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Research Article

Iterative Approximation of Common Fixed Points for Edge-Preserving Quasi-Nonexpansive Mappings in Hilbert Spaces along with Directed Graph

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We present iterative approximation results of an iterative scheme for finding common fixed points of edge-preserving quasi-nonexpansive self-maps in Hilbert spaces along with directed graph. We obtain weak as well as strong convergence of our scheme under various assumptions. That is, we impose several possible mild conditions on the domain, on the mapping, or on the parameters involved in our scheme to prove convergence results. We support numerically our main outcome by giving an example. Eventually, an application is provided for solving a variational inequality problem. Our result are new/generalized some recently announced results of the literature.

1. Introduction

Theory of fixed points is an important field of analysis that provides possibly efficient tools for solving nonlinear problems (see, e.g., [1, 2] and others). For a nonlinear problem, it is often very difficult to calculate the exact value of solution. In this case, the approximate value of the requested solution is always under the consideration [3, 4]. In 1922, Banach [5] suggested the well-known principle for a special class of operators called contractions in Banach spaces. He proved that any contraction on a Banach space admits one and only one fixed point and the sequence of successive approximations converges in the strong sense to this fixed point. The Banach result motivated several researchers and many generalizations of this principle have been investigated [4, 6]. The sequence of successive approximations does not converge in general when the mapping is nonexpansive. Thus, Mann [7] and Ishikawa [8] suggested new iterative schemes for fixed point approximations of nonexpansive and generalized nonexpansive mappings.

Now we consider a Hilbert space, namely, $\mathbb{X}^*$ and assume that $\mathbb{K}^*$ denote any nonempty convex as well as a closed subset of $\mathbb{X}^*$. Notice that an element $e \in \mathbb{X}^*$ is known as a fixed point for the self-map $P\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ if one has $P(e) = e$. In this research, the notation $F^*(P)$ will denote the fixed point set of P, i.e., $F^*(P) = \{e \in \mathbb{K}^*\colon Pe = e\}$. The self-map $P\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ is known as a nonexpansive map if $\|Pe - \Pr\| \le \|e - r\|$ for all $e, r \in \mathbb{K}^*$. Also, P is known as a quasi-nonexpansive mapping when P has at least one fixed point in $\mathbb{X}^*$, i.e., $F^*(P) \neq \varnothing$ and if $z^* \in F^*(P)$, then $\|Pe - z^*\| \le \|e - z^*\|$ for all $e \in \mathbb{X}^*$. Diaz and Metcalf [9] introduced the concept of quasi-nonexpansive mapping along with some related ideas. Note that the set of fixed point of a quasi-nonexpansive mapping is closed and convexed [10].

In [11], the authors suggested some applications of the fixed point theory of nonexpansive mappings in image recovery and signal processing. Many researchers developed different iteration schemes to find approximate fixed points of nonexpansive type mappings. In 1974, Senter and Dotson [12] established some fixed point results for quasi-nonexpansive mapping through Mann iteration in a uniformly convex Banach space. In a Hilbert space setting, by an elementary constructive method, Itoh and Takahashi [10] proved existence of common fixed points of a quasi-nonexpansive mapping in 1978. In 1992, Ghosh and Debnath [13] established convergence of Ishikawa iteration [8] to a unique fixed point of a quasi-nonexpansive mapping in a uniformly convex Banach space. In 2011, Jin and Tian [14] proved some fixed point results for quasi-nonexpansive mapping in a Hilbert space by using an iterative process involving Lipschitzian mapping. In 2013, Suantai and Bunyawat [15] established fixed points in a uniformly convex Banach space for a finite family of multi-valued quasi-nonexpansive mappings.

In the recent research in the theory of fixed points, several researchers studied theory of fixed points in the setting of graph theory and many fruitful generalizations of the classical results are obtained. For example, Echenique [16] studied theory of fixed points in the setting of graph theory and obtained several new results. Similarly, Espinola and Kirk [17] obtained some novel fixed point results in the setting of graph theory. Alfuraidan and Khamsi [18] studied theory of fixed points for nonexpansive mappings in the setting of a hyperbolic space with direct graph. Tiammee et al. [19] obtained the analog of the remarkable Browder's fixed point result in the setting of a Hilbert space with directed graph.

Suppose a directed graph G^*, $V^*(G^*)$ its vertices, and $E^*(G^*)$, the set of edges. Assume that $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ is convex and transitive. Suppose the mappings P_i: $(i = 1, 2)$ from $\mathbb{K}^*$ to $\mathbb{K}^*$ are $G-$ are edge-preserving with $F^*(P) = F^*(P_1) \cap F^*(P_2)$ nonempty.

In [20], Tripak defined the following iteration for finding fixed points which produce a sequence of iterates $\{e_k\}$ for a starting point $e_1 \in \mathbb{K}^*$:

$$\begin{cases} e_{k+1} = (1-\alpha_k)e_k + \alpha_k P_1 r_k, \\ r_k = (1-\beta_k)e_k + \beta_k P_2 e_k, \end{cases} \tag{1}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are sequences in [0, 1].

Motivated by the work of Tripak [20], in 2018, Suparatulatorn et al. [21] proved weak and strong convergence of the following iteration scheme for $G-$ nonexpansive mappings in Banach spaces endowed with graph:

$$\begin{cases} e_{k+1} = (1-\alpha_k)P_1 e_k + \alpha_k P_2 r_k, \\ r_k = (1-\beta_k)e_k + \beta_k P_1 e_k, k \geq 1, \end{cases} \tag{2}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are sequences in [0, 1].

Inspired by the work of Suparatulatorn et al. [21], we proved some weak and strong convergence of a sequence $\{e_k\}$ defined by (2) for edge-preserving quasi-nonexpansive mappings in a Hilbert space along with directed graph. With the help of an example, we show comparison between sequences $\{e_k\}$ defined by (1) and (2).

2. Preliminaries

This section start with some necessary concepts and some established results that are required to obtain the main results.

Definition 1. Consider a graph $G^* = (V^*(G^*), E^*(G^*))$. In this case, for any choice of vertices e and r in G^*, the path in G^* from e to r of the length $N \in \mathbb{N} \cup \{0\}$ is essentially a sequence, namely, $\{e_k\}_{k=0}^{N}$ of $N+1$ vertices with $e_0 = e$, $e_N = r$ and $(e_{k-1}, e_k) \in E^*(G^*)$ for any $k = 0, 1, \ldots, N-1$.

