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Research Article

A Context-Free Grammar Associated with Fibonacci and Lucas Sequences

Harold Ruilong Yang

School of Science, Tianjin University of Technology and Education, Tianjin 300222, China

Correspondence should be addressed to Harold Ruilong Yang; yangruilong18@outlook.com

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We introduce a context-free grammar $G = \{s \longrightarrow s + d, d \longrightarrow s\}$ to generate Fibonacci and Lucas sequences. By applying the grammar G, we give a grammatical proof of the Binet formula. Besides, we use the grammar G to provide a unified approach to prove several binomial convolutions about Fibonacci and Lucas numbers, which were given by Hoggatt, Carlitz, and Church. Meanwhile, we also obtain some new binomial convolutions.

1. Introduction

Recall that the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ are defined through the same recurrence relations: for $n \geq 2$,

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2}, \\ L_n &= L_{n-1} + L_{n-2}, \end{aligned} \tag{1}$$

with initial values $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$, respectively.

Fibonacci and Lucas numbers have close connections with the golden ratio. Through this paper, we use α to denote the golden ratio, that is, $\alpha = ((1 + \sqrt{5})/2)$. Let $\beta = -1/\alpha = ((1 - \sqrt{5})/2)$. α and β are two roots of the quadratic equation $x^2 - x - 1 = 0$. The famous Binet formulas for Fibonacci and Lucas numbers show that, for $n \in \mathbb{Z}$,

$$\begin{aligned} F_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \\ L_n &= \alpha^n + \beta^n. \end{aligned} \tag{2}$$

The binomial identities involving Fibonacci and Lucas numbers are studied widely in recent decades. The study of binomial identities involving Fibonacci and Lucas numbers is beginning from a group of identities of Hoggatt [1]. After then, Carlitz [2] and Church and Bicknell [3] enriched the binomial identities family. The details of these identities will be expanded in the next section.

In this paper, we introduce a context-free grammar to describe Fibonacci and Lucas numbers. Let

$$G_1 \coloneqq \{s \longrightarrow s + d, d \longrightarrow s\}. \tag{3}$$

We find that, for $n \geq 0$,

$$\begin{aligned} D^n(s - d)\big|_{s=d=1} &= F_n, \\ D^n(3d - s)\big|_{s=d=1} &= L_n. \end{aligned} \tag{4}$$

Here, D is the formal derivative associated with G. By applying this grammar, we give a grammatical framework to prove the binomial convolutions by Hoggatt, Carlitz, Church, and Bicknell and obtain some new binomial convolutions involving Fibonacci and Lucas numbers.

In Section 2, we posed the binomial identities given by Hoggatt, Layman, Carlitz, Church, and Bicknell. In Section 3, we give a new context-free grammar, which is called Fibonacci grammar. Based on the grammar, we give a grammatical expression of Fibonacci and Lucas numbers. As the application of this expression, a grammatical proof of some classic relations associated with Fibonacci and Lucas numbers are given, including the Binet formula. In Section 4, we provide a uniform framework to prove binomial identities by a grammatical manner. We prove all the identities given by Hoggatt, Carlitz, Church, and Bicknell and give some new binomial identities associated with Fibonacci and Lucas numbers.

2. Binomial Identities of Hoggatt, Carlitz, Church, and Bicknell

In this section, we recall the work of Hoggatt, Carlitz, Church, and Bicknell on binomial identities about Fibonacci and Lucas numbers.

In [1], Hoggatt found that, for $n \geq 1$,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} F_k &= F_{2n}, \\ \sum_{k=0}^{n} \binom{n}{k} F_{3k} &= 2^n F_{2n}, \\ \sum_{k=0}^{n} \binom{n}{k} F_{4k} &= 3^n F_{2n}. \end{aligned} \tag{5}$$

Carlitz [2] extent Hoggatt's series of identities to more general relations: for $0 \leq r \leq m, n \geq 0$ and $t \geq 0$,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-1)^{r(n-k)} F_{m-r}^{n-k} F_r^k F_{mk+t} &= F_m^n F_{rn+t}, \\ \sum_{k=0}^{n} \binom{n}{k} (-1)^{r(n-k)} F_{m-r}^{n-k} F_r^k L_{mk+t} &= F_m^n L_{rn+t}. \end{aligned} \tag{6}$$

Besides, Church and Bicknell provided several different binomial convolutions about Fibonacci and Lucas numbers. In [3], Church and Bicknell showed that, for $m, n, r \geq 0$,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} F_{\mathrm{mk}} F_{\mathrm{mn-mk}} &= \frac{1}{5} \left(2^n L_{\mathrm{mn}} - 2L_m^n\right), \\ \sum_{k=0}^{n} \binom{n}{k} L_{\mathrm{mk}} L_{\mathrm{mn-mk}} &= 2^n L_{\mathrm{mn}} + 2L_m^n, \\ \sum_{k=0}^{n} \binom{n}{k} F_{\mathrm{4mk+4mr}} &= L_{2m}^n F_{\mathrm{2mn+4mr}}. \end{aligned} \tag{7}$$

3. Tilings and the Fibonacci Grammar

The approach of studying combinatorial polynomials by using context-free grammars was introduced by Chen [4]. In this decade, many combinatorists have found the relations between combinatorial polynomials and context-free grammars; see [4–7], for example. A context-free grammar G is a set of substitution rules on a set of variables X. We can define a formal derivative D associated with a context-free grammar G as a differential operator on polynomials or Laurent polynomials in X. In precise, D is a linear operator satisfying the relation

$$D(\mathrm{uv}) = \mathrm{u}D(v) + D(u)v, \tag{8}$$

which can be in general given as Leibnitz formula

$$D^n(\mathrm{uv}) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v). \tag{9}$$

For the purpose of combinatorial enumeration, the variables are attached to combinatorial structures by grammatical labelings by Chen and Fu [5]. In order to provide a combinatorial expression of Fibonacci numbers, we need a corresponding combinatorial structure. Although many combinatorial interpretations of Fibonacci numbers exist (see exercises 1–9 in [8], p.14, for example), we use the tiling definition given as below.

