of the i.v.p. (35) can be expressed in the following form:

$$\theta(t) = \frac{\theta_0 \text{cn}(\sqrt{w}\,t|m) + \left(\dot{\theta}_0/\sqrt{w}\right)\text{dn}(\sqrt{w}\,t|m)\text{sn}(\sqrt{w}\,t|m)}{1 + \left(p + q\varphi_0^2 - w/2w\right)\text{sn}(\sqrt{w}\,t|m)^2}, \tag{36}$$

with

$$p = \lambda_0\kappa, q = -\lambda_1\kappa, w = \frac{p}{1-2m}, \text{and } m = \frac{1}{2}\left(1 - \frac{p}{\sqrt{\left(p + q\theta_0^2\right)^2 + 2q\dot{\theta}_0^{\,2}}}\right). \tag{37}$$

The function $f \equiv f(t)$ is to be determined later, sn $(\sqrt{w}\,t|m)$ is the elliptic sine, cn $(\sqrt{w}\,t|m)$ is elliptic cosine, dn $(\sqrt{w}\,t|m)$ is the delta amplitude, and $0 \le m \le 1$.

Inserting solution (8) into the i.v.p. (33), we have the following equation:

$$\beta^2 + \lambda_0\kappa(\dot{f})^2 + \lambda_0\varepsilon\omega_2\cos(\gamma t) - \lambda_0\omega_0^2 = 0. \quad (38)$$

The solution of (38) gives the value of $f(t)$:

$$f = \pm\frac{\sqrt{333}}{\gamma\sqrt{83\kappa}}\sqrt{\lambda_0\kappa - \beta^2}E\left(\frac{\gamma}{2}t, \frac{2\lambda_0\varepsilon\omega_2}{\beta^2 - \lambda_0\kappa}\right). \quad (39)$$

We can choose the solution with plus sign, then $f(0) = 0$ and $\dot{f}(0) = \sqrt{-(\beta^2/\lambda_0\kappa) + 1}$.

Now, we must determine the values of θ_0 and $\dot{\theta}_0$ using the conditions $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = \dot{\varphi}_0$. The required values are given by the following equation:

$$\dot{\theta}_0 = \frac{\beta\varphi_0 + \dot{\varphi}_0}{\sqrt{1 - \left(\beta^2/\lambda_0\kappa\right)}}, \theta_0 = \varphi_0. \quad (40)$$

Inserting $f(t)$ given in (39) into solutions (34) and (36), we get the solution of the i.v.p. (33):

$$\varphi|_{\text{Approx}(1)} = \exp(-\beta t)\theta(f(t)), \quad (41)$$

with

$$\theta(f(t)) = \frac{\theta_0 \text{cn}(\sqrt{w}f(t)|m) + \left(\dot{\theta}_0/\sqrt{w}\right)\text{dn}(\sqrt{w}f(t)|m)\text{sn}(\sqrt{w}f(t)|m)}{1 + \left(p + q\varphi_0^2 - w/2w\right)\text{sn}(\sqrt{w}f(t)|m)^2}. \quad (42)$$

4.2. The Second Formula: Jacobi Elliptic Solution. Here, we can use another approximation to $\sin\varphi \approx (\varphi - 2/13\varphi^3)$, then the i.v.p. (32) reduces to the damped Duffing i.v.p. with variable coefficients:

$$\begin{cases} \mathbb{Q} \equiv \ddot{\varphi} + 2\beta\dot{\varphi} + \phi(t)\left(\varphi - \dfrac{2}{13}\varphi^3\right) = 0, \\ \\ \varphi(0) = \varphi_0 \text{ and } \dot{\varphi}(0) = \dot{\varphi}_0. \end{cases} \quad (43)$$

Let us consider the solution of the i.v.p. (43) is defined by the following ansatz:

$$\varphi(t)|_{\text{Approx}(2)} = \varphi_0 e^{-t\beta}\text{cn}(\omega(t) + C, m). \quad (44)$$

Inserting the ansatz (44) in the i.v.p (43), we finally obtain the following equation:

$$\begin{cases} 13m\dot{\omega}(t)^2 + \varphi_0^2 e^{-2t\beta}\phi(t) = 0, \\ (2m-1)\dot{\omega}(t)^2 - \beta^2 + \phi(t) = 0. \end{cases} \quad (45)$$

Eliminating m from system (45), we obtain the following equation:

$$\begin{cases} \dot{\omega}(t) = \sqrt{\left(\phi(t)\left(13 - 2\varphi_0^2 e^{-2t\beta}\right) - 13\beta^2/13\right)}, \\ \omega(0) = 0. \end{cases} \quad (46)$$

Equation (46) is not integrable, thus it solved numerically in order to get the value of $\omega(t)$.