Definition 2. A given graph is referred to as a connected graph if one is able to find an edge (path) between any two given vertices of G^*. While, a directed graph (sometimes called digraph), is a graph such that the edges have a direction. For any two given vertices e and r in a graph G^*, the distance $d(e, r)$ from e to r is defined as a length of a shortest path from e to r.

Remark 1. In order for $d(e, r)$ to be defined for all pairs of e, r of vertices in G^*, the graph G^* must be connected.

Definition 3 (see [21]). Assume that, we have a non-empty subset, namely, $\mathbb{K}^*$ of any given Hilbert space $\mathbb{X}^*$. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ is a directed graph with $V^*(G^*) = \mathbb{K}^*$, and $E^*(G^*)$ contains all the loops, i.e., $E^*(G^*) \supseteq \{(e, e)\colon e \in V^*(G^*)\}$. The self-map $P\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ is known as an edge-preserving nonexpansive self-map (sometimes called $G-$ edge-preserving nonexpansive self-map) if

$$\text{for all } e, r \in \mathbb{K}^*,\ (e, r) \in E^*(G^*) \Rightarrow (P(e), P(r)) \in E^*(G^*). \tag{3}$$

Definition 4. Assume that, we have a non-empty subset, namely, $\mathbb{K}^*$ of any given Hilbert space $\mathbb{X}^*$. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ is a directed graph with $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ contains all the loops. The self-map $P\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ is known as an edge-preserving quasi-nonexpansive self-map, if one have

(i) P is edge-preserving

(ii) P is quasi-nonexpansive

Definition 5 (see [19]). Assume that, we have a directed graph, namely, $G^* = (V^*(G^*), E^*(G^*))$. In this case, one regards G^* as a transitive directed graph whenever for any pair of point, $e, r, s \in V^*(G^*)$ satisfying $(e, r) \in E^*(G^*)$ and $(r, s) \in E^*(G^*)$, it follows that $(e, s) \in E^*(G^*)$.

Definition 6 (see [19, 21]). Assume that, we have non-empty subset, namely, $\mathbb{K}^*$ of a given Hilbert space $\mathbb{X}^*$. Then, for any $G^* = (V^*(G^*), E^*(G^*))$, where $V^*(G^*) = \mathbb{K}^*$ is any given directed graph, the set $\mathbb{K}^*$ is referred to as a set with the property WG, whenever, we have a weakly convergent sequence $\{e_k\}$ in $\mathbb{K}^*$ and the point $e \in \mathbb{K}^*$ if its weak limit, then one can construct a subsequence, namely, $\{e_{k_n}\}$ of $\{e_k\}$ with the property $(e_{k_n}, e) \in E^*(G^*)$ for any choice of $k \in \mathbb{N}$.

Definition 7 (see [21]). Assume that, we have a non-empty subset, namely, $\mathbb{K}^*$ of a given Hilbert space $\mathbb{X}^*$ and $P\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ a self-map. The self-map P is known as a G–demiclosed at $r \in \mathbb{X}^*$ if, for each choice of the sequence $\{e_k\}$ in $\mathbb{K}^*$ with $\{e_k\}$ convergent weakly to $e \in \mathbb{K}^*$, $\{Pe_k\}$ convergent strongly to r and $(e_k, e_{k+1}) \in E^*(G^*)$, it follows that $Pe = r$.

Definition 8 (see [22]). A Banach space $\mathbb{X}^*$ satisfies Opial's property, if for any sequence $\{e_k\}$, $e_k \rightharpoonup e$ implies that

$$\limsup_{k \longrightarrow \infty} \|e_k - e\| < \limsup_{k \longrightarrow \infty} \|e_k - r\|, \tag{4}$$

for all $r \in \mathbb{X}^*$ such that $e \neq r$. It is known that every Hilbert space satisfies Opial's property.

Definition 9 (see [23]). Assume that, we have a Hilbert space, namely, $\mathbb{X}^*$. In this case, the sequence $\{e_k\}$ in $\mathbb{X}^*$ is referred to as a Fejer monotone sequence corresponding to the subset $\mathbb{K}^*$ of $\mathbb{X}^*$, if one has

$$\|e_{k+1} - p\| \leq \|e_k - p\|, \tag{5}$$

for any choice of $p \in \mathbb{K}^*$, $k \geq 1$.

Proposition 1 (see [23]). *Assume that, we have a non-empty subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{K}^*$. In this case, if the sequence $\{e_k\}$ is a Fejer monotone corresponding to the set $\mathbb{K}^*$. Then, one have*

(a) The sequence $\{e_k\}$ is essentially bounded

(b) For any choice of $e \in \mathbb{K}^*$, the sequence $\{\|e_k - e\|\}$ is convergent.

Suppose that $\mathbb{X}^*$ is a Hilbert space. We can construct a graph in $\mathbb{X}^*$ by taking $V^*(G^*) = \mathbb{X}^*$ or $V^*(G^*) =$ any subset of $\mathbb{X}^*$, and $E^*(G^*) \supseteq \{(e, e)\colon e \in V^*(G^*)\}$, i.e., $E^*(G^*)$ contains all the loops (for constructing graph in an arbitrary space, refer [19]).

Definition 10 (see [20, 21]). Assume that, we have a non-empty subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{X}^*$. In this case, the self-maps $P_i\,(i = 1, 2)$ on K^* are said to be with condition $(B$, if one has essentially a function that is nondecreasing and have the properties $f^*(0) = 0$, $f^*(r) > 0$ for all other $r > 0$ and for any choice of $e \in \mathbb{K}^*$,

$$\max\{\|e - P_1 e, \|e - P_2 e\|\} \geq f(d(e, F^*(P))), \tag{6}$$

where $d(e, F^*(P)) = \inf\{\|e - z^*\|\colon z^* \in F^*(P)\}$ *and* $F^*(P) = F^*(P_1) \cap F^*(P_2)$.