For $n \geq 1$, a tiling of length n is refer to a tiling of a rectangle with size $1 \times n$ by squares and rectangles with size 1×2. Here, we call the rectangle with size $1 \times n$ by an n-board, and call the rectangle with size 1×2 by a domino.

For example, there are 5 tilings of length 4 Figure 1:

A classical result shows that there are exactly F_{n+1} tilings with length n for all $n \geq 0$. Here, we let $F_1 = 1$ count the unique empty tiling of 0-board. By counting the number of blocks of a tiling, we give a generating function $F_n(q)$ as a q-analogue of Fibonacci numbers:

$$F_n(q) = \sum_{k=1}^{n} F(n,k) q^k, \text{for } n \geq 2, \tag{10}$$

where $F(n,k)$ denotes the number of tilings of an $(n-1)$-board with k blocks. Here, we define $F_1(q) = 1$ and $F_0(q) = 0$.

Lemma 1. *For $n \geq 2$, it holds that*

$$F_n(q) = q\left(F_{n-1}(q) + F_{n-2}(q)\right). \tag{11}$$

Proof. Consider the last block of a tiling T of length n. There are two classes: ending with a square and ending with a domino. The first class corresponds to the function $qF_{n-1}(q)$ since the left tiling has size $n-1$; and the second class corresponds to the function $qF_{n-2}(q)$. This completes the proof.

Let

$$G := \{s \longrightarrow \mathrm{sq}+d, d \longrightarrow \mathrm{sq}\}, \tag{12}$$

where q is a constant, that is, q. Since the grammar has close connections between the function $F_n(q)$, we call the grammar G as the q-Fibonacci grammar. Let $q = 1$ in G, the grammar G degenerates to be

$$G_1 := \{s \longrightarrow s + d, d \longrightarrow s\}, \tag{13}$$

which is called Fibonacci grammar. □

Theorem 2. *Let G be the q-Fibonacci grammar, and D be the formal derivative associated with the grammar G. For $n \geq 2$, we have*

$$D^n(s-d) = \mathrm{sqF}_{n-1}(q) + \mathrm{dqF}_{n-2}(q), \tag{14}$$

and for $n \geq 0$,

$$D^n(s-d)\big|_{s=d=1} = F_n(q). \tag{15}$$

Proof. It can be seen that (15) can be obtained from (14) by setting $s = d = 1$. To prove (14), we introduce a grammatical labeling of a tiling by labeling the blocks. We label each block of a tiling T by q, and label the last block extra by s if it is a square, and by d if it is a domino. Then, we define the weight of the tiling T to be the sum of labelings of all blocks in T, that is, for a tiling T having k blocks, $w(T) = \mathrm{sq}^k$ when ending at a square and $w(T) = \mathrm{dq}^k$ when ending at a domino. Then it is natural to say that

$$\sum_T w(T) = \mathrm{sqF}_{n-1}(q) + \mathrm{dqF}_{n-2}(q), \tag{16}$$

since the sum of the weights of tilings ending at a square equals $\mathrm{sqF}_{n-1}(q)$ and the sum of the weights of tilings ending at a domino equals $\mathrm{dqF}_{n-2}(q)$.

Now, we show by induction that, for $n \geq 2$, it holds

$$D^n(s-d) = \sum_T w(T), \tag{17}$$

where T runs over the set of tilings with length n.

For $n = 2$, $D^2(s-d) = D(d) = \mathrm{sq}$, which is equal to the weight of the unique tiling of length 1. Thus, (17) holds for $n = 2$. Assume that (17) holds for n. To show that (17) is valid for $n+1$, we consider the process to generate a tiling of length $n+1$ from a tiling of length n.

For a tiling T ending at a domino, the only way to add the length of T is adding a new square in the end of T. In order to label the new tiling consistently, we delete the labeling d for the last domino in old tiling and label the new square by sq. This corresponds to the substitution rule $d \longrightarrow \mathrm{sq}$.

For a tiling T ending at a square, we have two chooses. If we change the last square to a new domino, we change the labeling of the last part from sq to dq. If we add a new square at the end of T, we turn the labeling s from the old last square to the new last square and add a new labeling q to it. These two chooses correspond to the substitution rule $s \longrightarrow d + \mathrm{sq}$.

For example, the first tiling in Figure 1 is labeled by dq^2, and the unique corresponding tiling is labeled by sq^3.

Figure 2 And the second tiling in Figure 1 is labeled by sq^3, the two corresponding tilings are labeled by sq^4 and dq^3, respectively Figure 3.

Notice that we can generate all tilings of length $n+1$ as above. Thus,

$$D^{n+1}(s-d) = D\left(D^n(s-d)\right) = D\left(\sum_T w(T)\right) = \sum_{T'} w(T'), \tag{18}$$

where T' runs over the set of tilings with length $n+1$. Thus (17) holds for $n+1$. Now (17) holds for all $n \geq 2$ by induction. This completes the proof. □

Theorem 3. *Let D_1 be the formal derivative associated with the Fibonacci grammar. For $n \geq 0$,*

$$D_1^n(s-d) = \mathrm{dF}_{n-2} + \mathrm{sF}_{n-1}, \tag{19}$$

and

$$D_1^n\left(\mathrm{dF}_{n+2} - \mathrm{sF}_{n+1}\right) = (-1)^{n+1}(s-d). \tag{20}$$

Proof. Equation (19) can be deduced from (14) by setting $q = 1$.

