Hindawi
Journal of Mathematics
Volume 2023, Article ID 6512823, 14 pages
https://doi.org/10.1155/2023/6512823

Research Article

Global Existence and Decaying Rates of the Strong Solution for the Boussinesq System

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Received 17 February 2023; Revised 11 May 2023; Accepted 5 August 2023; Published 31 August 2023

Academic Editor: Qingkai Zhao

This paper focuses on the global existence and time-decay rates of the strong solution for the Boussinesq system with full viscosity in $\mathbb{R}^n$ for $n \geq 3$. Under the initial assumption of $(\theta_0, u_0) \in L^{n/3} \times L^n$ with a small norm, and $n > 3$ or $n = 3$ and $\theta_0 \in L^{r_0}$ for some $r_0 > 1$, global existence and uniqueness of the strong solution (θ, u) for the Boussinesq system is established. This solution is proven to obey the following estimates: $\|\theta(t)\|_r \leq Ct^{-(3-n/p)/2}$ for $n/3 \leq p < \infty$, $\|u(t)\|_p \leq Ct^{-(1-n/q)/2}$ for $n \leq q \leq \infty$, $\|\nabla\theta(t)\|_p \leq Ct^{-(3-n/p)/2-1/2}$ and $\|\nabla^2\theta(t)\|_p = O(t^{-n(1/r-1/p)/2-1})$ as $t \longrightarrow \infty$ for $r \leq p < n/2$, and $\|\nabla u(t)\|_q \leq Ct^{-(1-n/q)/2-1/2}$ and $\|\nabla^2 u(t)\|_q = O(t^{-n(1/r-1/q)/2-1})$ as $t \longrightarrow \infty$ for $n \leq q < 2n$, where $r = n/3$ if $n > 3$ and $1 < r < \min\{r_0, n/2\}$ if $n = 3$.

1. Introduction

This paper investigated the global existence, uniqueness, and time-decay rates of the strong solution for the Boussinesq system in $\mathbb{R}^n$ $(n \geq 3)$ as follows:

$$\begin{cases} \partial_t\theta - \nu\Delta\theta + u\cdot\nabla\theta = 0, & t > 0, x \in \mathbb{R}^n; \\ \partial_t u - \mu\Delta u + u\cdot\nabla u + \nabla\pi = \kappa\theta e_n, & t > 0, x \in \mathbb{R}^n; \\ \nabla\cdot u = 0, & t > 0, x \in \mathbb{R}^n; \\ \theta(0,x) = \theta_0(x), u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (1)$$

Here, the unknown vector-valued function $u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$ represents the velocity of the flow of a fluid, while the unknown scalar functions $\theta(x,t)$ and $\pi(x,t)$ represent their temperature and inner pressure at the place $x \in \mathbb{R}^n$ and time $t > 0$. Moreover, $\theta_0(x)$ and $u_0(x)$ denote the initial temperature and initial velocity, respectively.

The Boussinesq system is a simplified model to simulate the motion of the ocean or the atmosphere. By the Boussinesq approximation, we can neglect the variation of the density of the fluid but the var biation of temperature, which causes a vortex buoyancy force $\kappa\theta e_n$ in the system, where $e_n = (0, \ldots, 0, 1)$ denotes the unit vertical vector. There are three physical constants associated with the fluid: $\kappa > 0$ is the proportion coefficient, $\mu > 0$ is the viscous index of the fluid, and $\nu > 0$ is the thermal confusion number. For the sake of simplicity, by means of rescaling of the unknowns, we always assume that $\kappa = 1$.

Evidently, if $\theta = 0$, then system (1) reduces to the standard incompressible Navier–Stokes equations. This means that the classical theory for the Navier–Stokes equations may be extended to the Boussinesq system. Danchin and Paicu [1] and Cannon and DiBenedetto [2] showed that if the initial data (θ_0, u_0) of both lie in the energy space $L^2(\mathbb{R}^n)$, then the Boussinesq system (1) permits a global weak solution (θ, u) lying in

$$C\left([0,\infty), L^2\right) \times \left(L^\infty_{\text{loc}}\left(0,\infty; L^2\right) \cap L^2_{\text{loc}}\left(0,\infty; H^1\right)\right). \quad (2)$$

This is an extension of the results for the Leray–Hopf weak solutions of the Navier–Stokes equations (cf. [3]). Brandolese and Schonbek [4] and Han [5] investigated the time-decay property of the weak solutions of (1) in L^2 space under the extra assumption $\theta_0 \in L^1$.

Danchin and Paicu [6] readdressed the global existence of the weak solutions for (1) with a partial viscosity $(\nu = 0)$ under the initial conditions: $\theta_0 \in L^{n/3,\infty} \cap L^{p_0,\infty}$, $u_0 \in L^{n,\infty}$, and

$$\|\theta_0\|_{L^{n/3,\infty}} + \|u\|_{L^{n,\infty}} \le c\mu, \tag{3}$$

for some sufficient small constant $c>0$, where

$$\begin{cases} p_0 > \frac{3}{2}, & \text{if } n = 3, \\ p_0 > \frac{4}{3}, & \text{if } n = 4, \\ p_0 \ge \frac{n}{3}, & \text{if } n \ge 5, \end{cases} \tag{4}$$

and $L^{r,\infty} = (L^1, L^\infty)_{1-1/r,\infty}$ is the real interpolation space between L^1 and L^∞ (refer to [1], §7.24]), especially $L^{1,\infty} = L^1$ and $L^{r,\infty} = L^r_w$ for $1<r<\infty$, called the Lorenz space. Sokrani [7] and Wang et al. [8] focused on the global posedness of the 3-D Boussinesq system with the application of axisymmetric data and spherical coordinates, respectively.

As for the strong solvability of the Boussinesq system, the authors of reference [1] proved that if the initial temperature is $\theta_0 \in \dot{B}^0_{n,1} \cap L^{n/3}$ and $u_0 \in \dot{B}^{n/p-1}_{p,1} \cap L^{n,\infty}$ for some $p \ge n$ with the condition

$$\mu^{-1}\|\theta_0\|_{L^{n/3}} + \|u\|_{L^{n,\infty}} \le c\mu, \tag{5}$$

for $c>0$ that is small enough, then system (1) with partial viscosity has a unique and globally existing strong solution.

$$(\theta, u) \in C\Big([0,\infty); \dot{B}^0_{n,1}\Big) \times \Big(C\Big([0,\infty); \dot{B}^{n/p-1}_{p,1}\Big) \cap L^1_{\mathrm{loc}}\Big(0,\infty; \dot{B}^{n/p+1}_{n,1}\Big)\Big), \tag{6}$$

where $\dot{B}^s_{p,r}$ denotes the homogeneous Besov space (cf. [[2], §2.3]). A similar result was obtained in [4] for (1) with full viscosity and $n=3$. It was proved that under the initial assumption

$$\|\theta_0\|_{L^1} + \|\theta_0\|_{L^\infty(\mathbb{R}^3, |x|^3 dx)} + \|u_0\|_{L^\infty(\mathbb{R}^3, |x| dx)} \le \varepsilon, \tag{7}$$

for some sufficiently small number of $\varepsilon>0$, system (1) has a unique strong solution of (θ, u) in the space

$$\Big(L^\infty\big(0,\infty; L^1\big) \cap L^\infty\big((0,\infty)\times\mathbb{R}^3; (\sqrt{t} + |x|)^3 dt dx\big)\Big) \times L^\infty\big((0,\infty)\times\mathbb{R}^3; (\sqrt{t} + |x|) dt dx\big). \tag{8}$$