Furthermore, the solution (44) could be written as

$$\begin{aligned} \varphi|_{\text{Approx}(2)} &= \varphi_0 e^{-t\beta} cn(\omega(t) + C, m) \\ &= \varphi_0 e^{-t\beta}\frac{cn + b_1 sndn}{1 + b_2 sn^2}, \end{aligned} \quad (47)$$

with

$$\begin{cases} sn \equiv sn(\omega(t)|m) = \sqrt{1 - cn^2}, \\ cn \equiv cn(\omega(t)|m), \\ dn \equiv dn(\omega(t)|m) = \left[\sqrt{1 - m\left(1 - cn^2\right)}\right]. \end{cases} \quad (48)$$

By inserting (46) and (47) into the i.v.p. (43) and after straightforward and simply calculations, finally, the values of b_1 and b_2 could be obtained as follows:

$$b_1 = \frac{\sqrt{13}(\beta\varphi_0 + \dot{\varphi}_0)}{\sqrt{-13\beta^2 - \kappa S_1}}, \quad (49)$$

and for $\varphi_0 \neq 0$

$$b_2 = -\frac{\kappa}{2\varphi_0\left(13\beta^2 + \kappa S_1\right)^2}\begin{bmatrix} \varphi_0\begin{pmatrix} -169S_2 - 26\beta\dot{\varphi}_0\varphi_0 \\ +4\kappa\varphi_0^4 - 52\kappa\varphi_0^2 \end{pmatrix} \\ \\ +13\sin(\varphi_0)\left(13\beta^2 + \kappa S_1\right) \end{bmatrix}, \quad (50)$$

while $\varphi_0 = 0$, we obtain $b_2 = 0$, where $S_1 = (2\varphi_0^2 - 13)$ and $S_2 = (\beta^2 - \kappa)$.

By substituting equations (49)–(53) into the i.v.p. (43) and solving the obtained solution, we finally obtain the value of "m" for $\varphi_0 \neq 0$ as follows:

$$m = \frac{\begin{array}{l} -26\kappa\varphi_0\dot{\varphi}_0 \cos(\varphi_0)\left(13S_2 + 2\kappa\varphi_0^2\right)^2 \\ + 26\kappa \sin(\varphi_0)\left(26\varphi_0\beta S_2 + 3\dot{\varphi}_0\left(13S_2 + 2\kappa\varphi_0^2\right) - 2\beta\kappa\varphi_0^3\right)\left(13S_2 + 2\kappa\varphi_0^2\right) \\ +\varphi_0 \left[\begin{array}{c} -156\beta\kappa\varphi_0\dot{\varphi}_0^2\left(13S_2 + 2\kappa\varphi_0^2\right) \\ +\beta\varphi_0 \left(\begin{array}{c} -13\gamma^2\varepsilon\omega_2 S_1\left(13S_2 + 2\kappa\varphi_0^2\right) + \\ 8\kappa^2\varphi_0^2\left(78\varphi_0^2 S_2 - 507S_2 + 4\kappa\varphi_0^4\right) - 8788\kappa S_2^2 \end{array}\right) \\ -\dot{\varphi}_0 \left(\begin{array}{c} 8\kappa\varphi_0^2\left(507\left(\beta^4 - \kappa^2\right) + 2\kappa^2\left(39 - 2\varphi_0^2\right)\varphi_0^2\right) + \\ 13\gamma^2\varepsilon\omega_2 S_1\left(13S_2 + 2\kappa\varphi_0^2\right) + \\ 8788\kappa S_2^2 \end{array}\right) \end{array}\right] \end{array}}{8\varphi_0\left(\beta\varphi_0 + \dot{\varphi}_0\right)\left(13S_2 + 2\kappa\varphi_0^2\right)^3}. \tag{51}$$