As for (20), we have

$$\begin{aligned} D_1\left(\mathrm{dF}_n - \mathrm{sF}_{n-1}\right) &= D_1(d)F_n - D_1(s)F_{n-1} \\ &= \mathrm{sF}_n - (s+d)F_{n-1} \\ &= -\left(\mathrm{dF}_{n-1} - \mathrm{sF}_{n-2}\right). \end{aligned} \tag{21}$$

Applying the relation n times repeatedly, we obtain

$$\begin{aligned} D_1^n\left(\mathrm{dF}_{n+2} - \mathrm{sF}_{n+1}\right) &= D_1^{n-1}\left(\mathrm{dF}_{n+1} - \mathrm{sF}_n\right) = \cdots = (-1)^n\left(\mathrm{dF}_2 - \mathrm{sF}_1\right) \\ &= (-1)^{n+1}(s-d), \end{aligned} \tag{22}$$

which implies (20).

By setting $s = d = 1$ in (19), we get the following corollary as a grammatical expression about the Fibonacci numbers. □

Corollary 4. *For $n \geq 0$, it holds that*

$$D_1^n(s-d)\big|_{s=d=1} = F_n, \tag{23}$$

$$D_1^n(d)\big|_{s=d=1} = F_{n+1}, \tag{24}$$

$$D_1^n(s)\big|_{s=d=1} = F_{n+2}. \tag{25}$$

As an application of Fibonacci grammar, we give a grammatical proof of Binet's formula.

Theorem 5 (Binet's formula). *Let $\alpha = 1 + \sqrt{5}/2, \beta = 1 - \sqrt{5}/2$. For $n \geq 0$, it holds that*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{26}$$

Equivalently,

$$F(t) = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!} = \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}. \tag{27}$$

Proof. Let $u = s - d$, then $D(u) = d$. Since $\alpha + \beta = 1$ and $\alpha\beta = -1$, one can verify that

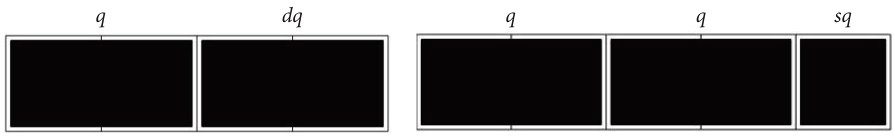
$$D_1(u + \alpha d) = d + \alpha s = \alpha u + (1+\alpha)s = \alpha(u + \alpha d). \tag{28}$$

Thus,

$$D_1^n(u + \alpha d) = \alpha^n(u + \alpha d), \tag{29}$$

which implies that

Figure 1: All five tilings of length 4.



(a) (b)

Figure 2: An example for the action of the substitution rule $d \longrightarrow$ sq. (a) A tiling labeled by dq^2. (b) $d \longrightarrow$ sq.

(a) (b) (c)

Figure 3: An example for the action of the substitution rule $s \longrightarrow d +$ sq. (a) A tiling labeled by sq^3. (b) $s \longrightarrow$ sq. (c) $s \longrightarrow d$.

$$\begin{aligned}\mathbf{Gen}(u+\alpha d) &= \sum_{n=0}^{\infty} D_1^n(u+\alpha d)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}(u+\alpha d)\frac{\alpha^n t^n}{n!} = (u+\alpha d)e^{\alpha t}.\end{aligned} \tag{30}$$

Similarly, we have

$$D_1(u+\beta d) = d+\beta s = (1+\beta)d+\beta u = \beta(u+\beta d), \tag{31}$$

and

$$\mathbf{Gen}(u+\beta d) = (u+\beta d)e^{\beta t}. \tag{32}$$

Combining (30) and (32), we obtain

$$\mathbf{Gen}(u) = \frac{1}{\beta-\alpha}\left(\beta(u+\alpha d)e^{\alpha t} - \alpha(u+\beta d)e^{\beta t}\right). \tag{33}$$

By setting $s = d = 1$ in (33), we obtain

$$F(t) = \mathbf{Gen}(u)|_{s=d=1} = \frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}. \tag{34}$$

Now, we complete the proof.

By using the same grammar, we can also generate Lucas numbers in a grammatical manner, whose proof is omitted. □

Theorem 6. *For $n \geq 0$,*

$$\begin{aligned}D_1^n(3d-s) &= \mathrm{d}L_{n-2} + \mathrm{s}L_{n-1},\\ D_1^n(3d-s)|_{s=d=1} &= L_n.\end{aligned} \tag{35}$$

Following properties of Fibonacci and Lucas numbers are classic and useful in this paper. For the sake of completeness, we provide a grammatical proof.

Lemma 7. *For $n \geq 1$ and $m \geq 1$,*

$$F_{n+m-1} = F_n F_m + F_{n-1}F_{m-1}, \tag{36}$$

$$L_{n+m-1} = L_n F_m + L_{n-1}F_{m-1}, \tag{37}$$

$$L_n = F_{n+1} + F_{n-1}. \tag{38}$$

Proof. Notice that

$$\begin{aligned}&D_1^{n+m-1}(s-d)\\ &= D_1^{m-2}\left(D_1^{n+1}(s-d)\right)\\ &= D_1^{m-2}(dF_{n-1}+sF_n)\\ &= F_{n-1}D_1^{m-2}(d) + F_n D_1^{m-2}(s).\end{aligned} \tag{39}$$

Now, setting $s = d = 1$ implies (36) by (23)–(25).

We have the equation

$$\begin{aligned}&D_1^{n+m-1}(3d-s)\\ &= D_1^{m-2}\left(D_1^{n+1}(3d-s)\right)\\ &= D_1^{m-2}(\mathrm{d}L_{n-1}+\mathrm{s}L_n)\\ &= L_{n-1}D_1^{m-2}(d) + L_n D_1^{m-2}(s),\end{aligned} \tag{40}$$

which implies (37) by setting $s = d = 1$.