Considering that the associated function spaces are invariant under the following scaling transformation:

$$\begin{aligned} &\theta(x,t) \longmapsto \lambda^3\theta\big(\lambda x, \lambda^2 t\big), u(x,t) \longmapsto \lambda u\big(\lambda x, \lambda^2 t\big), \\ &\theta_0(x) \longmapsto \lambda^3\theta_0(\lambda x), u_0(x) \longmapsto \lambda u_0(\lambda x), \end{aligned} \tag{9}$$

respectively, the initial spaces $L^{n/3}$, $L^\infty(\mathbb{R}^3, |x|^3 dx)$ and L^n, $\dot{B}^{n/p-1}_{p,1}$, and $L^\infty(\mathbb{R}^3, |x| dx)$ are called critical. In this sense, results obtained in [1, 4] can be viewed as the extensions of those in [9–11] for the Navier–Stokes equations, where the critical space $\dot{H}^{1/2}$ or L^n for the initial velocity was employed.

This paper also addresses the global solvability of the Boussinesq system (1) with full viscosity. Motivated by [1, 4, 6], we first establish the existence and uniqueness of results for the global integral solution of (1) under the hypothesis $(\theta_0, u_0) \in L^{n/3} \times L^n$, and

$$\|\theta_0\|_{L^{n/3}} + \|u\|_{L^n} \le \varepsilon, \tag{10}$$

for some $\varepsilon>0$ that is small enough. Compared to the literature, though the initial space L^n for u_0 is stronger than $L^{n,\infty}$, the initial assumption $\theta_0 \in L^{n/3}$ is much cheaper.

By using the smoothing action of the heat semigroup $e^{t\Delta}$ on L^p spaces for all $p\ge 1$, and the estimates of the operators $e^{t\Delta}P$, $e^{t\Delta}\nabla$, and $e^{t\Delta}P\nabla$, this paper also describes the higher regularity of the weak solution. We will show that under the additional condition

$$n>3, \text{or } n=3 \text{ and } \theta_0 \in L^{r_0} \text{ for some } r_0>1. \tag{11}$$

The global integral solution of (1) has enough regularity to become a strong solution. Compared with [1, 4, 8], here, the extra assumption $\theta_0 \in \dot{B}^0_{n,1}, L^\infty(\mathbb{R}^3, |x|^3 dx)$ or H^2 is replaced with a simpler one. Noticing that there is no any other restriction on the lower bounds of r_0, compared with (4), it gives a partial answer to [6] for the minimal conditions to guarantee the solvability of (1) in the case $n=3$. In addition, time-decay properties of the higher order norms of the strong solution are also investigated in this paper.

To end this section, we give an outline of the paper. Preliminaries and main results are stated in Section 2. Section 3 is devoted to the proof of the existence and uniqueness of the weak and strong solution of the Boussinesq system with full viscosity, together with the estimates for the solution and its gradient. In Section 4, time-decaying properties of the temporal and higher-order spatial norms of the strong solution are presented.

2. Preliminaries and Main Results

For $1\le p\le\infty$ and $\alpha>0$, let L^p and $W^{\alpha,p}$ be the usual Lebesgue and Sobolev space of scalar- or vector-valued functions defined on $\mathbb{R}^n$ with the norm denoted by $\|\cdot\|_p$ and $\|\cdot\|_{\alpha,p}$, respectively. Let C_0^∞ be the collection of all scalar- or vector-valued functions whose components are smooth with compact supports, $C^\infty_{0,\sigma}$ be its subset containing all the divergence free vector fields, and let L^p_σ be the completion of $C^\infty_{0,\sigma}$ in the space L^p for $1<p<\infty$. We use P to denote the Helmholtz projection from L^p onto L^p_σ defined by

$I + \nabla(-\Delta)^{-1}\mathrm{div}$. We also use C to denote a universal positive constant which may change from line to line, but does not depend on the involved functions.

It is observed that every classical solution (θ, u) to (1) solves the following system of integral equations:

$$\begin{cases} \theta(t) = e^{t\nu\Delta}\theta_0 - \int_0^t e^{(t-\tau)\nu\Delta}\mathrm{div}(\theta(\tau)u(\tau))\mathrm{d}\tau, \\ u(t) = e^{t\mu\Delta}u_0 - \int_0^t e^{(t-\tau)\mu\Delta}P\mathrm{div}(u(\tau)\otimes u(\tau))\mathrm{d}\tau \\ + \int_0^t e^{(t-\tau)\mu\Delta}P(\theta(\tau)e_n)\mathrm{d}\tau. \end{cases} \tag{12}$$

To deal with system (12), let us introduce some function spaces with temporal weights. Suppose that $\alpha > 0$ and X is a Banach space, then we define the equation as

$$\begin{aligned} L_\alpha^\infty(0,\infty;X) &= \left\{ f \in L_{\mathrm{loc}}^\infty(0,\infty;X)\colon \|f\|_{L_\alpha^\infty(X)} := \operatorname*{esssup}_{t>0} t^\alpha \|f(t)\|_X < \infty \right\}, \\ C_\alpha((0,\infty),X) &= \left\{ f \in C((0,\infty),X)\colon \|f\|_{C_\alpha(X)} := \sup_{t>0} t^\alpha \|f(t)\|_X < \infty \right\}, \\ C_{\alpha,0}((0,\infty),X) &= \left\{ f \in C_\alpha((0,\infty),X)\colon \lim_{t\longrightarrow 0} t^\alpha \|f(t)\|_X = 0 \right\}. \end{aligned} \tag{13}$$

For the sake of convenience, in the following discussions, kernels of the operators $e^{t\nu\Delta}$, $e^{t\mu\Delta}P$, $e^{t\nu\Delta}\nabla$, and $e^{t\mu\Delta}P\nabla$ are denoted by $G_\nu(x,t)$, $\widetilde{G}_\mu(x,t)$, $K_\nu(x,t)$, and $\widetilde{K}_\mu(x,t)$, respectively, where $G_\nu(x,t) = (4\pi\nu t)^{-n/2}\exp\{-|x|^2/4\nu t\}$ is the Gaussian kernel, and others are vector-valued functions associated with $G_\nu(x,t)$ or $G_\mu(x,t)$. There are some estimates for these kernels for $1 \le p \le \infty$ and $t > 0$ (refer to [4]) referred to as

$$\|G_\nu(t)\|_p \le C(\nu t)^{-n/2p'}, \|K_\nu(t)\|_p \le C(\nu t)^{-n/2p'-1/2}, \tag{14}$$

and

$$\left\|\widetilde{G}_\mu(t)\right\|_p \le C(\mu t)^{-n/2p'}\ (p>1), \left\|\widetilde{K}_\mu(t)\right\|_p \le C(\mu t)^{-n/2p'-1/2}, \tag{15}$$

where $C = C(n,p) > 0$ and p' is the conjugate number of p, i.e., $1/p + 1/p' = 1$.