For $\varphi_0 = 0$, the value of the modulus "m" given in (51) reduces to

$$m = \frac{\gamma^2\varepsilon\omega_2}{8\left(\beta^2 - \kappa\right)^2}. \tag{52}$$

4.3. Third Formula: Trigonometric Approach. Another approximation in terms of trigonometric function to the i.v.p. (43) can be determined by inserting the following ansatz:

$$\varphi|_{\mathrm{Approx}(3)} = A\exp(-\rho t)\cos\left(f + \arccos\left(\frac{\varphi_0}{A}\right)\right), \tag{53}$$

into the i.v.p. (46) which leads to

$$\begin{aligned} \mathbb{Q} &\equiv \ddot{\varphi} + 2\beta\dot{\varphi} + \phi(t)\left(\varphi - \frac{2}{13}\varphi^3\right) \\ &= -\frac{2}{13}A^3\cos^3(\theta)e^{-3\rho t}\phi(t) + Ae^{-\rho t}\cos(\theta) \\ &\quad \cdot\left(\rho^2 - 2\beta\rho + \phi(t) - (\dot{f})^2\right) \\ &\quad + Ae^{-\rho t}\sin(\theta)\left(2(\beta - \rho)\dot{f} + \ddot{f}\right), \end{aligned} \tag{54}$$

where $f \equiv f(t)$, $f(0) = 0$, and ρ is an optimal parameter.

For vanishing the coefficient of $\cos(\theta)$, we obtain the following equation:

$$\left(\rho^2 - 2\beta\rho + \phi(t) - (\dot{f})^2\right) = 0, \text{and } f(0) = 0. \tag{55}$$

Solving (55) with $f(0) = 0$ yields

$$f = \int_0^t \sqrt{\rho^2 - 2\beta\rho + \phi(t)}\, d\tau = \frac{2\sqrt{\Pi_0}}{\gamma} E\left(\frac{\gamma}{2}t, -\frac{2Q_0}{\Pi_0}\right), \tag{56}$$

with $\Pi_0 = (\rho^2 - 2\beta\rho + \omega_0^2 - Q_0)$ and $E((\gamma/2)t, -(2Q_0/\Pi_0))$ represents the elliptic integral of the second kind.

Applying the ICs $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = \dot{\varphi}_0$, we obtain the following equation:

$$A = \pm\frac{\sqrt{2\rho\varphi_0 \cdot \dot{\varphi}_0 + \varphi_0^2\left(\Pi_0 + \rho^2\right) + \dot{\varphi}_0^{\ 2}}}{\sqrt{\Pi_0}}. \tag{57}$$

The number ρ is a free arbitrary parameter that we choose in order to get as small residual error as possible.

5. Results and Discussion

In this section, we proceed to analyze all obtained approximations. The discussion can be divided into several cases as follows.

Case 1. For the following numerical example $(\beta, \omega_0, \omega_2, \varepsilon, \gamma, \varphi(0), \dot{\varphi}(0)) = (0.1, 1, 0.5, 0.2, 0.2, 0, 0.1)$

$$\begin{cases} \mathbb{R} \equiv \ddot{\varphi} + 0.2\dot{\varphi} + \phi(t)\left(\lambda_1\varphi - \lambda_2\varphi^3\right) = 0, \\ \phi(t) = \omega_0^2 - \varepsilon\omega_2\cos(\gamma t) = 1 - 0.1\cos(0.2t), \\ \varphi(0) = 0 \text{ and } \dot{\varphi}(0) = 0.1. \end{cases} \tag{58}$$

The analytical approximations to the damped disturbed NPO (58) according to the first formula (44), second formula (50), and third formula (56) are compared with the RK4 numerical solution and the He's HPM approximation as illustrated in Figure 3. Also, the MDE L_∞ for the mentioned approximations is estimated:

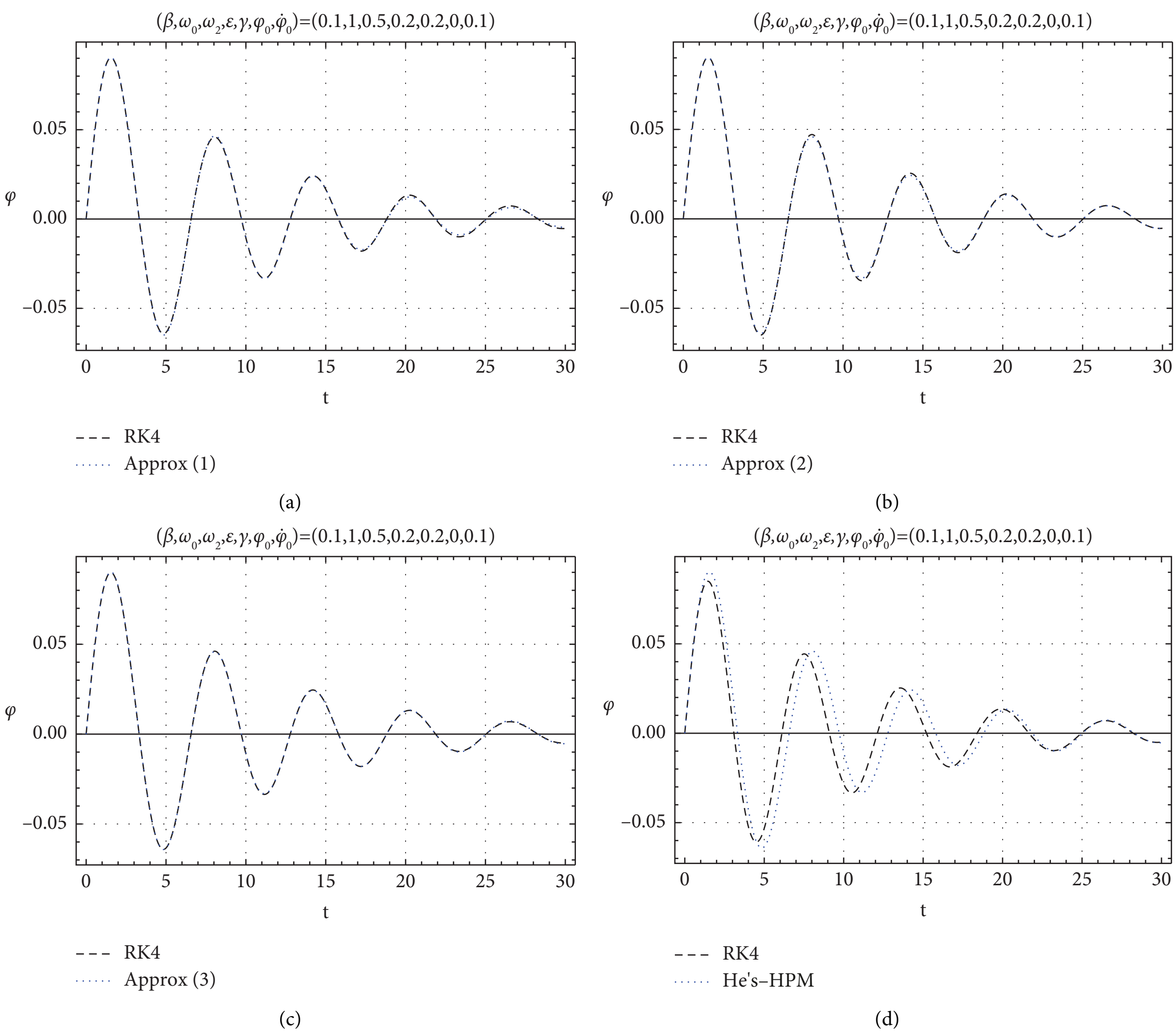

Figure 3: A comparison between the RK numerical approximation and the different types of the analytical approximations to the i.v.p. (33): (a) the Jacobi elliptic solution (44), (b) the Jacobi elliptic solution (50), (c) the trigonometric solution (56), and (d) the He's HPM approximation. Here, $(\omega_2, \varphi_0) = (1, 0)$.