Definition 11 (see [20, 21]). Assume that, we have a non-empty subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{X}^*$ and $G^* = (V^*(G^*), E^*(G))$ a direct graph with $V^*(G^*) = K^*$. In this case, the set E^* is said to be with WG property if for any choice of $\{e_k\}$ in K^* with weak limit p and $(e_k, e_{k+1}) \in E^*(G^*)$, one has a subsequence $\{e_{k_i}\}$ *of* $\{e_k\}$ with $(e_{k_i}, p) \in E^*(G^*)$

The following results will be used to reach our main results:

Lemma 1 (see [24]). *Assume that, we have a Banach space, namely, $\mathbb{X}^*$. In this case, if $\mathbb{X}^*$ is with Opial's property, then for any choice of a sequence, namely, $\{e_k\}$ in $\mathbb{X}^*$ and $e, r \in \mathbb{X}^*$, if one have $\lim_{k \longrightarrow \infty}\|e_k - e\|$ and $\lim_{k \longrightarrow \infty}\|e_k - r\|$ exists and the subsequences, namely, $\{e_{k_n}\}$ and $\{e_{k_m}\}$ of $\{e_k\}$ that are weakly convergent to e and r, respectively, then subsequently, it follows that $e = r$.*

Lemma 2 (see [25]). *Assume that, we have a uniformly convex Banach space, namely, $\mathbb{X}^*$ and there is a sequence $\{\alpha_k\}$ in $[\delta^*, 1 - \delta^*]$ where $\delta^* \in (0, 1)$. If one have a real number, $c^* \geq 0$ such that for any $\{e_k\}$ and $\{r_k\}$ in $\mathbb{X}^*$ with $\limsup_{k \longrightarrow \infty}\|e_k\| \leq c^*$, $\limsup_{k \longrightarrow \infty}\|r_k\| \leq c^*$, and $\limsup_{k \longrightarrow \infty}\|\alpha_k e_k + (1 - \alpha_k) r_k\| = c^*$. Subsequently, one has $\lim_{k \longrightarrow \infty}\|e_k - r_k\| = 0$.*

Lemma 3 (see [26]). *Assume that, we have a Banach space, namely, $\mathbb{X}^*$. In this case, for any real constant $R > 1$, the space $\mathbb{X}^*$ is uniformly convex $\Leftrightarrow$ one can find a function g that is strictly increasing, convex and continuous and admit the properties, $g(0) = 0$ and*

$$\begin{aligned}\|\lambda^* e + (1 - \lambda^*) r\|^2 \leq \lambda^*\|e\|^2 + (1 - \lambda^*)\|r\|^2 \\ - \lambda^*(1 - \lambda^*) g(\|e - r\|),\end{aligned} \tag{7}$$

for any choice of $e \in B_R(0) = \{e \in \mathbb{X}\colon \|e\| \leq R\}$ and $\lambda^* \in [0, 1]$.

Lemma 4 (see [27]). *Assume that, we have a non-empty subset, namely, K^* of a given Hilbert space. In this case, for any choice of $x \in H^*$ and $y \in K^*$, it follows that the following are essentially equivalent:*

(i) $\|x - y\| = d(x, K^*)$

(ii) $<x - y, z - y> \geq 0$, for every choice of $z \in K^*$

3. Main Results

This section proceeds with the following lemmas:

Lemma 5. *Assume that, we have a non-empty convex closed subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{K}^*$. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ a directed transitive graph with $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ is convex. Assume that $P_i\,(i = 1, 2)\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ is an edge-preserving quasi-nonexpansive self-map. Fix any point $e_1 \in \mathbb{K}^*$ with $(e_1, P_1 e_1) \in E^*(G^*)$ and $(e_1, P_2 r_1) \in E^*(G^*)$. Let $\{e_k\}$ be a sequence in $V^*(G^*)$ defined by (2) and $z^* \in F^*(P)$ with (e_1, z^*), $(z^*, e_1) \in E^*(G^*)$. Then, we have the followings:*

(a) $(e_k, e_{k+1}) \in E^*(G^*)$ *and* $(r_k, r_{k+1}) \in E^*(G^*)$ *for every choice of* $k \geq 1$

(b) $(e_k, P_1 e_k) \in E^*(G^*)$ *and* $(e_k, P_2 r_k) \in E^*(G^*)$ *for every choice of* $k \geq 1$

(c) $(e_{k+1}, P_1 e_k) \in E^*(G^*)$ *and* $(e_{k+1}, P_2 r_k) \in E^*(G^*)$ *for every choice of* $k \geq 1$

Proof

(a) Since we have $(e_1, z^*) \in E^*(G^*)$, and P_1 is essentially an edge-preserving, it follows that $(P_1 e_1, z^*) \in E^*(G^*)$. Now thanks to the convexity of $E^*(G^*)$, one has

$$\left((1-\beta_1)e_1 + \beta_1 P_1 e_1, (1-\beta_1)z^* + \beta_1 z^*\right) \in E^*(G^*). \tag{8}$$

i.e., $(r_1, z^*) \in E^*(G^*)$. By edge-preserving of P_2, we have $(P_2 r_1, z^*) \in E^*(G^*)$. Since $(P_1 e_1, z^*) \in E^*(G^*)$, $(P_2 r_1, z^*) \in E^*(G^*)$, by convexity of $E^*(G^*)$, we have

$$\left((1-\alpha_1)P_1 e_1 + \alpha_1 P_2 r_1, (1-\alpha_1)z^* + \alpha_1 z^*\right) \in E^*(G^*). \tag{9}$$

i.e., $(e_2, z^*) \in E^*(G^*)$. Since $(e_1, z^*) \in E^*(G^*)$, $(e_2, z^*) \in E^*(G^*)$, by transitivity of G^*, we have $(e_1, e_2) \in E^*(G^*)$. Continuing this process, we get $(e_k, e_{k+1}) \in E^*(G^*)$.

Note that $(e_2, z^*) \in E^*(G^*)$, hence $(P_1 e_2, z^*) \in E^*(G^*)$. Now thanks to the convexity of $E^*(G^*)$, it follows that $(r_2, z^*) \in E^*(G^*)$ and also using the transitivity of G^*, it follows that $(r_1, r_2) \in E^*(G^*)$. If we go in the similar way, we get $(r_k, r_{k+1}) \in E^*(G^*)$.