As a direct consequence of (14), it follows that for $1 \le p \le q \le \infty$, there is a constant $C = C(n,p,q) > 0$ such that

$$\left\|e^{t\nu\Delta}f\right\|_q \le C(\nu t)^{-n(1/p-1/q)/2}\|f\|_p. \tag{16}$$

Remark 1. By the boundedness and differentiability of the heat semigroup $e^{t\nu\Delta}$ on L^p, we assert that for all $f \in L^p$ and for all $k \ge 1$, we have

$$e^{t\nu\Delta}f \in C_b([0,\infty),L^p)\cap C\left((0,\infty),W^{k,p}\right)\cap C^\infty((0,\infty),L^p), \tag{17}$$

where $C_b([0,\infty),X)$ denotes the space of all continuous and bounded X– valued functions on $[0,\infty)$.

Furthermore, for $q > p$, by the density of $L^p \cap L^q$ in L^p, we can check that $e^{t\nu\Delta}f \in C((0,\infty),L^q)$, and in addition to (16), it holds that

$$\lim_{t\longrightarrow 0} t^{n(1/p-1/q)/2}\left\|e^{t\nu\Delta}f\right\|_q = 0. \tag{18}$$

In other words, $e^{t\nu\Delta}f \in C_{n(1/p-1/q)/2,0}((0,\infty),L^q)$.

Given that $0 < \gamma \le 1$, we consider the fractional power of negative Laplacian with the index γ defined by (cf. [12])

$$(-\Delta)^\gamma f(x) = C(\gamma,n)\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{f(x)-f(y)}{|x-y|^{n+2\gamma}}\mathrm{d}y. \tag{19}$$

If $1 < p \le q < \infty$, then there is a corresponding constant $C = C(n,\gamma,p,q) > 0$ such that

$$\left\|(-\Delta)^\gamma e^{t\Delta}f\right\|_q \le Ct^{-\gamma-n(1/p-1/q)/2}\|f\|_p, \tag{20}$$

for $0 \le \gamma \le 1$ and for a constant $C = C(n,\gamma,\delta,q) > 0$ such that

$$\left\|(-\Delta)^\gamma\left(I-e^{t\Delta}\right)u\right\|_q \le Ct^{\delta-\gamma}\left\|(-\Delta)^\delta u\right\|_q, \tag{21}$$

for all $u \in W^{2\delta,q}$ and $0 \le \gamma < \delta \le 1$ (refer to [9, 11]), where $(-\Delta)^0 := I$ naturally.

The following lemma plays an important role in checking the continuity of a function in L^q space.

Lemma 2. *Assume that $1 \le p_i, q, r_i < \infty$ such that $1 + 1/q = 1/r_i + 1/p_i$, and $\alpha_i = n/2r_i' + (i-1)/2 \in (0,1)$ and $1 - \alpha_i \le \beta_i < 1$ for $i = 1,2$. Given a scalar field f_1, two vector fields f_2 and $\widetilde{f}_1$ and a tensor field $\widetilde{f}_2$ lying in $L_{\beta_i}^\infty(0,\infty;L^{p_i})$ verifies that*

$$\|f_i(t)\|_{p_i}, \left\|\widetilde{f}_i(t)\right\|_{p_i} = o\left(t^{-\beta_i}\right)\ \text{as}\ t \longrightarrow 0, \tag{22}$$

where $p_1, r_1 > 1$ *for* $\widetilde{f}_1$, *and let*

$$\begin{aligned}F_1(t) &= \int_0^t e^{(t-\tau)\nu\Delta} f_1(\tau)\mathrm{d}\tau, F_2(t) = \int_0^t e^{(t-\tau)\nu\Delta}\mathrm{div} f_2(\tau)\mathrm{d}\tau,\\ \widetilde{F}_1(t) &= \int_0^t e^{(t-\tau)\nu\Delta} P\widetilde{f}_1(\tau)\mathrm{d}\tau, \widetilde{F}_2(t) = \int_0^t e^{(t-\tau)\nu\Delta} P\mathrm{div}\widetilde{f}_2(\tau)\mathrm{d}\tau.\end{aligned} \tag{23}$$

Then, the following conclusions hold true:

(i) *If* $\alpha_i + \beta_i = 1$, then $F_i, \widetilde{F}_i \in C_b([0,\infty), L^q)$, *and* $F_i(0) = 0$, $\widetilde{F}_i(0) = 0$

(ii) *If* $\alpha_i + \beta_i > 1$, *then* $F_i, \widetilde{F}_i \in C_{\alpha_i+\beta_i-1,0}(0,\infty; L^q)$

The derivation of the estimates for $\|F_i(t)\|_q$ and $\|\widetilde{F}_i(t)\|_q$ is left in the proof of Theorem 4 with concrete exponents. Verification of the continuity of $\|F_2(t)\|_1$ with respect to $t > 0$ is performed in the proof of ([13], Lemma 2.5), and others can be made in a much similar way, so we omit the whole reasoning process here.

Lemma 3 (see [4]). *Suppose that* $\widehat{X}$ *and* $\widehat{Y}$ *are two Banach spaces,* $B_1: \widehat{X}\times\widehat{X} \longrightarrow \widehat{X}$ *and* $B_2: \widehat{Y}\times\widehat{X} \longrightarrow \widehat{Y}$ *are two bilinear operators, and* $L: \widehat{Y} \longrightarrow \widehat{X}$ *is a linear operator, then corresponding to these operators there are three constants* $k_i > 0$, $i = 1, 2, 3$ *such that*

$$\begin{aligned}&\|B_1(u,v)\|_{\widehat{X}} \le k_1\|u\|_{\widehat{X}}\|v\|_{\widehat{X}}, \|B_2(\theta,u)\|_{\widehat{Y}} \le k_2\|\theta\|_{\widehat{Y}}\|u\|_{\widehat{X}},\\ &\|L(\theta)\|_{\widehat{X}} \le k_3\|\theta\|_{\widehat{Y}}.\end{aligned} \tag{24}$$

Then, for any $\widehat{\theta}_0 \in \widehat{Y}$ and $\widehat{u}_0 \in \widehat{X}$, if

$$2k_3\|\widehat{\theta}_0\|_{\widehat{Y}} + \|\widehat{u}_0\|_{\widehat{X}} \le \frac{k_1}{4(2k_1+k_2)^2}, \tag{25}$$

then, the following equation system

$$\begin{cases}\theta = \widehat{\theta}_0 + B_2(\theta,u),\\ u = \widehat{u}_0 + B_1(u,u) + L(\theta),\end{cases} \tag{26}$$

has a unique solution $(\theta,u) \in \widehat{Y}\times\widehat{X}$ which verifies that

$$2k_3\|\theta\|_{\widehat{Y}} + \|u\|_{\widehat{X}} \le \frac{k_1}{(2k_1+k_2)^2}. \tag{27}$$

We now present the main results of the paper.