Finally, (38) can be deduced from

$$D_1^n(3d-s) = D_1^{n-1}(2s-d) = D_1^{n-1}(s) + D_1^{n-1}(s-d), \tag{41}$$

by setting $s = d = 1$. □

Lemma 8. *For $m \geq n \geq k \geq 0$,*

$$F_m F_n - F_{m+k}F_{n-k} = (-1)^{n-k}F_{m+k-n}F_k. \tag{42}$$

Proof. From (20), it holds that

$$D_1^{n-2}(d)F_n - D_1^{n-2}(s)F_{n-1} = (-1)^{n-1}(s-d). \tag{43}$$

Acting the operator D_1^{m-n+1} on the two hand sides of the above equation, we obtain

$$D_1^{m-1}(d)F_n - D_1^{m-1}(s)F_{n-1} = (-1)^{n-1}D_1^{m-n+1}(s-d), \tag{44}$$

which is changed to

$$F_mF_n - F_{m+1}F_{n-1} = (-1)^{n-1}F_{m+1-n}, \tag{45}$$

by setting $s = d = 1$. This is the special case of (42) in $k = 1$.

Notice that

$$\begin{aligned} D_1^m(s-d)F_n - D_1^{m+k}(s-d)F_{n-k} &= D_1^m\big((s-d)F_n - D_1^k(s-d)F_{n-k}\big) \\ &= D_1^m\big((s-d)F_n - (dF_{k-2} - sF_{k-1})F_{n-k}\big) \\ &= D_1^m(s)\big(F_n - F_{n-k}F_{k-1}\big) - D_1^m(d)\big(F_n + F_{n-k}F_{k-2}\big) \\ &= D_1^m(s)F_{n-k+1}F_k - D_1^m(d)F_{n-k+2}F_k. \end{aligned} \tag{46}$$

The last equation holds from (36). Setting $s = d = 1$ in the above relation, we obtain

$$F_mF_n - F_{m+k}F_{n-k} = F_k\big(F_{m+2}F_{n-k+1} - F_{m+1}F_{n-k+2}\big), \tag{47}$$

which equals the right hand side of (42) by (45). □

Lemma 9. *For $n \geq 1$,*

$$F_{4n+1} + 1 = F_{2n+1}L_{2n}, \tag{48}$$

$$F_{4n} = F_{2n}L_{2n}, \tag{49}$$

$$F_{4n-1} + 1 = F_{2n-1}L_{2n}. \tag{50}$$

Proof. We can verify that

$$\begin{aligned} D_1^{2n}(3d-s)F_{2n+1} - D_1^{4n}(d) &= D_1^{2n}\big((3d-s)F_{2n+1} - D_1^{2n}(d)\big) \\ &= D_1^{2n}\big((3d-s)F_{2n+1} - (\mathrm{d}F_{2n-1} + \mathrm{s}F_{2n})\big) \\ &= D_1^{2n}\big(d(3F_{2n+1} - F_{2n-1}) - s(F_{2n+1} + F_{2n})\big). \end{aligned} \tag{51}$$

Since

$$3F_{2n+1} - F_{2n-1} = 2F_{2n+1} + F_{2n} = F_{2n+1} + F_{2n+2} = F_{2n+3}, \tag{52}$$

it holds that

$$\begin{aligned} D_1^{2n}(3d-s)F_{2n+1} - D_1^{4n}(d) &= D_1^{2n}\big(\mathrm{d}F_{2n+3} - \mathrm{s}F_{2n+2}\big) \\ &= (-1)^{2n}(2d-s), \end{aligned} \tag{53}$$

which is reduced to (48) by setting $s = d = 1$.

Similarly, equations (49) and (50) can be obtained by simplifying the expression

$$D_1^{2n}(3d-s)F_{2n} - D_1^{4n-1}(d), \tag{54}$$

and the expression

$$D_1^{2n}(3d-s)F_{2n-1} - D_1^{4n-2}(d), \tag{55}$$

in the same manner as above and then setting $s = d = 1$. □

4. m-th Order Fibonacci Grammar, Binomial Fibonacci, and Lucas Identities

In this section, we provide a framework to prove the identities involving Fibonacci and Lucas numbers associated with binomial coefficients.

Let G be a context-free grammar with an alphabet X, and let k be a constant for G. We define the product of the grammar G and k to be the context-free grammar in which each letter $a \in X$ corresponds the substitution rule $a \longrightarrow \mathrm{kG}(a)$, denoted as kG. A grammar G on an alphabet X is defined to be linear if for each letter $a \in X$, $G(a)$ is a linear function on X.

Lemma 10. *Let G be a linear context-free grammar with an alphabet X, and let k be a constant for G. For $n \geq 0$ and each linear function f, it holds that*

$$D_{kG}^n(f) = k^n D_G^n(f). \tag{56}$$

Proof. Because of the linearity of the operator D, it is enough to show (56) holds for every letter $a \in X$. We prove the assertion by induction on n. The case for $n = 1$ is evident. Assume that the assertion holds for n. Now, consider the case for $n + 1$, since

$$D_{\mathrm{kG}}^{n+1}(a) = D_{\mathrm{kG}}\big(D_{\mathrm{kG}}^n(a)\big) = \mathrm{kD}_G\big(D_{\mathrm{kG}}^n(a)\big) = \mathrm{kD}_G\big(k^nD_G^n(a)\big), \tag{57}$$

which equals $k^{n+1}D_G^{n+1}(a)$. This completes the proof.

Consider the following context-free grammar G_m:

$$G_m \coloneqq \{d \longrightarrow D_1^m(d), s \longrightarrow D_1^m(s)\}. \tag{58}$$

We call G_m the m-th order Fibonacci grammar. Let D_m be the formal derivative associated with the grammar G_m. According to (56), D_m is equivalent to D_1^m when acting on a linear functions of s and d. When $m = 1$, 1-th order Fibonacci grammar is just the Fibonacci grammar.