(b) Now we go the prove $(e_k, P_1 e_k) \in E^*(G^*)$ for every choice of $k \geq 1$. For this, we use the method of induction on k. As assumed, $(e_1, P_1 e_1) \in E^*(G^*)$, it follows that the induction method is hold for the value $k = 1$. Next we assume that $(e_k, P_1 e_k) \in E^*(G^*)$ for any choice of $k \geq 2$. Notice that $(e_k, e_{k+1}) \in E^*(G^*)$, $(e_k, P_1 e_k) \in E^*(G^*)$, it follows that $(e_{k+1}, P_1 e_k) \in E^*(G^*)$ due to the transitive property of G^*. Also $(e_k, e_{k+1}) \in E^*(G^*)$, it follows that $(P_1 e_k, P_1 e_{k+1}) \in E^*(G^*)$ because P_1 is edge-preserving map. Again, notice that $(e_{k+1}, P_1 e_k) \in E^*(G^*)$, and $(P_1 e_k, P_1 e_{k+1}) \in E^*(G^*)$, that lead us to the fact $(e_{k+1}, P_1 e_{k+1}) \in E^*(G^*)$.

In a similarly way, $(e_1, P_2 r_1) \in E^*(G^*)$, this proves that the induction is hold for the value $k = 1$. We now suppose $(e_k, P_2 r_k) \in E^*(G^*)$ for any choice of $k \geq 2$. Since $(r_k, r_{k+1}) \in E^*(G^*)$, it follows that $(P_2 r_k, P_2 r_{k+1}) \in E^*(G^*)$. Also $(e_k, e_{k+1}) \in E^*(G^*)$, $(e_k, P_2 r_k) \in E^*(G^*)$, due to the transitive property of G^*, it follows that $(e_{k+1}, P_2 r_k) \in E^*(G^*)$. Now $(e_{k+1}, P_2 r_k) \in E^*(G^*)$, $(P_2 r_k, P_2 r_{k+1}) \in E^*(G^*)$, one has $(e_{k+1}, P_2 r_{k+1}) \in E^*(G^*)$.

(c) By part (b), we have $(e_{k+1}, P_1 e_k) \in E^*(G^*)$ and $(e_{k+1}, P_2 r_k) \in E^*(G^*)$ for any $k \geq 1$. □

Lemma 6. *Assume that, we have a non-empty convex closed subset, namely,* $\mathbb{K}^*$ *of any Hilbert space* $\mathbb{X}^*$*. In this case, suppose* $G^* = (V^*(G^*), E^*(G^*))$ *is a directed transitive graph with* $V^*(G^*) = \mathbb{K}^*$ *and* $E^*(G^*)$ *is convex. Assume that* $P_i\,(i = 1, 2)\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ *is an edge-preserving quasi-nonexpansive self-maps. Fix any point* $e_1 \in \mathbb{K}^*$ *with* $(e_1, P_1 e_1) \in E^*(G^*)$ *and* $(e_1, P_2 r_1) \in E^*(G^*)$*. Let* $\{e_k\}$ *be a sequence in* $V(G)$ *defined by (2) and* $z^* \in F^*(P)$ *such that* $(e_1, z^*), (z^*, e_1) \in E^*(G^*)$*. Let* $\{\alpha_k\}$, $\{\beta_k\}$ *are sequences in* $[\delta^*, 1-\delta^*]$*, for some* $\delta^* \in (0, 1)$*. Then, we have followings:*

(a) $(e_k, z^*), (z^*, e_k), (r_k, z^*)$*, and* (z^*, r_k) *are in* $E^*(G^*)$ *for* $k \geq 2$

(b) $(e_k, r_k) \in E^*(G^*)$ *for* $k \geq 1$

Proof

(a) We proceed by induction on k. Since $(e_1, z^*) \in E^*(G^*)$, hence induction is true for $k = 1$. By Lemma 5(a), we have $(e_2, z^*) \in E^*(G^*)$. Next we assume that $(e_k, z^*) \in E^*(G^*)$ for $k \geq 3$. As $(e_k, z^*) \in E^*(G^*)$, we have $(P_1 e_k, z^*) \in E^*(G^*)$. Note that (e_k, z^*), $(P_1 e_k, z^*) \in E^*(G^*)$, by convexity of $E^*(G^*)$, we have

$$\left((1-\beta_k)e_k + \beta_k P_1 e_k, (1-\beta_k)z^* + \beta_k z^*\right) \in E^*(G^*). \tag{10}$$

i.e., $(r_k, z^*) \in E^*(G^*)$. Note that $(r_k, z^*) \in E^*(G^*)$, we have $(P_2 r_k, z^*) \in E^*(G^*)$ (as P is an edge-preserving mapping). Since $(P_1 e_k, z^*) \in E^*(G^*)$, $(P_2 r_k, z^*) \in E^*(G^*)$, by convexity of $E^*(G^*)$, we have

$$\left((1-\alpha_k)P_1 e_k + \alpha_k P_2 r_k, (1-\alpha_k)z^* + \alpha_k z^*\right) \in E^*(G^*). \tag{11}$$

i.e., $(e_{k+1}, z^*) \in E^*(G^*)$. Therefore, $(e_k, z^*) \in E^*(G^*)$ for $k \geq 2$. Using a similar argument, we can show that $(z^*, e_k) \in E^*(G^*)$ and $(z^*, r_k) \in E^*(G^*)$.

(b) By part (a), we have $(e_k, z^*) \in E^*(G^*)$ and $(r_k, z^*) \in E^*(G^*)$, by transitivity of G^*, we have $(e_k, r_k) \in E^*(G^*)$. □

Lemma 7. *Assume that, we have a non-empty convex closed subset, namely,* $\mathbb{K}^*$ *of any Hilbert space* $\mathbb{X}^*$*. In this case, suppose* $G^* = (V^*(G^*), E^*(G^*))$ *is a directed transitive graph with* $V^*(G^*) = \mathbb{K}^*$ *and* $E^*(G^*)$ *is convex. Assume that* $P_i\,(i = 1, 2)\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ *are edge-preserving quasi-nonexpansive self-maps. Fix any point* $e_1 \in \mathbb{K}^*$ *such that* $(e_1, P_1 e_1) \in E^*(G^*)$ *and* $(e_1, P_2 r_1) \in E^*(G^*)$*. Let* $\{e_k\}$ *be a sequence in* $V^*(G^*)$ *defined by (2) and* $z^* \in F^*(P)$ *such that* $(e_1, z^*), (z^*, e_1) \in E^*(G^*)$*. Then,* $\lim_{k \longrightarrow \infty}\|e_k - z^*\|$ *exists.*