Theorem 4. *There is a small number* $\varepsilon = \varepsilon(\lambda,\mu) > 0$ *and a constant* $C = C(\lambda,\mu) \ge 1$ *such that for all* $(\theta_0,u_0) \in L^{n/3}\times L^n_\sigma$ *with the restriction (10), system (12) has a unique global solution* (θ,u) *fulfilling the equation that follows:*

$$\begin{aligned}&\theta \in C_b\left([0,\infty), L^{n/3}\right)\cap C\left((0,\infty), L^p\right) \quad \text{for } n/3 < p \le n/2,\\ &u \in C_b\left([0,\infty), L^n_\sigma\right)\cap C\left((0,\infty), L^q\right) \quad \text{for } n < q \le 2n,\end{aligned} \tag{28}$$

and

$$\|\theta(t)\|_{n/3} + \|u(t)\|_n \le C\varepsilon, \quad t \ge 0, \tag{29}$$

$$\begin{aligned}\|\theta(t)\|_p &\le C\varepsilon t^{-(3-n/p)/2}, \|u(t)\|_q\\ &\le C\varepsilon t^{-(1-n/q)/2}, \quad t > 0.\end{aligned} \tag{30}$$

Here, $C_b([0,\infty), L^{n/3}) = C([0,\infty), L^{n/3}) \cap L^\infty(0,\infty; L^{n/3})$ endowed with the norm $\|\cdot\|_{L^\infty(L^{n/3})}$.

Remark 5. It is easy to check that every solution (θ,u) of system (12) verifies the following integral equations:

$$\begin{cases}\displaystyle\int_0^T\int_{\mathbb{R}^n}\left(-\theta\partial_t\varphi - \nu\theta\Delta\varphi + \theta u\cdot\nabla\varphi\right)(x,t)\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^n}\theta_0(x)\varphi(x,0)\mathrm{d}x,\\ \displaystyle\int_0^T\int_{\mathbb{R}^n}\left(-u\cdot\partial_t\psi - \mu u\cdot\Delta\psi + (u\otimes u)\colon\nabla\psi\right)(x,t)\mathrm{d}x\mathrm{d}t,\\ \displaystyle= \int_{\mathbb{R}^n}u_0(x)\cdot\psi(x,0)\mathrm{d}x + \int_0^T\int_{\mathbb{R}^n}\left(\theta u_n\right)(x,t)dxdt.\end{cases} \tag{31}$$

Here, $0 < T < \infty$ is arbitrary, and $\varphi \in C^1_0([0,T); C^\infty_0)$ and $\psi \in C^1_0([0,T); C^\infty_{0,\sigma})$ are also arbitrary. Symbol $a\otimes b = (a_ib_j)_{n\times n}$ denotes the tensor product of two vectors, while $A\colon B = \sum_{i,j=1}^n A_{ij}B_{ij}$ denotes the scalar product of the two tensors. In this sense, (θ,u) is called the very weak solution of the Bossinesq system (1) (refer to [14]).

Theorem 6. *In addition to (10), we assume that condition (11) holds, then the global solution* (θ,u) *for (12) obtained in Theorem 4 satisfies the following properties:*

$$\theta \in W^{1,2}_{\mathrm{loc}}\left(0,\infty; L^{2n/5}\right)\cap L^2_{\mathrm{loc}}\left(0,\infty; W^{2,2n/5}\right), \tag{32}$$

$$u \in W^{1,2}_{\mathrm{loc}}\left(0,\infty; L^n\right)\cap L^2_{\mathrm{loc}}\left(0,\infty; W^{2,n}\right), \tag{33}$$

and

$$\begin{cases}\partial_t\theta - \nu\Delta\theta + u\cdot\nabla\theta = 0, \text{in } L^{2n/5},\\ \partial_t u - \mu\Delta u + P(u\cdot\nabla u) = P(\theta e_n), \text{in } L^n_\sigma,\end{cases} \tag{34}$$

for a.e. $t > 0$.

Furthermore, for $n/2 < p < \infty$ and $2n < q \le \infty$, we have

$$\theta \in C_{(3-n/p)/2}\left(0,\infty; L^p\right), u \in C_{(1-n/q)/2}\left(0,\infty; L^q\right), \quad (35)$$

with the same estimates as in (30). For $r \le p < n/2$ and $n \le q < 2n$, we also have

$$\nabla\theta \in C_{\varpi}\left(0,\infty; L^p\right), \nabla u \in C_{1-(2q/)}\left(0,\infty; L^q\right), \quad (36)$$

with the estimates

$$\begin{aligned} \|\nabla\theta(t)\|_p &\le C_0 t^{-\varpi}, \\ \|\nabla u(t)\|_q &\le C\varepsilon t^{-(1-n2/q)}, \end{aligned} \quad (37)$$

where $C_0 = 2C\|\theta_0\|_r$ for some $C > 0$, $1 < r < \min\{r_0, n/2\}$ if $n = 3$, and $C_0 = C\varepsilon$, $r = n/3$ if $n > 3$, and $\varpi = n(1/r - 1/p)/2 + 1/2$.

Remark 7. The function pair (θ, u), which verifies (32), (33), and (34), where $2n/5$ is replaced with some $p \ge r$, is called strong solution of the Boussinesq system (1).

By using the decomposition method for the integral representation of the solution developed in [15, 13], we can also derive the asymptotic behavior and growth of the temporal derivative exponents and spatial Hessian of θ and u, that is,

Theorem 8. *Under the assumptions on the initial data (10) and (11), the strong solution (θ, u) also satisfies*

$$\begin{aligned} &\partial_t\theta, \nabla^2\theta \in C\left((0,\infty); L^p\right), r \le p < \frac{n}{2}, \\ &\partial_t u, \nabla^2 u, \nabla\pi \in C\left((0,\infty); L^q\right), n \le q < 2n, \end{aligned} \quad (38)$$

and

$$\left\|\partial_t\theta(t)\right\|_p + \left\|\nabla^2\theta(t)\right\|_p \le C\varepsilon t^{-n(1/r-n/p)/2-1}, \quad (39)$$

$$\left\|\partial_t u(t)\right\|_q + \left\|\nabla^2 u(t)\right\|_q + \|\nabla\pi(t)\|_q \le C\varepsilon t^{-n(1/r-n/q)/2-1}, \quad (40)$$

for $t > 2T_0$, where $T_0 = 1$ for $n = 3$ and $T_0 = 0$ for $n > 3$, and π is the associated inner pressure of the fluid.

3. Global Existence, Uniqueness, and Regularity of the Solution

We first give a proof of the existence and uniqueness of the integral solution for the Bossinesq system (1).

Proof of Theorem 4: we define two intersection spaces

$$\begin{aligned} \widehat{Y} &= C_b\left([0,\infty), L^{n/3}\right) \cap C_{1/2,0}\left((0,\infty), L^{n/2}\right), \\ \widehat{X} &= C_b\left([0,\infty), L^n_\sigma\right) \cap C_{1/4,0}\left((0,\infty), L^{2n}\right), \end{aligned} \quad (41)$$

with the norms

$$\begin{aligned} \|\theta\|_{\widehat{Y}} &= \|\theta\|_{L^\infty\left(L^{n/3}\right)}\theta + \|\theta\|_{C_{1/2}\left(L^{n/2}\right)}, \\ \|u\|_{\widehat{X}} &= \|\theta\|_{L^\infty\left(L^n\right)} + \|\theta\|_{C_{1/4}\left(L^{2n}\right)}. \end{aligned} \quad (42)$$

For $(\theta, u) \in \widehat{Y} \times \widehat{X}$, we define that

$$\begin{aligned} L(\theta)(t) &= \int_0^t e^{(t-\tau)\mu\Delta} P\left(\theta(\tau)e_n\right)\mathrm{d}\tau, \\ B_1(u,u)(t) &= -\int_0^t e^{(t-\tau)\mu\Delta} P\mathrm{div}(u(\tau)\otimes u(\tau))\mathrm{d}\tau, \\ B_2(\theta,u)(t) &= -\int_0^t e^{(t-\tau)\nu\Delta}\mathrm{div}(\theta(\tau)u(\tau))\mathrm{d}\tau. \end{aligned} \quad (43)$$

We also define that $\widehat{\theta}_0 \coloneqq e^{t\nu\Delta}\theta_0$ and $\widehat{u}_0 \coloneqq e^{t\mu\Delta}u_0$. Under this setting, equation system (12) can be abstracted to (26).