$$\begin{cases} L_\infty\big|_{\text{Approx}(1)} = \max_{0\le t\le 30}\left|\varphi|_{\text{Approx}(1)} - \varphi_{\text{RK}}\right| = 0.00114366, \\ L_\infty\big|_{\text{Approx}(2)} = \max_{0\le t\le 30}\left|\varphi|_{\text{Approx}(2)} - \varphi_{\text{RK}}\right| = 0.00139714, \\ L_\infty\big|_{\text{Approx}(3)} = \max_{0\le t\le 30}\left|\varphi|_{\text{Approx}(3)} - \varphi_{\text{RK}}\right| = 0.000432514, \\ L_\infty\big|_{\text{He's HPM}} = \max_{0\le t\le 30}\left|\varphi|_{\text{He's–HPM}} - \varphi_{\text{RK}}\right| = 0.0246647. \end{cases} \tag{59}$$

It is observed that the exact congruence between the analytical approximations (41), (47), (53), and the RK4 numerical solution. Also, it is clear that the derived approximations show high accuracy as compared to the He's HPM approximation. Henceforth, for He's HPM approximation, we used $\lambda_1 = 1$ and $\lambda_2 = 1/6$.

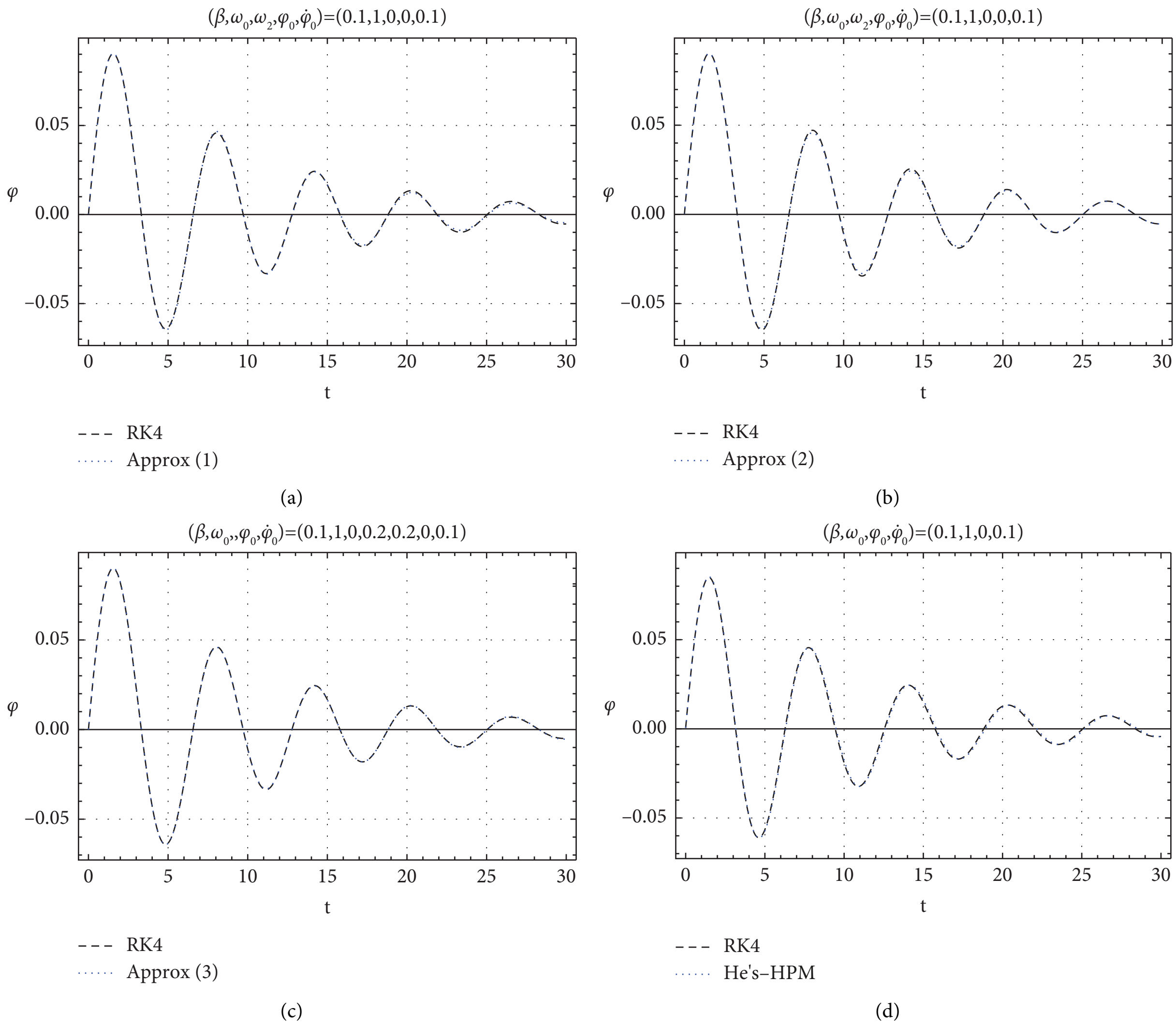

Figure 4: A comparison between the RK numerical approximation and the different types of the analytical approximations to the i.v.p. (33): (a) the Jacobi elliptic solution (44), (b) the Jacobi elliptic solution (50), (c) the trigonometric solution (56), and (d) the He's HPM approximation. Here, $(\omega_2, \varphi_0) = (0, 0)$.