Proof. Using quasi-nonexpansiveness of $P_i\,(i = 1, 2)$, we have

$$\begin{aligned}\|r_k - z^*\| &= \|(1-\beta_k)e_k + \beta_k P_1 e_k - z^*\| \\ &\le (1-\beta_k)\|e_k - z^*\| + \beta_k\|P_1 e_k - z^*\| \\ &\le (1-\beta_k)\|e_k - z^*\| + \beta_k\|e_k - z^*\| \\ &\le \|e_k - z^*\|,\end{aligned} \tag{12}$$

$$\begin{aligned}\|e_{k+1} - z^*\| &= \|(1-\alpha_k)P_1 e_k + \alpha_k P_2 r_k - z^*\| \\ &\le (1-\alpha_k)\|P_1 e_k - z^*\| + \alpha_k\|P_2 r_k - z^*\| \\ &\le (1-\alpha_k)\|e_k - z\| + \alpha_k\|e_k - z\| \\ &\le \|e_k - z^*\|.\end{aligned} \tag{13}$$

It follows that the sequence $\{e_k\}$ is Fejer monotone with respect to $F^*(P)$. Hence, by the Proposition 1, sequence $\{e_k\}$ is bounded and $\{\|e_k - z^*\|\}$ converges, i.e., $\lim_{k\longrightarrow\infty}\|e_k - z^*\|$ exists. □

Lemma 8. *Assume that, we have a non-empty convex closed subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{X}^*$. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ is a directed transitive graph with $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ is convex. Assume that $P_i\,(i = 1, 2)\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ are edge-preserving quasi-nonexpansive self-maps and $\{\alpha_k\}, \{\beta_k\}$ are sequences in $[\delta^*, 1-\delta^*]$ for some $\delta^* \in (0,1)$. Fix any point $e_1 \in \mathbb{K}^*$ such that $(e_1, P_1 e_1) \in E^*(G^*)$ and $(e_1, P_2 r_1) \in E^*(G^*)$. Let $\{e_k\}$ be a sequence in $\mathbb{K}^*$ defined by (2) and $z^* \in F^*(P)$ such that $(e_1, z^*), (z^*, e_1) \in E^*(G^*)$. Then,*

$$\lim_{k\longrightarrow\infty}\|P_1 e_k - e_k\| = 0 = \lim_{k\longrightarrow\infty}\|P_2 e_k - e_k\|. \tag{14}$$

Proof. By Lemma 7, $\lim_{k\longrightarrow\infty}\|e_k - z^*\|$ exists. Let $\lim_{k\longrightarrow\infty}\|e_k - z^*\| = c^*$.

Suppose $c^* = 0$, so due to the quasi-nonexpansiveness of the mappings P_i, one has

$$\begin{aligned}\|e_k - P_i e_k\| &\le \|e_k - z^*\| + \|z^* - P_i e_k\| \\ &\le \|e_k - z^*\| + \|z^* - e_k\|.\end{aligned} \tag{15}$$

Hence, the required result is hold in this case.

Next we prove the result for $c^* > 0$. Since $\lim_{k\longrightarrow\infty}\|e_k - z^*\| = c^*$, it follows that $\lim_{k\longrightarrow\infty}\|e_k - z^*\| \le c^*$. Also $\|r_k - z^*\| \le \|e_k - z^*\|$, this implies that $\lim_{k\longrightarrow\infty}\|r_k - z^*\| \le c^*$. Since P_1 is quasi-nonexpansive, we have

$$\begin{aligned}&\limsup_{k\longrightarrow\infty}\|P_1 e_k - z^*\| \le c^*. \\ &\limsup_{k\longrightarrow\infty}\|\beta_k(P_1 e_k - z^*) + (1-\beta_k)(e_k - z^*)\| \\ &\le \beta_k \limsup_{k\longrightarrow\infty}\|P_1 e_k - z^*\| \\ &\quad + (1-\beta_k)\limsup_{k\longrightarrow\infty}\|(e_k - z^*)\| \\ &\le c^*.\end{aligned} \tag{16}$$

Therefore, from Lemma 2, we have

$$\lim_{k\longrightarrow\infty}\|(P_1 e_k - z^*) - (e_k - z^*)\| = 0 \Rightarrow \lim_{k\longrightarrow\infty}\|P_1 e_k - e_k\| = 0. \tag{17}$$

Note that

$$\|e_k - r_k\| \le \beta_k\|e_k - P_1 e_k\| \Rightarrow \|e_k - r_k\| \le \delta^*\|e_k - P_1 e_k\|. \tag{18}$$

Using Lemma 3, we have

$$\begin{aligned}\|e_{k+1} - z^*\|^2 &= \|(1-\alpha_k)P_1 e_k + \alpha_k P_2 r_k - z^*\|^2 \\ &\le \alpha_k\|r_k - z^*\|^2 + (1-\alpha_k)\|e_k - z^*\|^2 - \alpha_k(1-\alpha_k)g(\|P_2 r_k - e_k\|) \\ &\le \alpha_k\|e_k - z^*\|^2 + (1-\alpha_k)\|e_k - z^*\|^2 - \alpha_k(1-\alpha_k)g(\|P_2 r_k - e_k\|) \\ &\le \|e_k - z^*\|^2 - \alpha_k(1-\alpha_k)g(\|P_2 r_k - e_k\|) \\ &\le \|e_k - z^*\|^2 - \delta^{*2} g(\|P_2 r_k - e_k\|) \\ &\Rightarrow \delta^{*2} g(\|P_2 r_k - e_k\|) \le \|e_{k+1} - z^*\|^2 - \|e_k - z^*\| \\ &\quad \Rightarrow \lim_{k\longrightarrow\infty} g(\|P_2 r_k - e_k\|) = 0 \\ &\quad \Rightarrow \lim_{k\longrightarrow\infty}\|P_2 r_k - e_k\| = 0.\end{aligned} \tag{19}$$

Since,

$$\begin{aligned}\|P_2 e_k - e_k\| &\le \|P_2 e_k - P_2 r_k\| + \|P_2 r_k - e_k\| \\ &\le \|e_k - z^*\| + \|z^* - r_k\| + \|P_2 r_k - e_k\| \\ &\le \|e_k - r_k\| + \|P_2 r_k - e_k\|.\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}\lim_{k\longrightarrow\infty}\|P_2 e_k - e_k\| &\le \lim_{k\longrightarrow\infty}\delta^*\|e_k - P_1 e_k\| + \lim_{k\longrightarrow\infty}\|P_2 r_k - e_k\| \\ &\le \delta^*\lim_{k\longrightarrow\infty}\|e_k - P_1 e_k\| + \lim_{k\longrightarrow\infty}\|P_2 r_k - e_k\| \\ &\Rightarrow \lim_{k\longrightarrow\infty}\|P_2 e_k - e_k\| = 0.\end{aligned} \tag{21}$$