We now investigate the boundedness of L, B_1 and B_2 one by one. First, via (15) and Young's inequality, we have

$$\begin{aligned} \|L(\theta)(t)\|_n &\le \int_0^t \left\|\widetilde{G}_\mu(t-\tau)\right\|_{n/(n-1)}\|\theta(\tau)\|_{n/2}\mathrm{d}\tau \\ &\le C\mu^{-1/2}\int_0^t (t-\tau)^{-1/2}\tau^{-1/2}\mathrm{d}\tau \cdot \|\theta\|_{C_{1/2}\left((0,t),L^{n/2}\right)} \\ &\le C\mu^{-1/2}\|\theta\|_{\widehat{Y}}. \end{aligned} \quad (44)$$

Moreover, by taking $r_1 = 2n/(2n-3)$, we can deduce that

$$\begin{aligned} \|L(\theta)(t)\|_{2n} &\le \int_0^t \left\|\widetilde{G}_\mu(t-\tau)\right\|_{r_1}\|\theta(\tau)\|_{n/2}\mathrm{d}\tau \\ &\le C\mu^{-3/4}\int_0^t (t-\tau)^{-3/4}\tau^{-1/2}\mathrm{d}\tau \cdot \|\theta\|_{C_{1/2}\left((0,t),L^{n/2}\right)} \\ &\le C\mu^{-3/4}t^{-1/4}\|\theta\|_{\widehat{Y}}. \end{aligned} \quad (45)$$

As a direct consequence of the inclusion $\theta \in C_{1/2,0}((0,\infty), L^{n/2})$, we have

$$\lim_{t\longrightarrow 0^+}\|\theta\|_{C_{1/2}\left((0,t),L^{n/2}\right)} = 0. \quad (46)$$

Thus, by invoking Lemma 2, we conclude that $L(\theta) \in \widehat{X}$, and

$$\|L(\theta)\|_{\widehat{X}} \le C_3\left(\mu^{-1/2} + \mu^{-3/4}\right)\|\theta\|_{\widehat{Y}}. \quad (47)$$

Here, the constant $C_3 > 0$ is independent of ν, μ and θ. Similarly, for $u, v \in \widehat{X}$, we have

$$\begin{aligned}\|B_1(u,v)(t)\|_n &\le \int_0^t \|\widetilde{K}_\mu(t-\tau)\|_{2n/(2n-1)}\|u(\tau)\|_{2n}\|v(\tau)\|_n \mathrm{d}\tau\\ &\le C\mu^{-3/4}\int_0^t (t-\tau)^{-3/4}\tau^{-1/4}d\tau\cdot\|u\|_{C_{1/4}((0,t),L^{2n})}\|v\|_{L^\infty(L^n_\sigma)}\\ &\le C\mu^{-3/4}\|u\|_{\widehat{X}}\|v\|_{\widehat{X}},\\ \|B_1(u,v)(t)\|_{2n} &\le \int_0^t \|\widetilde{K}_\mu(t-\tau)\|_{2n/(2n-1)}\|u(\tau)\|_{2n}\|v(\tau)\|_{2n}\mathrm{d}\tau\\ &\le C\mu^{-3/4}\int_0^t (t-\tau)^{-3/4}\tau^{-1/2}d\tau\cdot\|u\|_{C_{1/4}((0,t),L^{2n})}\|v\|_{C_{1/4}((0,t),L^{2n})}\\ &\le C\mu^{-3/4}t^{-1/4}\|u\|_{\widehat{X}}\|v\|_{\widehat{X}}.\end{aligned} \tag{48}$$

Since

$$\lim_{t\longrightarrow 0^+}\|u\|_{C_{1/4}((0,t),L^{2n})}=0, \tag{49}$$

by invoking Lemma 2 again, we conclude that $B_1(u,v)\in\widehat{X}$, and

$$\|B_1(u,v)\|_{\widehat{X}}\le C_1\mu^{-3/4}\|u\|_{\widehat{X}}\|v\|_{\widehat{X}}, \tag{50}$$

for some constant $C_1>0$ which is independent of ν,μ and u,v.

Yet for $\theta\in\widehat{Y}$ and $u\in\widehat{X}$, we have

$$\begin{aligned}\|B_2(\theta,u)(t)\|_{n/3} &\le \int_0^t \|K_\nu(t-\tau)\|_1\|\theta(\tau)\|_{n/2}\|u(\tau)\|_n \mathrm{d}\tau\\ &\le C\nu^{-1/2}\int_0^t (t-\tau)^{-1/2}\tau^{-1/2}\mathrm{d}\tau\cdot\|\theta\|_{C_{1/2}((0,t),L^{n/2})}\|u\|_{L^\infty(L^n_\sigma)}\\ &\le C\nu^{-1/2}\|\theta\|_{\widehat{Y}}\|u\|_{\widehat{X}},\\ \|B_2(\theta,u)(t)\|_{n/2} &\le \int_0^t \|K_\nu(t-\tau)\|_{2n/(2n-1)}\|\theta(\tau)\|_{n/2}\|u(\tau)\|_{2n}\mathrm{d}\tau\\ &\le C\nu^{-3/4}\int_0^t (t-\tau)^{-3/4}\tau^{-3/4}\mathrm{d}\tau\\ &\quad\cdot\|\theta\|_{C_{1/2}((0,t),L^{n/2})}\|u\|_{C_{1/4}((0,t),L^{2n})}\\ &\le C\nu^{-3/4}t^{-1/2}\|\theta\|_{\widehat{Y}}\|u\|_{\widehat{X}}.\end{aligned} \tag{51}$$

Due to (46), by invoking Lemma 2 the third time, we have $B_2(\theta,u)\in\widehat{Y}$, and

$$\|B_2(\theta,u)\|_{\widehat{Y}}\le C_2\left(\nu^{-1/2}+\nu^{-3/4}\right)\|\theta\|_{\widehat{Y}}\|u\|_{\widehat{X}}, \tag{52}$$

for some constant $C_2>0$ which is independent of ν,μ and θ,u.

In addition, since $\theta_0\in L^{n/3}$ and $u_0\in L^n_\sigma$, by virtue of (16) and Remark 1, it follows that $\widehat{\theta}_0\in\widehat{Y}$ and $\widehat{u}_0\in\widehat{X}$, and

$$\|\widehat{\theta}_0\|_{\widehat{Y}}\le c_0\left(1+\nu^{-1/2}\right)\|\theta_0\|_{n/3},\ \|\widehat{u}_0\|_{\widehat{X}}\le d_0\left(1+\mu^{-1/4}\right)\|u_0\|_n, \tag{53}$$

where the constants $c_0,d_0\ge 1$ depend only on n.