Case 2. The approximations (41), (47), and (53) can recover the solutions to the damped undisturbed NPO (32) for $\omega_2 = 0$, i.e., the i.v.p. (7). Now, by considering the values $(\beta, \omega_0, \omega_2, \varphi(0), \dot{\varphi}(0)) = (0.1, 1, 0, 0, 0.1)$, the i.v.p. (7)

$$\begin{cases} \ddot{\varphi} + 0.2\dot{\varphi} + p\varphi + q\varphi^3 = 0, \\ \varphi(0) = 0 \text{ and } \dot{\varphi}(0) = 0.1. \end{cases} \tag{60}$$

The analytical approximations (8), (41), (47), (53), and the RK4 approximation and the He's HPM approximation to the i.v.p. (60) are graphically introduced in Figure 4. Also, the MDE L_∞ is computed for all mentioned approximations

$$\begin{cases} L_\infty\big|_{\text{Sol. (8)}} = \max_{0\le t\le 30}\left|\varphi(t)\right|_{\text{Sol. (8)}} - \varphi_{\text{RK}}(t)\big| = 0.000555229, \\ L_\infty\big|_{\text{Approx (1)}} = \max_{0\le t\le 30}\left|\varphi(t)\right|_{\text{Approx (1)}} - \varphi_{\text{RK}}(t)\big| = 0.000714541, \\ L_\infty\big|_{\text{Approx (2)}} = \max_{0\le t\le 30}\left|\varphi(t)\right|_{\text{Approx (2)}} - \varphi_{\text{RK}}(t)\big| = 0.000121574, \\ L_\infty\big|_{\text{Approx (3)}} = \max_{0\le t\le 30}\left|\varphi(t)\right|_{\text{Approx (3)}} - \varphi_{\text{RK}}(t)\big| = 0.000120303, \\ L_\infty\big|_{\text{He's HPM}} = \max_{0\le t\le 30}\left|\varphi\right|_{\text{He's–HPM}} - \varphi_{\text{RK}}\big| = 0.00189472. \end{cases} \tag{61}$$