The upcoming theorems show the strongly convergent of the sequence defined by (2). □

Theorem 1. *Assume that, we have a non-empty convex closed subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{X}^*$. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ is a directed transitive graph such that $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ is convex. Assume that $P_i\,(i = 1,2)\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ are edge-preserving quasi-nonexpansive self-maps that are with the Condition (B). Fix $e_1 \in \mathbb{K}^*$ such that $(e_1, P_1e_1) \in E^*(G^*)$ and $(e_1, P_2r_1) \in E^*(G^*)$. Let $\{e_k\}$ be a sequence in $\mathbb{K}^*$ defined by (2) and $z^* \in F^*(P)$ such that $(e_1, z^*), (z^*, e_1) \in E^*(G^*)$. Let $\{\alpha_k\}$, $\{\beta_k\}$ are sequences in $[\delta^*, 1-\delta^*]$, for some $\delta^* \in (0,1)$. Then, the sequence $\{e_k\}$ defined by (2) converges strongly to a common fixed points of P_i.*

Proof. Let $z^* \in F^*(P)$. From Lemma 7, we have

$$\left\|e_{k+1} - z^*\right\| \le \left\|e_k - z^*\right\|, \tag{22}$$

it gives that

$$d(e_{k+1}, F^*(P)) \le d(e_k, F^*(P)). \tag{23}$$

Thus, $\lim_{k\longrightarrow\infty} d(e_k, F^*(P))$ exists. Since, P_i satisfies Condition (B) and from Lemma 8, we have $\lim_{k\longrightarrow\infty}\|P_ie_k - e_k\| = 0$, it follows that $\lim_{k\longrightarrow\infty} f(d(e_k, F^*(P))) = 0$ and thus $\lim_{k\longrightarrow\infty} d(e_k, F^*(P)) = 0$.

Next, we prove that $\{e_k\}$ is a Cauchy sequence in $V^*(G^*)$. Since $\lim_{k\longrightarrow\infty} d(e_k, F^*(P)) = 0$, for $\varepsilon^* > 0$, there exists a constant k_0 such that for all $k \ge k_0$, we have

$$d(e_k, F^*(P)) < \frac{\varepsilon^*}{4}. \tag{24}$$

Hence, there must exists a $p^* \in F^*(P)$ such that

$$\left\|e_{k_0} - p^*\right\| < \frac{\varepsilon^*}{2}. \tag{25}$$

Now for $m^*, n \ge k_0$, we have

$$\begin{aligned}\left\|e_{n+m^*} - e_n\right\| &\le \left\|e_{n+m^*} - p^*\right\| + \left\|p^* - e_n\right\| \\ &\le 2\left\|e_{k_0} - p^*\right\| \\ &< \varepsilon^*.\end{aligned} \tag{26}$$

It follows that $\{e_k\}$ is a Cauchy sequence in $\mathbb{K}^*$. Since $\mathbb{K}^*$ is closed subset of a Hilbert space $\mathbb{X}^*$, so there exists a point say $e \in \mathbb{K}^*$ such that $\|e_k - e\| \longrightarrow 0$ as $k \longrightarrow \infty$. Next, we prove that e is common fixed point of P_i. For this,

$$\begin{aligned}0 \le \left\|P_ie - e\right\| &\le \left\|P_ie - e_{k+1}\right\| + \left\|e_{k+1} - e\right\| \\ &\le (1-\alpha_k)\left\|P_ie - P_1e_k\right\| + \alpha_k\left\|P_ie - P_2r_k\right\| + \left\|e_{k+1} - e\right\| \\ &\le (1-\alpha_k)\left(\left\|P_ie - z^*\right\| + \left\|z^* - P_ie_k\right\|\right) \\ &\quad + \alpha_k\left(\left\|P_ie - z^*\right\| + \left\|z^* - P_2r_k\right\|\right) + \left\|e_{k+1} - e\right\| \\ &\le (1-\alpha_k)\left(\left\|e - z^*\right\| + \left\|z^* - r_k\right\|\right) \\ &\quad + \alpha_k\left(\left\|e - z^*\right\| + \left\|z^* - r_k\right\|\right) + \left\|e_{k+1} - e\right\| \\ &= (1-\alpha_k)\left\|e - r_k\right\| + \alpha_k\left\|e - r_k\right\| + \left\|e_{k+1} - e\right\| \\ &= \left\|e - r_k\right\| + \left\|e_{k+1} - e\right\| \\ &\le \left\|e - e_k\right\| + \delta^*\left\|e_k - P_1e_k\right\| + \left\|e_{k+1} - e\right\| \longrightarrow \quad o \quad \text{as} \quad k \longrightarrow \infty.\end{aligned} \tag{27}$$

This shows that e is a common fixed point of P_i. □

Theorem 2. *Assume that, we have a non-empty convex closed subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{X}^*$. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ is a directed transitive graph with $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ is convex. Assume that $P_i\,(i = 1,2)\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ are the edge-preserving quasi-nonexpansive self-maps. Fix any point $e_1 \in \mathbb{K}^*$ such that $(e_1, P_1e_1) \in E^*(G^*)$ and $(e_1, P_2r_1) \in E^*(G^*)$. Let $\{e_k\}$ be a sequence in $\mathbb{K}^*$ defined by (2) and $z^* \in F^*(P)$ such that $(e_1, z^*), (z^*, e_1) \in E^*(G^*)$. Then, the sequence $\{e_k\}$ defined by (2) converges strongly to a common fixed point of P_i if and only if $\lim_{k\longrightarrow\infty} d(e_k, F^*(P)) = 0$, where $d(e_k, F^*(P)) = \inf\{\|e_k - z^*\|\colon z^* \in F^*(P)\}$.*

Proof. Suppose, the sequence $\{e_k\}$ converges strongly to a common fixed point of P_i, then clearly $\lim_{k\longrightarrow\infty} d(e_k, F^*(P)) = 0$.