From (19), it can be verified that

$$\begin{aligned} D_1^m(s) &= D_1^{m+2}(s-d) = \mathrm{dF}_m + \mathrm{sF}_{m+1}, \\ D_1^m(d) &= D_1^{m+1}(s-d) = \mathrm{dF}_{m-1} + \mathrm{sF}_m. \end{aligned} \tag{59}$$

Thus,

$$G_m \coloneqq \{d \longrightarrow \mathrm{dF}_{m-1} + \mathrm{sF}_m, s \longrightarrow \mathrm{dF}_m + \mathrm{sF}_{m+1}\}. \tag{60}$$

It should be noticed that D_m is not equivalent to D_1^m. For example,

$$D_2\big(d^2\big) = 2\mathrm{dD}_2(d) = 2d(d+s), \tag{61}$$

yet

$$D_1^2\left(d^2\right) = D_1(2\mathrm{ds}) = 2d(d+s) + 2s^2. \tag{62}$$

Let

$$\begin{aligned}\alpha_1 &= -d^2 + 2\mathrm{ds},\\ \alpha_2 &= s^2 - 2\mathrm{sd} + 2d^2,\\ \alpha_3 &= -s^2 + \mathrm{sd} + d^2.\end{aligned} \tag{63}$$

One can easy to check that

$$\begin{aligned}&D_1(\alpha_1) = 2\alpha_2, D_1(\alpha_2) = 2(\alpha_1 + \alpha_2),\\ &D_1(\alpha_3) = \alpha_3.\end{aligned} \tag{64}$$

Following assertion is critical for the proof. □

Lemma 11. *For $n \geq 1$, it holds that*

$$D_m^n(\alpha_1 - \alpha_2)\big|_{s=d=1} = 2^n F_{\mathrm{mn}}, \tag{65}$$

$$\begin{aligned}D_m^n(\alpha_1)\big|_{s=d=1} &= 2^n F_{\mathrm{mn}+2},\\ D_m^n(\alpha_2)\big|_{s=d=1} &= 2^n F_{\mathrm{mn}+1}.\\ D_m^n(\alpha_3)\big|_{s=d=1} &= L_m^n.\end{aligned} \tag{66}$$

Proof. We can verify that

$$\begin{aligned}D_m(\alpha_1) &= D_m\left(-d^2 + 2\mathrm{sd}\right)\\ &= -2\mathrm{dD}_m(d) + 2\mathrm{sD}_m(d) + 2\mathrm{dD}_m(s)\\ &= 2(s-d)(\mathrm{sF}_m + \mathrm{dF}_{m-1}) + 2d(\mathrm{sF}_{m+1} + \mathrm{dF}_m)\\ &= \left(2\mathrm{sd} - 2d^2\right)F_{m-1} + \left(2s^2 - 2\mathrm{ds} + 2d^2\right)F_m + 2\mathrm{dsF}_{m+1}\\ &= \left(2\mathrm{sd} - 2d^2\right)(F_{m+1} - F_m) + \left(2s^2 - 2\mathrm{ds} + 2d^2\right)F_m + 2\mathrm{dsF}_{m+1}\\ &= 2\left(-d^2 + 2\mathrm{sd}\right)F_{m+1} + 2\left(2d^2 - 2\mathrm{sd} + s^2\right)F_m\\ &= 2F_{m+1}\alpha_1 + 2F_m\alpha_2.\end{aligned} \tag{67}$$

Similarly, it holds that $D_m(\alpha_2) = 2F_m\alpha_1 + 2F_{m-1}\alpha_2$. According to Lemma 10,

$$\begin{aligned}D_m^n(\alpha_1) &= 2^n\widetilde{D}^n(\alpha_1),\\ D_m^n(\alpha_2) &= 2^n\widetilde{D}^n(\alpha_2),\end{aligned} \tag{68}$$

where $\widetilde{D}$ is the formal derivative associated with the grammar

$$\widetilde{G} = \{\alpha_1 \longrightarrow \alpha_1 F_{m+1} + \alpha_2 F_m, \alpha_2 \longrightarrow \alpha_1 F_m + \alpha_2 F_{m-1}\}. \tag{69}$$

Notice that $\widetilde{G}$ is as same as the m-th order Fibonacci grammar by setting $\alpha_1 = s$ and $\alpha_2 = d$. Thus,

$$\begin{aligned}D_m^n(\alpha_1)\big|_{s=d=1} &= 2^n\widetilde{D}^n(\alpha_1)\big|_{s=d=1} = 2^n D_m^n(d)\big|_{s=d=1} = 2^n D_1^{\mathrm{mn}}(d)\big|_{s=d=1} = 2^n F_{\mathrm{mn}+1},\\ D_m^n(\alpha_2)\big|_{s=d=1} &= 2^n\widetilde{D}^n(\alpha_2)\big|_{s=d=1} = 2^n D_m^n(s)\big|_{s=d=1} = 2^n D_1^{\mathrm{mn}}(s)\big|_{s=d=1} = 2^n F_{\mathrm{mn}+2}.\end{aligned} \tag{70}$$

Thus,

$$D_m^n(\alpha_1 - \alpha_2)\big|_{s=d=1} = D_m^n(\alpha_1)\big|_{s=d=1} - D_m^n(\alpha_2)\big|_{s=d=1} = 2^n F_{\mathrm{mn}}. \tag{71}$$

As for α_3, one can verify that

$$\begin{aligned}D_m(\alpha_3) &= -2\mathrm{sD}_m(s) + \mathrm{sD}_m(d) + \mathrm{dD}_m(s) + 2\mathrm{dD}_m(d)\\ &= (d - 2s)(\mathrm{sF}_{m+1} + \mathrm{dF}_m) + (s + 2d)(sF_m + \mathrm{dF}_{m-1})\\ &= \alpha_3(F_{m-1} + F_{m+1}) = \alpha_3 L_m.\end{aligned} \tag{72}$$

The last equation holds from (38). This completes the proof.