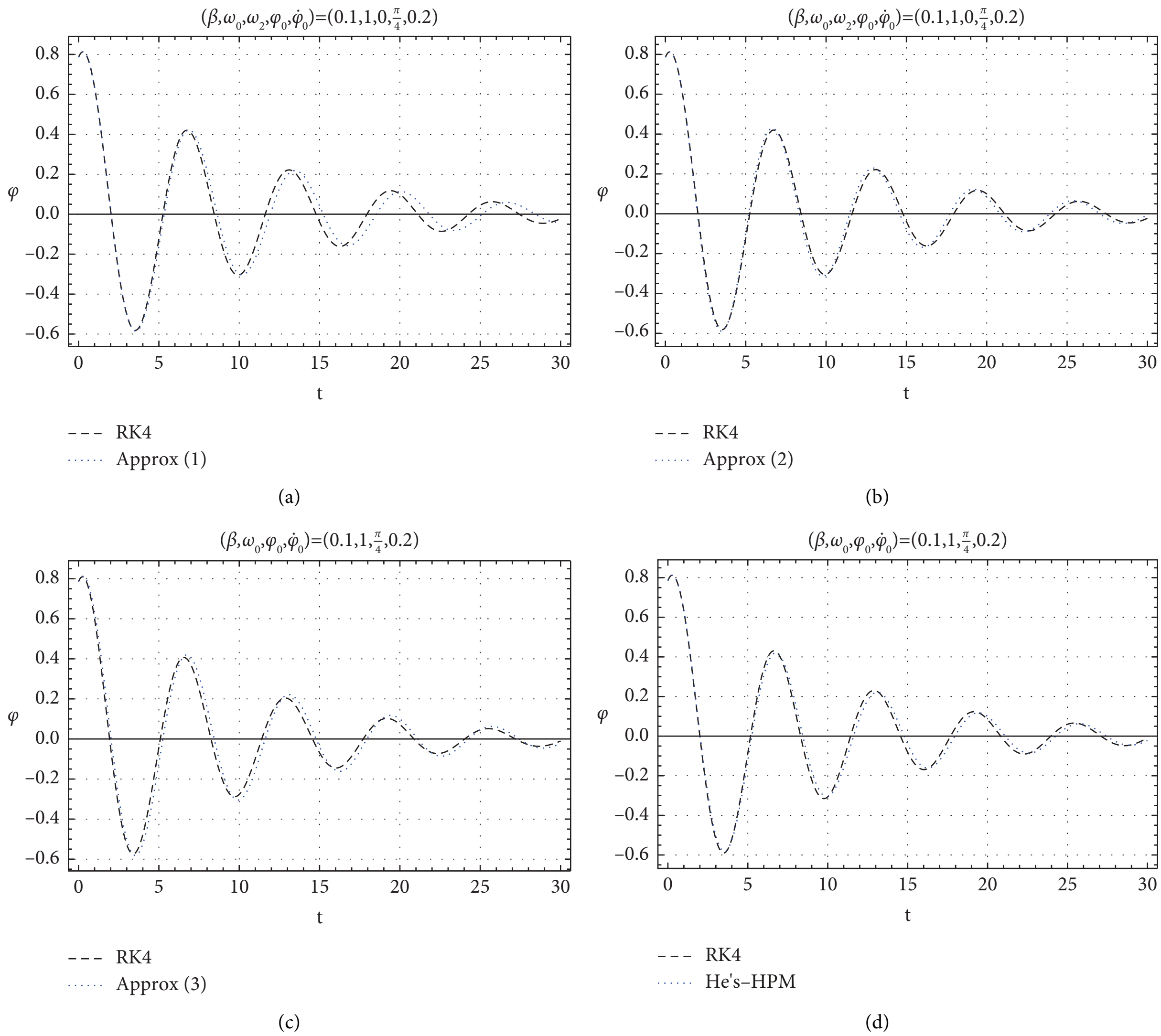

Figure 5: A comparison between the RK numerical approximation and the different types of the analytical approximations to the i.v.p. (62): (a) the Jacobi elliptic solution (44), (b) the Jacobi elliptic solution (50), (c) the trigonometric solution (56), and (d) the He's HPM approximation. Here, $(\omega_2, \varphi_0) = (0, \pi/4)$.

Case 3. For $(\beta, \omega_0, \omega_2, \varphi(0), \dot{\varphi}(0)) = (0.1, 1, 0, \pi/4, 0.2)$, the numerical form to the i.v.p. (7) reads

$$\begin{cases} \ddot{\varphi} + 0.2\dot{\varphi} + p\varphi + q\varphi^3 = 0, \\ \varphi(0) = \dfrac{\pi}{4} \text{ and } \dot{\varphi}(0) = 0.2. \end{cases} \tag{62}$$

The trigonometric solution (8) as well as the Jacobi elliptic solutions (41) and (47) in addition to the RK4 numerical solution and the He's HPM approximation to the i.v.p. (62) are graphically plotted as shown in Figure 5 and their MDE L_∞ is estimated as follows:

$$\begin{cases} L_\infty|_{\text{Sol. (8)}} = \max_{0\le t\le 30}\left|\varphi(t)|_{\text{Sol. (8)}} - \varphi_{\text{RK}}(t)\right| = 0.0116904, \\ L_\infty|_{\text{Approx (1)}} = \max_{0\le t\le 30}\left|\varphi(t)|_{\text{Approx (1)}} - \varphi_{\text{RK}}(t)\right| = 0.0806987, \\ L_\infty|_{\text{Approx (2)}} = \max_{0\le t\le 30}\left|\varphi(t)|_{\text{Approx (2)}} - \varphi_{\text{RK}}(t)\right| = 0.0422837, \\ L_\infty|_{\text{Approx (3)}} = \max_{0\le t\le 30}\left|\varphi(t)|_{\text{Approx (3)}} - \varphi_{\text{RK}}(t)\right| = 0.0807087, \\ L_\infty|_{\text{He}'\text{sHPM}} = \max_{0\le t\le 30}\left|\varphi|_{\text{He}'\text{s}-\text{HPM}} - \varphi_{\text{RK}}\right| = 0.0528277. \end{cases} \tag{63}$$