Now, we apply Lemma 3 with $k_1=C_1\mu^{-3/4}$, $k_2=C_2(\nu^{-1/2}+\nu^{-3/4})$, and $k_3=C_3(\mu^{-1/2}+\mu^{-3/4})$ to assert that under the hypothesis

$$\begin{aligned}&2C_3c_0\left(1+\nu^{1/2}\right)\|\theta_0\|_{n/3}+d_0\mu^{1/2}\nu^{1/2}\|u_0\|_n\\ &\le\frac{C_1\mu^{3/2}\nu^2}{4\left[2C_1\nu^{3/4}+C_2\mu^{3/4}\left(1+\nu^{1/4}\right)\right]^2\left(1+\mu^{1/4}\right)}=:h(\mu,\nu),\end{aligned} \tag{54}$$

condition (25) is satisfied, and consequently, the integral equation team (12) admits a unique solution $(\theta,u)\in\widehat{Y}\times\widehat{X}$ verifying (27). Thus, the global solvability and uniqueness of the integral solution to (12) are reached, where the small number $\varepsilon(\mu,\nu)$ and the constant $C(\lambda,\mu)$ appearing in (29) take the values

$$\varepsilon(\mu,\nu)=\min\left\{1,\frac{h(\mu,\nu)}{2C_3c_0\left(1+\nu^{1/2}\right)+d_0\mu^{1/2}\nu^{1/2}}\right\},$$
$$C(\lambda,\mu)=\max\left\{1,\frac{4\left(2C_3c_0\left(1+\nu^{1/2}\right)+d_0\mu^{1/2}\nu^{1/2}\right)\left(1+\mu^{1/4}\right)}{\nu^{1/2}\min\left\{2C_3,\mu^{3/4}\right\}}\right\}, \tag{55}$$

respectively. Finally, by means of interpolation we obtain the two estimates in (30).

Remark 9. Note that, $h(\mu,\nu)\longrightarrow 0$ and consequently $\varepsilon(\mu,\nu)\longrightarrow 0$, while $C(\lambda,\mu)\longrightarrow\infty$ as $\nu\longrightarrow 0$. Hence, methods employed here are not feasible any more to deal with the global existence of the weak solutions of (1) in the case $\nu=0$.

Then, we turn to show the higher regularity of the integral solution (θ,u) to make it become a strong solution for (1). Proof of Theorem 6: for the sake of convenience, herein after, the two viscosity indices ν and μ are both normalised to 1. Without loss of generality, we assume that $1<r<6/5$ in the case of $n=3$. We also assume that $\|\theta_0\|_{n/3}\le\varepsilon/2$ and $\|u_0\|_n\le\varepsilon/2$. We take an exponent $q\in(n,2n)$, and let $\alpha=(1-n/q)/2$ and $\beta=\alpha+1/2$, then we have $0<\alpha<1/4$ and $1/2<\beta<3/4$. Note that the global solution (θ,u) for (12) is obtained as the limit of the approximate solutions $\left\{(\theta^k,u^k)\right\}_{k=1}^{\infty}$, where (θ^{k+1},u^{k+1}) solves the following equation system:

$$\begin{cases}\theta^{k+1}(t)=e^{t\Delta}\theta_0-\displaystyle\int_0^t e^{(t-\tau)\nu\Delta}\mathrm{div}\left(\theta^k(\tau)u^k(\tau)\right)\mathrm{d}\tau,\\ u^{k+1}(t)=e^{t\Delta}u_0-\displaystyle\int_0^t e^{(t-\tau)\mu\Delta}P\mathrm{div}\left(u^k(\tau)\otimes u^k(\tau)\right)\mathrm{d}\tau+\int_0^t e^{(t-\tau)\mu\Delta}P\left(\theta^k(\tau)e_n\right)\mathrm{d}\tau,\end{cases} \tag{56}$$

and $\theta^0=\widehat{\theta}_0$ and $u^0=\widehat{u}_0$. Consequently, we may derive all the desired estimates of (θ,u) by taking limits of the estimates of (θ^{k+1},u^{k+1}), while the latter can be made by the arguments of iteration.

First of all, by reviewing the proof of Theorem 4, we assert that (30) with $n/3\le p\le n/2$ and $n\le q\le 2n$ holds uniformly for (θ^k,u^k), $k=1,2,\cdots$.

As for the reasoning basis, we can deduce that

$$\begin{aligned}&\widehat{\theta}_0\in C_{3/4}\left((0,\infty);L^{2n/3}\right)\cap C_1\left((0,\infty);L^n\right),\nabla\widehat{\theta}_0\in C_\eta\left((0,\infty);L^{2n/5}\right),\\&\nabla\widehat{u}_0\in C_{1/2}\left((0,\infty);L^n\right)\cap C_\beta\left((0,\infty);L^q\right).\end{aligned} \tag{57}$$

Then, they verify the following estimates:

$$\begin{aligned}\max\left\{\|\widehat{\theta}_0\|_{C_{3/4}\left(L^{2n/3}\right)},\|\widehat{\theta}_0\|_{C_1\left(L^n\right)}\right\}&\le C\|\theta_0\|_{n/3}\le\frac{C\varepsilon}{2},\\ \|\nabla\widehat{\theta}_0\|_{C_\eta\left(L^{2n/5}\right)}&\le C\|\theta_0\|_r\le\frac{C_0}{2},\end{aligned} \tag{58}$$

and

$$\max\left\{\|\nabla\widehat{u}_0\|_{C_\beta\left(L^q\right)},\|\nabla\widehat{u}_0\|_{C_{1/2}\left(L^n\right)}\right\}\le C\|u_0\|_n\le\frac{C\varepsilon}{2}, \tag{59}$$

where inequality (20) with $\gamma=1/2$ and the fact

$$\|\nabla f\|_r\le\left\|(-\Delta)^{1/2}f\right\|_r\le C\|\nabla f\|_r, \tag{60}$$

for $1<r<\infty$ are both employed, and

$$\eta=\begin{cases}\dfrac{3(1/r-1/2)}{2}, & \text{if } n=3,\\[2mm] \dfrac{3}{4}, & \text{if } n>3.\end{cases} \tag{61}$$

The estimate of $\|\theta^{k+1}(t)\|_{2n/3}$ is based on the inequality (14) for the kernel K_1 and the estimate in (30) for $\|u^k(\tau)\|_q$, that is,

$$\begin{aligned}\left\|\theta^{k+1}(t)\right\|_{2n/3} &\le \left\|\widehat{\theta}_0(t)\right\|_{2n/3} + \int_0^t \left\|K_1(t-\tau)\right\|_{q'}\left\|\theta^k(\tau)\right\|_{2n/3}\left\|u^k(\tau)\right\|_q \mathrm{d}\tau \\ &\le \left\|\widehat{\theta}_0(t)\right\|_{2n/3} + C_\alpha \int_0^t (t-\tau)^{\alpha-1}\tau^{-3/4-\alpha}\mathrm{d}\tau \\ &\quad \cdot \left\|u^k\right\|_{L^\infty_\alpha(L^q)}\left\|\theta^k\right\|_{L^\infty_{3/4}(L^{2n/3})} \\ &\le \left\|\widehat{\theta}_0(t)\right\|_{2n/3} + C_\alpha C\varepsilon t^{-3/4}\left\|\theta^k\right\|_{L^\infty_{3/4}(L^{2n/3})},\end{aligned} \tag{62}$$