Now, we begin to proof binomial convolutions about Fibonacci and Lucas numbers. □

Theorem 12. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}F_{\mathrm{mk}}L_{\mathrm{mn-mk}} = 2^n F_{\mathrm{mn}}, \tag{73}$$

$$\sum_{k=0}^{n}\binom{n}{k}F_{\mathrm{mk}}F_{\mathrm{mn-mk}} = \frac{2^{\mathrm{mn}}}{5}(2F_{\mathrm{mn}+m} - F_{\mathrm{mn}}) - \frac{2}{5}, \tag{74}$$

$$\sum_{k=0}^{n}\binom{n}{k}L_{\mathrm{mk}}L_{\mathrm{mn-mk}} = 2^n L_{\mathrm{mn}} + 2L_m^n. \tag{75}$$

Proof. According to (56),

$$\begin{aligned}D_m^k(s-d)\big|_{s=d=1} &= D_1^{\mathrm{mk}}(s-d)\big|_{s=d=1} = F_{\mathrm{mk}},\\ D_m^k(3d-s)\big|_{s=d=1} &= D_1^{\mathrm{mk}}(3d-s)\big|_{s=d=1} = L_{\mathrm{mk}}.\end{aligned} \tag{76}$$

This deduces that the left hand side of (73) equals the summation:

$$\begin{aligned}L.H.S. &= \sum_{k=0}^{n}\binom{n}{k}F_{\mathrm{mk}}L_{\mathrm{mn-mk}}\\ &= \sum_{k=0}^{n}\binom{n}{k}D_m^k(s-d)\big|_{s=d=1}D_m^{n-k}(3d-s)_{s=d=1}\\ &= D_m^n((s-d)(3d-s))\big|_{s=d=1}\\ &= D_m^n(\alpha_1 - \alpha_2)\big|_{s=d=1},\end{aligned} \tag{77}$$

which equals $2^n F_{\mathrm{mn}}$ by (65). This completes the proof of (73).

The left hand side of (74) can be calculated by a similar manner. We have

$$\begin{aligned} L.H.S. &= \sum_{k=0}^{n}\binom{n}{k}F_{\mathrm{mk}}F_{\mathrm{mn-mk}} \\ &= \sum_{k=0}^{n}\binom{n}{k}D_m^k(s-d)\big|_{s=d=1}D_m^{n-k}(s-d)\big|_{s=d=1} \\ &= D_m^n\left((s-d)^2\right)\big|_{s=d=1}. \end{aligned} \tag{78}$$

By using Gaussian elimination, $d^2 - 2\mathrm{sd} + s^2$ can be represented as the linear combination of α_1, α_2 and α_3, namely,

$$d^2 - 2\mathrm{sd} + s^2 = -\frac{1}{5}\alpha_1 + \frac{2}{5}\alpha_2 - \frac{2}{5}\alpha_3. \tag{79}$$

Thus,

$$D_m^n\left((s-d)^2\right) = -\frac{1}{5}D_m^n(\alpha_1) + \frac{2}{5}D_m^n(\alpha_2) - \frac{2}{5}D_m^n(\alpha_3). \tag{80}$$

So,

$$D_m^n\left((s-d)^2\right)\big|_{s=d=1} = -\frac{1}{5}2^nF_{\mathrm{mn}} + \frac{2}{5}2^nF_{\mathrm{mn}+1} - \frac{2}{5}L_m^n. \tag{81}$$

This completes the proof of (74).

As for (75), we need consider the following grammatical convolution $D_m^n((3d-s)^2)$. It follows by Leibniz formula that

$$\begin{aligned} D_m^n\left((3d-s)^2\right) &= \sum_{k=0}^{n}\binom{n}{k}D_m^k(3d-s)D_m^{n-k}(3d-s) \\ &= \sum_{k=0}^{n}\binom{n}{k}D_1^{\mathrm{mk}}(3d-s)D_1^{m(n-k)}(3d-s), \end{aligned} \tag{82}$$

which reduces by setting $s = d = 1$ to be

$$\sum_{k=0}^{n}\binom{n}{k}L_{\mathrm{mk}}L_{\mathrm{mn-mk}}. \tag{83}$$

Now, let us calculate $D_m^n((3d-s)^2)$. By using Gaussian elimination, $(3d-s)^2$ can be represented as the linear combination of α_1, α_2 and α_3, namely,

$$(3d-s)^2 = -\alpha_1 + 2\alpha_2 + 2\alpha_3. \tag{84}$$

Thus,

$$\begin{aligned} D_m^n(3d-s)^2\big|_{s=d=1} &= -D_m^n(\alpha_1)\big|_{s=d=1} + 2D_m^n(\alpha_2)\big|_{s=d=1} \\ &\quad + 2D_m^n(\alpha_3)\big|_{s=d=1} \\ &= -2^nF_{\mathrm{mn}} + 2^{n+1}F_{\mathrm{mn}+1} + 2L_m^n \\ &= 2^nL_{\mathrm{mn}} + 2L_m^n. \end{aligned} \tag{85}$$

The last equation holds from (38). This completes the proof.

Equations (74) and (75) are given by Church and Bicknell [3]. As far as we know, (73) is new. □

Theorem 13. *For $n, m \geq 1$ and $r \geq 0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}F_{4\mathrm{mk}+4\mathrm{mr}} = L_{2m}^nF_{2\mathrm{mn}+4\mathrm{mr}}. \tag{86}$$

Proof. According to (19),

$$\begin{aligned} F_{4\mathrm{mk}+4\mathrm{mr}} &= D_1^{4\mathrm{mk}+4\mathrm{mr}}s - d\big|_{s=d=1} \\ &= D_1^{4\mathrm{mk}}\left(D_1^{4\mathrm{mr}}(s-d)\right)\big|_{s=d=1} \\ &= D_{4m}^k(\mathrm{s}F_{4\mathrm{mr}-1} + dF_{4\mathrm{mr}-2})\big|_{s=d=1} \\ &= F_{4\mathrm{mr}-1}D_{4m}^k(s)\big|_{s=d=1} + F_{4\mathrm{mr}-2}D_{4m}^k(d)\big|_{s=d=1}. \end{aligned} \tag{87}$$

Thus,

$$\begin{aligned} \sum_{k=0}^{n}\binom{n}{k}F_{4\mathrm{mk}+4\mathrm{mr}} &= F_{4\mathrm{mr}-1}\sum_{k=0}^{n}\binom{n}{k}D_{4m}^k(s)\big|_{s=d=1} \\ &\quad + F_{4\mathrm{mr}-2}\sum_{k=0}^{n}\binom{n}{k}D_{4m}^k(d)\big|_{s=d=1}. \end{aligned} \tag{88}$$