It is observed that the trigonometric approximation (8) is better than all other approximations for arbitrary angles with the pivot vertical. Despite this, all the obtained approximations give satisfactory results and have good accuracy.

Furthermore, all analytical approximations (41), (47), and (53) can be recovered the undamped disturbed nonlinear pendulum oscillation ($\beta = 0\backslash\&\omega_2 \neq 0$) and the undamped undisturbed nonlinear pendulum oscillation ($\beta = \omega_2 = 0$) for arbitrary angles with the vertical pivot. In addition, the obtained solutions can be used for investigation the nonlinear oscillations in different plasma models which most evolutions equations that governed the dynamics of plasma waves and oscillations can be reduced to a pendulum equations (25) and (26).

6. Conclusions

In the current work, the parametric pendulum oscillatory equation and some related oscillatory equations have been solved using different analytical and numerical techniques. In this investigation, two-cases called the damped undisturbed nonlinear pendulum equation/oscillator (NPE/NPO) and the damped disturbed NPE have been discussed. The following list provides a concise summary of the most significant findings:

(i) For the first oscillator, i.e., damped undisturbed NPO, this oscillator has been reduced to the constant coefficients damped Duffing equation, and its analytical approximations have been derived in terms of the trigonometric functions

  (a) Both ansatz method and He's frequency formulation were employed to find some approximations for the damped undisturbed NPE
  (b) The outcomes of comparing the obtained approximations and the numerical solutions revealed the great correctness of the obtained solutions

(ii) In the second oscillator, i.e., damped disturbed NPO, this oscillator has been reduced to the variable coefficients damped Duffing oscillator in order to facilitate the solution process

  (a) Three different formulas for the analytical approximations to the damped Duffing equation with variable coefficients in terms of Jacobi elliptic and trigonometric functions have been derived in detail.
  (b) In the first-formula, the modulus of Jacobi elliptic solution has been taken as zero while in the second formula, the modulus of Jacobi elliptic solution was taken as arbitrary value.
  (c) In the third formula, a new ansatz in terms of the trigonometric function was employed to find a high-accurate approximation in terms of trigonometric function.
  (d) It was found that the three-formulas to the analytical approximations to the damped disturbed NPO can be recovered different cases for the pendulum oscillators. Consequently, we discussed several cases for the nonlinear pendulum oscillators for small and arbitrary angles with the vertical pivot, e.g., the damped disturbed NPE ($\beta \neq 0\&\omega_2 \neq 0$) and the damped undisturbed nonlinear pendulum oscillation ($\beta \neq 0\&\omega_2 = 0$). Also, the obtained approximations can be recovered the undamped disturbed nonlinear pendulum oscillation ($\beta = 0 \,\&\, \omega_2 \neq 0$) and the undamped undisturbed nonlinear pendulum oscillation ($\beta = \omega_2 = 0$) for arbitrary angles with the vertical pivot.

Finally, the obtained results were compared with the RK4 numerical approximation and the He's HPM approximation. It was found that the derived anlaytical approximations are distinguished by their great precision and more stable across the whole time domain, especially the third formula. Many equations of motions that govern the various pendulum oscillations can be solved using all proposed techniques. In addition, the obtained solutions aid in the investigation of nonlinear oscillations in different plasma physics.

Data Availability

All data generated or analyzed during this study are included within the article and more details are available from the corresponding author upon request.

Additional Points

Future Work. The study of stability analysis to the present pendulum oscillator is one of the most important topics, but it is out of the scope of the present study and it will be addressed in the next work. Also, the differential transform method (DTM) with the Padé approximation in addition can be used for analyzing the present oscillator [34].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and approved the final manuscript.

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