where constraint $n < q < 2n$ is needed. This result, together with the iterative method, leads to the following inequalities:

$$\begin{aligned}\left\|\theta^{k+1}\right\|_{L^\infty_{3/4}(L^{2n/3})} &\le \left\|\widehat{\theta}_0\right\|_{L^\infty_{3/4}(L^{2n/3})} + C_\alpha C\varepsilon\left\|\theta^k\right\|_{L^\infty_{3/4}(L^{2n/3})} \\ &\le \left\|\widehat{\theta}_0\right\|_{L^\infty_{3/4}(L^{2n/3})} + C_\alpha C\varepsilon\left(\left\|\widehat{\theta}_0\right\|_{L^\infty_{3/4}(L^{2n/3})} + C_\alpha C\varepsilon\left\|\theta^k\right\|_{L^\infty_{3/4}(L^{2n/3})}\right) \\ &\le \cdots\cdots \le \left\|\widehat{\theta}_0\right\|_{L^\infty_{3/4}(L^{2n/3})}\sum_{j=0}^{k+1}(C_\alpha C\varepsilon)^j.\end{aligned} \tag{63}$$

Let >0, which is chosen in Theorem 4 is so small that $C_\alpha C\varepsilon \le 1/2$, then by using (58), we obtain

$$\left\|\theta^{k+1}\right\|_{L^\infty_{3/4}(L^{2n/3})} \le 2\left\|\widehat{\theta}_0\right\|_{L^\infty_{3/4}(L^{2n/3})} \le C\varepsilon, \tag{64}$$

uniformly for $k \in \mathbb{N}$.

The treatment of $\|\theta^{k+1}\|_{L^\infty_1(L^n)}$ is slightly different. Due to the singularity of t^{-1} at $t = 0$, we divide the integral interval $[0, t]$ into two equal parts to derive the following equation:

$$\begin{aligned}\left\|\theta^{k+1}(t)\right\|_n &\le \left\|\widehat{\theta}_0(t)\right\|_n + \int_0^{t/2}\left\|K_1(t-\tau)\right\|_{2n/(2n-3)}\left\|\theta^k(\tau)\right\|_{2n/3}\left\|u^k(\tau)\right\|_n \mathrm{d}\tau \\ &\quad + \int_{t/2}^t \left\|K_1(t-\tau)\right\|_{q'}\left\|\theta^k(\tau)\right\|_n\left\|u^k(\tau)\right\|_q \mathrm{d}\tau \\ &\le \left\|\widehat{\theta}_0(t)\right\|_n + C\int_0^{t/2}(t-\tau)^{-5/4}\tau^{-3/4}\mathrm{d}\tau \cdot \left\|\theta^k\right\|_{L^\infty_{3/4}(L^{2n/3})}\left\|u^k\right\|_{L^\infty(L^n)} \\ &\quad + C_\alpha\int_{t/2}^t (t-\tau)^{\alpha-1}\tau^{-\alpha-1}\mathrm{d}\tau \cdot \left\|\theta^k\right\|_{L^\infty_1(L^n)}\left\|u^k\right\|_{L^\alpha(L^q)} \\ &\le \left\|\widehat{\theta}_0(t)\right\|_n + C\varepsilon t^{-1} + C_\alpha C\varepsilon t^{-1}\left\|\theta^k\right\|_{L^\infty_1(L^n)}.\end{aligned} \tag{65}$$

Also, by the arguments of iteration, we obtain

$$\begin{aligned}\left\|\theta^{k+1}\right\|_{L_1^\infty(L^n)} &\leq \left\|\widehat{\theta}_0\right\|_{L_1^\infty(L^n)} + C\varepsilon + C_\alpha C\varepsilon\left\|\theta^k\right\|_{L_1^\infty(L^n)} \\ &\leq \cdots \leq \left(\left\|\widehat{\theta}_0\right\|_{L_1^\infty(L^n)} + C\varepsilon\right)\sum_{j=0}^{k+1}\left(C_\alpha C\varepsilon\right)^j \\ &\leq 2\left(\left\|\widehat{\theta}_0\right\|_{L_1^\infty(L^n)} + C\varepsilon\right) \leq 3C\varepsilon,\end{aligned} \tag{66}$$

for all $k \in \mathbb{N}$ provided that $C_\alpha C\varepsilon \leq 1/2$.

We now turn to deal with $\|\nabla u^{k+1}(t)\|_n$. By employing (20) with $\gamma = 1/2$ and (29) and (60) for $\|u^k\|_n$ and (64), it can be deduced that

$$\begin{aligned}\left\|\nabla L\left(\theta^k\right)(t)\right\|_n &\leq \int_0^t \left\|\nabla G_1(t-\tau)\right\|_{(2n/(2n-1))}\left\|\theta^k(\tau)\right\|_{(2n/)}\mathrm{d}\tau \\ &\leq C\int_0^t (t-\tau)^{-3/4}\tau^{-3/4}\mathrm{d}\tau \cdot \left\|\theta^k\right\|_{L_{3/4}^\infty(L^{2n/3})} \leq C\varepsilon t^{-1/2},\end{aligned} \tag{67}$$

and

$$\begin{aligned}\left\|\nabla B_1\left(u^k, u^k\right)(t)\right\|_n &\leq \int_0^t \left\|\nabla \widetilde{G}_1(t-\tau)\right\|_{q'}\left\|u^k(\tau)\right\|_q\left\|\nabla u^k(\tau)\right\|_n\mathrm{d}\tau \\ &\leq C_\alpha\int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha-1/2}\mathrm{d}\tau \cdot \left\|u^k\right\|_{L_\alpha^q(L^n)}\left\|\nabla u^k\right\|_{L_{1/2}^\infty(L^n)} \\ &\leq C_\alpha C\varepsilon t^{-1/2}\left\|\nabla u^k\right\|_{L_{1/2}^\infty(L^n)}.\end{aligned} \tag{68}$$

We place the abovementioned two estimates together to obtain

$$\begin{aligned}\left\|\nabla u^{k+1}(t)\right\|_n &\leq \left\|\nabla \widehat{u}_0(t)\right\|_n + \left\|\nabla L\left(\theta^k\right)(t)\right\|_n + \left\|\nabla B\left(u^k, u^k\right)(t)\right\|_n \\ &\leq \left\|\nabla \widehat{u}_0(t)\right\|_n + C\varepsilon t^{-1/2} + C_\alpha C\varepsilon t^{-1/2}\left\|\nabla u^k\right\|_{L_{1/2}^\infty(L^n)}.\end{aligned} \tag{69}$$

Then, following the same derivation as in (66), we have

$$\begin{aligned}\left\|\nabla u^{k+1}\right\|_{L_{1/2}^\infty(L^n)} &\leq \left(\left\|\nabla \widehat{u}_0\right\|_{L_{1/2}^\infty(L^n)} + C\varepsilon\right)\sum_{j=0}^{k+1}\left(C_\alpha C\varepsilon\right)^j \\ &\leq 2\left(\left\|\nabla \widehat{u}_0\right\|_{L_{1/2}^\infty(L^n)} + C\varepsilon\right) \leq 3C\varepsilon, k \in \mathbb{N},\end{aligned} \tag{70}$$

for the case where $C_\alpha C\varepsilon \leq 1/2$.