Conversely, suppose that $\lim_{k\longrightarrow\infty} d(e_k, F^*(P)) = 0$. For $\varepsilon^* > 0$, there exists $k_0 \in \mathbb{N}$, such that for all $k \ge k_0$,

$$d(e_k, F^*(P)) < \frac{\varepsilon^*}{4}. \tag{28}$$

In, particular, there must $p^* \in F^*(P)$ such that

$$\left\|e_{k_0} - p^*\right\| < \frac{\varepsilon^*}{2}. \tag{29}$$

For $k, m^* \ge k_0$, we have

$$\begin{aligned}\|e_{k+m^*} - e_k\| &\le \|e_{k+m^*} - p^*\| + \|p^* - e_k\| \\ &= 2\|e_{k_0} - p^*\| < \varepsilon^*.\end{aligned} \tag{30}$$

It follows that $\{e_k\}$ is a Cauchy sequence in K^*. Since $\mathbb{K}^*$ is closed subset of a Hilbert space $\mathbb{X}^*$, so there exists a point say $e \in \mathbb{K}^*$ such that $\|e_k - e\| \longrightarrow 0$ as $k \longrightarrow \infty$. By our assumption $\lim_{k \longrightarrow \infty} d(e_k, F^*(P)) = 0$, it gives that

$$d(e, F^*(P)) = 0 \Rightarrow e \in F^*(P), \tag{31}$$

i.e., e is the common fixed point of Pi. □

Theorem 3. *Assume that, we have a non-empty convex compact subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{X}^*$. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ is a directed transitive graph such that $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ is convex. Assume that $P_i (i = 1, 2)$: $\mathbb{K}^* \longrightarrow \mathbb{K}^*$ are the edge-preserving quasi-nonexpansive self-maps. Fix any point $e_1 \in \mathbb{K}^*$ such that $(e_1, P_1 e_1) \in E^*(G^*)$ and $(e_1, P_2 r_1) \in E^*(G^*)$. Let $\{e_k\}$ be a sequence in $V^*(G^*)$ defined by (2) and $z^* \in F^*(P)$ such that $(e_1, z^*), (z^*, e_1) \in E^*(G^*)$. Let $\{\alpha_k\}$, $\{\beta_k\}$ are sequences in $[\delta^*, 1 - \delta^*]$, for some $\delta^* \in (0, 1)$. Then, the sequence $\{e_k\}$ defined by (2) converges strongly to a common fixed point of P_i.*

Proof. Since from Lemma 8, we have $\lim_{k \longrightarrow \infty}\|P_i e_k - e_k\| = 0$, and $\mathbb{K}^*$ is compact, there exists a subsequence $\{e_{k_n}\}$ of $\{e_k\}$ such that $e_{k_n} \longrightarrow z^*$ strongly for some $z^* \in \mathbb{K}^*$. Note that

$$\begin{aligned}\|e_{k_n} - P_i z^*\| &\le \|e_{k_n} - P_i e_{k_n}\| + \|P_i e_{k_n} - P_i z^*\| \\ &\le 2\|e_{k_n} - P_i e_{k_n}\| + \|e_{k_n} - z^*\|.\end{aligned} \tag{32}$$

This shows that $e_{k_n} \longrightarrow P_i z^*$ as $k \longrightarrow \infty$, i.e., $z^* = P_i z^*$. Also from Lemma 7, $\lim_{k \longrightarrow \infty}\|e_k - z^*\|$ exists, thus the sequence $\{e_k\}$ converges strongly to a common fixed point z^* of P_i.

Next lemma is related to the demiclosed property of $I - P_i$. □

Lemma 9. *Assume that, we have a non-empty convex closed subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{X}^*$ such that $\mathbb{K}^*$ is with the property WG. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ is a transitive graph with $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ is convex. Assume that $P_i (i = 1, 2)$: $\mathbb{K}^* \longrightarrow \mathbb{K}^*$ are the edge-preserving quasi-nonexpansive self-maps. Let $z^* \in F^*(P)$ such that $(e_1, z^*), (z^*, e_1) \in E^*(G^*)$. Then, $I - P_i (i = 1, 2)$ are G–demiclosed at 0.*

Proof. Let $\{e_k\}$ be a sequence in $\mathbb{K}^*$ such that $e_k \rightharpoonup e$ with $(e_k, e_{k+1}) \in E^*(G^*)$. From Lemma 7, we have $\lim_{k \longrightarrow \infty}\|P_i e_k - e_k\| = 0$. Since $\mathbb{K}^*$ has property WG, there is a subsequence $\{e_{k_n}\}$ of $\{e_k\}$ such that $(e_{k_n}, e) \in E^*(G^*)$ for all $k \in \mathbb{N}$. We claim that $e = P_i e$.

Suppose not. By quasi-nonexpansiveness of P_i and Opial's property, we have

$$\begin{aligned}\limsup_{k \longrightarrow \infty}\|e_{k_n} - e\| &< \limsup_{k \longrightarrow \infty}\|e_{k_n} - P_i e\| \\ &\le \limsup_{k \longrightarrow \infty}\left(\|e_{k_n} - P_i e_{k_n}\| + \|P_i e_{k_n} - P_i e\|\right) \\ &\le \limsup_{k \longrightarrow \infty}\|P_i e_{k_n} - P_i e\|,\end{aligned} \tag{33}$$

which is a contradiction. Therefore, $e = P_i e$.

The following theorem proves that the sequence $\{e_k\}$ defined by (2) converges weakly: □

Theorem 4. *Assume that, we have a non-empty convex closed subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{X}^*$ such that $\mathbb{K}^*$ is with the property WG. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ is a directed transitive graph with $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ is convex. Assume that $P_i (i = 1, 2)$: $\mathbb{K}^* \longrightarrow \mathbb{K}^*$ are the edge-preserving quasi-nonexpansive self-maps. Let $z^* \in F^*(P)$ such that $(e_1, z^*), (z^*, e_1) \in E^*(G^*)$. Let $\{\alpha_k\}, \{\beta_k\}$ are sequences in $[\delta^*, 1 - \delta^*]$ for some $\delta^* \in (0, 1)$. Then, the sequence $\{e_k\}$ defined by (2) converges weakly to a common fixed points of P_i.*

Proof. Let $z^* \in F^*(P)$. From Lemma 7, $\lim_{k \longrightarrow \infty}\|e_k - z^*\|$ exists, and by Proposition 1, $\{e_k\}$ is bounded. Let $\{e_{k_n}\}$ and $\{e_{k_m}\}$ be the subsequences of the sequence $\{e_k\}$ with weak limits p_1^* and p_2^*, respectively. Since from Lemma 8, we have $\|P_i e_{k_n} - e_{k_n}\|$ and $\|P_i e_{k_m} - e_{k_m}\|$ approach to 0 as $n, m \longrightarrow \infty$. By Lemma 9, we have $P_i p_1^* = p_1^*$ and $P_i p_2^* = p_2^*$. Therefore, $p_1^*, p_2^* \in F^*(P)$. By Lemma 1, we have $p_1^* = p_2^*$. Therefore, sequence $\{e_k\}$ converges weakly to a common fixed point of P_i. □

4. Numerical Example

In support of our main results, we proceed with the help of the following example. Also, we show the fastness of the iteration scheme (2) by comparing it with the iteration scheme (1).