Let $\overline{G}_{4m}$ denote the context-free grammar

$$\overline{G}_{4m} \coloneqq \{d \longrightarrow D_{4m}(d), s \longrightarrow D_{4m}(s), a \longrightarrow a\}, \tag{89}$$

and let $\overline{D}_{4m}$ be the formal derivative associated with $\overline{G}_{4m}$. Then,

$$\begin{aligned} \sum_{k=0}^{n}\binom{n}{k}F_{4\mathrm{mk}+4\mathrm{mr}} &= F_{4\mathrm{mr}-1}\overline{D}_{4m}^n(\mathrm{as})\big|_{a=s=d=1} \\ &\quad + F_{4\mathrm{mr}-2}\overline{D}_{4m}^n(\mathrm{ad})\big|_{a=s=d=1}. \end{aligned} \tag{90}$$

Next, let us turn to calculate $\overline{D}_{4m}(\mathrm{ad})$ and $\overline{D}_{4m}(\mathrm{as})$. Let $A = \mathrm{as}, B = \mathrm{ad}$, It can be verified that

$$\begin{aligned} \overline{D}_{4m}(A) &= \mathrm{as} + a\left(F_{4m}d + F_{4m+1}s\right) \\ &= A\left(F_{4m+1} + 1\right) + BF_{4m}, \\ \overline{D}_{4m}(B) &= \mathrm{ad} + a\left(F_{4m-1}d + F_{4m}s\right) \\ &= \mathrm{AF}_{4m} + B\left(F_{4m-1} + 1\right). \end{aligned} \tag{91}$$

According to (48)–(50), we have

$$\begin{aligned} \overline{D}_{4m}(A) &= L_{2m}\left(\mathrm{AF}_{2m+1} + \mathrm{BF}_{2m}\right), \\ \overline{D}_{4m}(B) &= L_{2m}\left(\mathrm{AF}_{2m} + \mathrm{BF}_{2m-1}\right). \end{aligned} \tag{92}$$

According to (56),

$$\begin{aligned} \overline{D}_{4m}^n(A) &= L_{2m}^n\widetilde{D}^n(A), \\ \overline{D}_{4m}^n(B) &= L_{2m}^n\widetilde{D}^n(B), \end{aligned} \tag{93}$$

where $\widetilde{D}$ is the formal derivative associated with the following grammar

$$\widetilde{G} \coloneqq \{A \longrightarrow \mathrm{AF}_{2m+1} + \mathrm{BF}_{2m}, B \longrightarrow \mathrm{AF}_{2m} + \mathrm{BF}_{2m-1}\}. \tag{94}$$

Notice that $\widetilde{G}$ is as same as $2m$-th Fibonacci grammar D_{2m} by substituting s, d into A, B, respectively. Thus,

$$\overline{D}_{4m}^{n}(A)\big|_{A=B=1}=L_{2m}^{n}D_{2m}^{n}(s)\big|_{s=d=1}=L_{2m}^{n}D_{1}^{2\mathrm{mn}}(s)\big|_{s=d=1}=L_{2m}F_{2\mathrm{mn}+2},$$
$$\overline{D}_{4m}^{n}(B)\big|_{A=B=1}=L_{2m}^{n}D_{2m}^{n}(d)\big|_{s=d=1}=L_{2m}^{n}D_{1}^{2\mathrm{mn}}(d)\big|_{s=d=1}=L_{2m}F_{2\mathrm{mn}+1}. \quad (95)$$

This deduces that

$$\sum_{k=0}^{n}\binom{n}{k}F_{4mk+4mr}=F_{4\mathrm{mr}-1}L_{2m}^{n}F_{2\mathrm{mn}+2}+F_{4\mathrm{mr}-2}L_{2m}^{n}F_{2\mathrm{mn}+1}=L_{2m}^{n}F_{2\mathrm{mn}+4\mathrm{mr}}. \quad (96)$$

The last equation holds from (36). This completes the proof. □

Theorem 14. *For $n,m\geq 1$ and $r,t\geq 0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{r(n-k)}F_{m-r}^{n-k}F_{r}^{k}F_{\mathrm{mk}+t}=F_{m}^{n}F_{rn+t}, \quad (97)$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{r(n-k)}F_{m-r}^{n-k}F_{r}^{k}L_{\mathrm{mk}+t}=F_{m}^{n}L_{rn+t}. \quad (98)$$

Proof. Consider the grammar G':

$$G':=\{s\longrightarrow F_{r}D_{m}(s),d\longrightarrow F_{r}D_{m}(d),a\longrightarrow(-1)^{r}F_{m-r}a\}. \quad (99)$$

Then, it can be easily see that

$$D'^{k}\left(D_{1}^{t}(s-d)\right)\big|_{s=d=1}=F_{r}^{k}D_{m}^{k}\left(D_{1}^{t}(s-d)\right)\big|_{s=d=1}=F_{r}^{k}D_{1}^{\mathrm{mk}+t}(s-d)\big|_{s=d=1}=F_{r}^{k}F_{mk+t},$$
$$D'^{k}(a)\big|_{a=1}=(-1)^{rk}F_{m-r}^{k}. \quad (100)$$

According to Leibnitz formula, the left hand side of (97) can be obtained from $D'^{n}(\mathrm{aD}_{1}^{t}(s-d))$ by setting $a=s=d=1$. Now, the proof of (97) can be reduced to the calculation of $D'^{n}(\mathrm{aD}_{1}^{t}(s-d))$.