Example 1. Assume that, we have the Hilbert space $\mathbb{X}^* = \mathbb{R}$, and its subset $\mathbb{K}^* = [0, 1]$. Suppose $G^* = (V^*(G^*), E^*(G^*))$ is a directed graph with $V^*(G^*) = \mathbb{K}^*$. Assume that $e, r \in V^*(G^*)$ are such that $\|e\| \le 1/8$, $\|r\| \le 1/8$ and $E^*(G^*) = \{(e, r): \|e - r\| \le 1/2\}$. Now we set P_1, P_2: $\mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$P_1 e = \begin{cases} \dfrac{e}{2}\sin\left(\dfrac{e}{2}\right), e \ne 0, \\ 0, e = 0. \end{cases} \tag{34}$$

and

$$P_2 e = \begin{cases} \dfrac{e}{4}, e \ne 0, \\ 0, e = 0. \end{cases} \tag{35}$$

Table 1: Numerical values obtained from different iterations.

k	Tripak	Suantai
0	0.5000000000000	0.5000000000000
1	0.2851156868633	0.0660411817701
2	0.1616434471742	0.0047407895839
3	0.0913322541496	0.0002994599049
4	0.0515046848607	0.0000187288547
5	0.0290128295992	0.0000011706020
6	0.0163328684427	0.0000000731628
7	0.0091914066181	0.000000004.5726
8	0.0051714862486	0.00000000028570
9	0.0029093788936	0.00000000001780
10	0.0016366578851	0.00000000000110

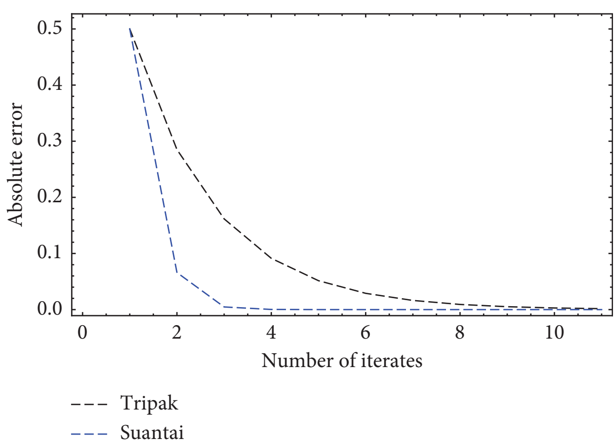

Figure 1: Behavior of iterates for the mappings P_1 and P_2.

It is clear that P_1 and P_2 both are edge-preserving quasi-nonexpansive mappings and they have a common fixed point 0. In this case, the convergence is shown in Table 1 and Figure 1.

5. Application in Variational Inequality

Theorem 5. *Assume that, we have a non-empty compact convex subset, namely, $\mathbb{K}^*$ of any Hilbert space $\mathbb{X}^*$. In this case, suppose $G^* = (V^*(G^*), E^*(G^*))$ is a directed transitive with $V^*(G^*) = \mathbb{K}^*$ and $E^*(G^*)$ is convex. Assume that $P\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ is an edge-preserving quasi-nonexpansive self-map and $\psi\colon \mathbb{K}^* \longrightarrow \mathbb{K}^*$ is a contraction with a contraction coefficient in [0, 1). Fix any point $e_1 \in \mathbb{K}^*$ such that $(e_1, Pe_1) \in E^*(G^*)$ and $(e_1, \Pr_1) \in E^*(G^*)$. Let $\{\alpha_k\}$ and $\{\beta_k\}$ are sequences in [0, 1]. Then, the sequence $\{e_k\}$ defined by (2) converges strongly to a fixed point $q \in P$, which is also the unique solution of the following variational inequality:*

$$\langle (I-\psi)q, x-q\rangle \geq 0. \tag{36}$$

Proof. From Proposition 1, sequence $\{e_k\}$ is bounded and from Lemma 8, we have $\lim_{k\longrightarrow\infty}\|Pe_k - e_k\| = 0$. We claim that

$$\limsup_{k\longrightarrow\infty} < (I-\psi)q, x-q > \geq 0, \tag{37}$$

where $q \in F(P)$ is unique fixed point of ψ.

Since $\mathbb{K}^*$ is compact, there exists a subsequence $\{e_{k_p}\}$ of $\{e_k\}$ such that $e_{k_p} \longrightarrow p$ for some $p \in \mathbb{K}^*$. By using Lemma 4, we have

$$\begin{aligned}\limsup_{k\longrightarrow\infty} < \psi(q)-q, e_k - q > &= \limsup_{k\longrightarrow\infty} < \psi(q)-q, e_{k_p} - q > \\ &= < \psi(q)-q, p-q > \geq 0.\end{aligned} \tag{38}$$

Now we claim that $e_k \longrightarrow q \in F(P)$. By doing similar procedure as in the proof of Theorem 3, $e_k \longrightarrow q \in F(P)$. □

6. Conclusions

We provided the following new outcome:

(i) We provided weak and strong convergence results of common fixed points for edge-preserving quasi-nonexpansive self-map in Hilbert spaces along with directed graph

(ii) We considered the larger class of edge-preserving nonexpansive mappings that includes the class of edge-preserving nonexpansive mappings

(iii) The main outcome is numerically supported by a numerical example

(iv) As an application, we solve a variational inequality problem

(v) Our results improve/extend the classical results of the literature form edge-preserving nonexpansive mappings to the general class of edge-preserving quasi-nonexpansive mappings

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors provided equal contribution to this paper.

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