Let $A=\mathrm{as},B=\mathrm{ad}$. One can check that

$$D'(A)=(-1)^{r}F_{m-r}\mathrm{as}+a\left(F_{r}F_{m+1}s+F_{r}F_{m}d\right)=A\left((-1)^{r}F_{m-r}+F_{r}F_{m+1}\right)+\mathrm{BF}_{r}F_{m},$$
$$D'(B)=(-1)^{r}F_{m-r}\mathrm{ad}+a\left(F_{r}F_{m}s+F_{r}F_{m-1}d\right)=\mathrm{AF}_{r}F_{m}+B\left((-1)^{r}F_{m-r}+F_{r}F_{m-1}\right). \quad (101)$$

In (42), by setting $n=r+1$ and $k=1$, we obtain

$$F_{m+1}F_{r}+(-1)^{r}F_{m-r}=F_{r+1}F_{m}. \quad (102)$$

And by setting $n=r-1$ and $k=-1$, we obtain

$$F_{m-1}F_{r}+(-1)^{r}F_{m-r}=F_{r-1}F_{m}. \quad (103)$$

Thus,

$$D'(A)=F_{m}\left(\mathrm{AF}_{r+1}+\mathrm{BF}_{r}\right),$$
$$D'(B)=F_{m}\left(\mathrm{AF}_{r}+\mathrm{BF}_{r-1}\right). \quad (104)$$

Then, D' can be viewed as the formal derivative associated with $F_{m}G_{r}$ with the alphabet $\{A,B\}$. According to (56),

$$D'^{n}(A)\big|_{A=B=1}=F_{m}^{n}D_{r}^{n}(s)\big|_{s=d=1}=F_{m}^{n}D_{1}^{rn}(s)\big|_{s=d=1}=F_{m}^{n}F_{nr+2},$$
$$D'^{n}(B)\big|_{A=B=1}=F_{m}^{n}D_{r}^{n}(d)\big|_{s=d=1}=F_{m}^{n}D_{1}^{rn}(d)\big|_{s=d=1}=F_{m}^{n}F_{nr+1}. \quad (105)$$

Thus,

$$D'^{n}\left(\mathrm{aD}_{1}^{t}(s-d)\right)\big|_{a=s=d=1}=D'^{n}\left(F_{t-1}\mathrm{ad}+F_{t-2}\mathrm{as}\right)\big|_{a=s=d=1}=F_{t-1}D'^{n}(B)\big|_{A=B=1}+F_{t-2}D'^{n}(A)\big|_{A=B=1}=F_{t-1}F_{m}^{n}F_{\mathrm{nr}+1}+F_{t-2}F_{m}^{n}F_{\mathrm{nr}+2}=F_{m}^{n}\left(F_{t-1}F_{\mathrm{nr}+1}+F_{t-2}F_{\mathrm{nr}+2}\right), \quad (106)$$

which equals $F_{m}^{n}F_{\mathrm{nr}+t}$ by (36). This complete the proof of (97).

As for (98), we consider the following Leibnitz relation:

$$D'^{n}\left(\mathrm{aD}_{1}^{t}(3d-s)\right)=\sum_{k=0}^{n}D'^{k}\left(D_{1}^{t}(3d-s)\right)D'^{n-k}(a). \quad (107)$$

Now, (98) can be obtained from (107) by setting $a=s=d=1$ since

$$D'^{k}\left(D_{1}^{t}(3d-s)\right)\big|_{s=d=1}=F_{r}^{k}L_{\mathrm{mk}+t},D'^{k}(a)=(-1)^{rk}F_{m-r}^{k}a,$$
$$D'^{n}\left(\mathrm{aD}_{1}^{t}(3d-s)\right)\big|_{a=s=d=1}=D'^{n}\left(L_{t-1}\mathrm{ad}+L_{t-2}\mathrm{as}\right)\big|_{a=s=d=1}=L_{t-1}D'^{n}(B)\big|_{A=B=1}+L_{t-2}D'^{n}(A)\big|_{A=B=1}=L_{t-1}F_{m}^{n}F_{\mathrm{nr}+1}+L_{t-2}F_{m}^{n}F_{\mathrm{nr}+2}=F_{m}^{n}\left(L_{t-1}F_{\mathrm{nr}+1}+L_{t-2}F_{\mathrm{nr}+2}\right), \quad (108)$$

which equals $F_{m}^{n}L_{\mathrm{nr}+t}$ from (37). This complete the proof of (98). □

Remark 15. It is easily to see that the technique of proving identities by using the simple Fibonacci grammar can be extended to study corresponding binomial convolutions involving Fibonacci polynomials $F_{n}(q)$, just considering the Fibonacci grammar. Meanwhile, one can extend the Fibonacci grammar to be

$$G:=\{s\longrightarrow\mathrm{ps}+d,d\longrightarrow s\}, \quad (109)$$

to get generalized Fibonacci and generalized Lucas numbers, who are defined as the linear recurrence

$$U_{n}=\mathrm{pU}_{n-1}+U_{n-2},V_{n}=\mathrm{pV}_{n-1}+V_{n-2}, \quad (110)$$

and the initial conditions $U_{0}=0,U_{1}=1$ and $V_{0}=2,V_{1}=p$.

Besides, the identities proved here are all from Leibnitz formula, hence in form

$$\sum_{k=0}^{n} \binom{n}{k} A_k B_{n-k}. \tag{111}$$

Therefore, there are many identities involving Fibonacci and Lucas numbers who are not in the standard binomial form. For example, Kilic and Tasdemir [9] provided several binomial double summations in the form

$$\sum_{0 \le i, j \le k} \binom{i}{j} U_{\mathrm{ri+4jt}}, \tag{112}$$

as well as the alternating binomial double summations in the form

$$\begin{aligned} &\sum_{0 \le i, j \le k} (-1)^{i} \binom{i}{j} U_{\mathrm{ri+kjt}}, \\ &\sum_{0 \le i, j \le k} (-1)^{j} \binom{i}{j} U_{\mathrm{ri+kjt}}, \\ &\sum_{0 \le i, j \le k} (-1)^{i+j} \binom{i}{j} U_{\mathrm{ri+kjt}}, \end{aligned} \tag{113}$$

for $k = 2, 4$. It's an interesting question to find a universal grammatical proof of these relations.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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