A similar procedure can be performed for $\|\nabla u^{k+1}(t)\|_q$, that is

$$\begin{aligned}\left\|\nabla L\left(\theta^k\right)(t)\right\|_q &\leq C\int_0^t (t-\tau)^{-n(3/2n-1/q)/2-1/2}\left\|\theta^k(\tau)\right\|_{2n/3}\mathrm{d}\tau \\ &\leq C\int_0^t (t-\tau)^{-n(3/2n-1/q)/2-1/2}\tau^{-3/4}\mathrm{d}\tau \leq C\varepsilon t^{-\beta},\end{aligned} \tag{71}$$

$$\begin{aligned}\left\|\nabla B_1\left(u^k, u^k\right)(t)\right\|_q &\leq C_\alpha\int_0^t (t-\tau)^{\alpha-1}\left\|u^k(\tau)\right\|_q\left\|\nabla u^k(\tau)\right\|_q\mathrm{d}\tau \\ &\leq C_\alpha C\varepsilon\int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha-\beta}\mathrm{d}\tau \cdot \left\|\nabla u^k\right\|_{L_\beta^\infty(L^q)} \\ &\leq C_\alpha C\varepsilon t^{-\beta}\left\|\nabla u^k\right\|_{L_\beta^\infty(L^q)},\end{aligned} \tag{72}$$

which jointly produce

$$\begin{aligned}\left\|\nabla u^{k+1}(t)\right\|_q &\leq \left\|\nabla\widehat{u}_0(t)\right\|_q + \left\|\nabla L\left(\theta^k\right)(t)\right\|_q + \left\|\nabla B_1\left(u^k,u^k\right)(t)\right\|_q \\ &\leq \left\|\nabla\widehat{u}_0(t)\right\|_q + C\varepsilon t^{-\beta} + C_\alpha C\varepsilon t^{-\beta}\left\|\nabla u^k\right\|_{L^\infty_\beta(L^q)},\end{aligned} \tag{73}$$

and consequently

$$\left\|\nabla u^{k+1}\right\|_{L^\infty_\beta(L^q)} \leq \left(\left\|\nabla\widehat{u}_0\right\|_{L^\infty_\beta(L^q)} + C\varepsilon\right)\sum_{j=0}^{k+1}\left(C_\alpha C\varepsilon\right)^j \leq 3C\varepsilon, \quad k\in\mathbb{N}, \tag{74}$$

for the same restriction on ε as mentioned above.

An estimate of $\|\nabla\theta^{k+1}(t)\|_{2n/5}$ can be made in two cases. In the case of $n=3$, we have

$$\begin{aligned}\left\|\nabla\theta^{k+1}(t)\right\|_{6/5} &\leq \left\|\nabla\widehat{\theta}_0(t)\right\|_{6/5} + \left\|\nabla B_2\left(\theta^k,u^k\right)(t)\right\|_{6/5} \\ &\leq \left\|\nabla\widehat{\theta}_0(t)\right\|_{6/5} + C\int_0^t (t-\tau)^{-3/4}\left\|u^k(\tau)\right\|_6\left\|\nabla\theta^k(\tau)\right\|_{6/5}\mathrm{d}\tau \\ &\leq \left\|\nabla\widehat{\theta}_0(t)\right\|_{6/5} + C\varepsilon t^{-3(1/r-1/2)/2}\left\|\nabla\theta^k\right\|_{L^\infty_{3(1/r-1/2)/2}\left(L^{6/5}\right)}.\end{aligned} \tag{75}$$

In the case of $n>3$, we have

$$\begin{aligned}\left\|\nabla\theta^{k+1}(t)\right\|_{2n/5} &\leq \left\|\nabla\widehat{\theta}_0(t)\right\|_{2n/5} + C_\alpha\int_0^t (t-\tau)^{\alpha-1}\left\|u^k(\tau)\right\|_q\left\|\nabla\theta^k(\tau)\right\|_{2n/5}\mathrm{d}\tau \\ &\leq \left\|\nabla\widehat{\theta}_0(t)\right\|_{2n/5} + CC_\alpha\varepsilon t^{-3/4}\left\|\nabla\theta^k\right\|_{L^\infty_{3/4}\left(L^{2n/5}\right)}.\end{aligned} \tag{76}$$

Both cases lead to the same conclusion

$$\left\|\nabla\theta^{k+1}\right\|_{L^\infty_\eta\left(L^{2n/5}\right)} \leq 2\left\|\nabla\widehat{\theta}_0\right\|_{L^\infty_\eta\left(L^{2n/5}\right)} \leq C_0, \quad k\in\mathbb{N}, \tag{77}$$

provided $\varepsilon>0$ is small enough.

Based on the estimates derived above, we assert that the weak-star cluster points of the sequences $\{\theta^{k+1}\}$ and $\{u^{k+1}\}$ and functions θ and u verify the estimates (64)–(77), respectively. Moreover, in light of Lemma 2, it follows that $\theta\in C_{1/4}((0,\infty);L^{2n/5})\cap C_{3/4}((0,\infty);L^{2n/3})\cap C_1((0,\infty);L^n)$, $\nabla\theta\in C_{3/4}((0,\infty),L^{2n/5})$, and $\nabla u\in C_{1/2}((0,\infty),L^n)\cap C_\beta((0,\infty),L^q)$. Besides, by the Gagliardo–Nirenberg inequality, we have

$$\|u(t)\|_\infty \leq C\|u(t)\|_q^{1-n/q}\|\nabla u(t)\|_q^{n/q} \leq C\varepsilon t^{-1/2}, \tag{78}$$

which combined with the continuity of $u(t)$ in $W^{1,q}$ and yields $u\in C_{1/2}((0,\infty),L^\infty)$, and consequently $u\cdot\nabla\theta\in C((0,\infty),L^{2n/5})$ and $u\cdot\nabla u\in C((0,\infty),L^n)$.

Now, we can check all the conditions for the regularity of the solution (θ,u). For $1<p<\infty$, we define $B_p=-\Delta$ in L^p and $A_p=-P\Delta$ in L^p_σ. We recall that, both B_p and A_p are sectorial operators with the domain $D(B_p)=W^{2,p}$ and $D(A_p)=W^{2,p}\cap L^p_\sigma$, and their native operators generate two uniformly bounded and exponentially decaying, analytic semigroups e^{-tB_p} and e^{-tA_p}, respectively. For all $0<\gamma<1$, by interpolation, we have $D(B_p^\gamma)=W^{2\gamma,p}$ and $D(A_p^\gamma)=W^{2\gamma,p}\cap L^p_\sigma$. Thus, for every $0<\delta<1$, we have $\theta(\delta)\in D(B_{2n/5}^{1/2})$ and $u(\delta)\in D(A_n^{1/2})$. Furthermore, by the uniqueness of the integral solution, (θ,u) can be represented by

$$\begin{cases}\theta(t) = e^{-(t-\delta)B_{2n/5}}\theta(\delta) - \displaystyle\int_\delta^t e^{-(t-\tau)B_{2n/5}}(\tau)u\cdot\nabla\theta(\tau)\mathrm{d}\tau, \\ u(t) = e^{-(t-\delta)A_n}u(\delta) - \displaystyle\int_\delta^t e^{-(t-\tau)A_n}P(u(\tau)\cdot\nabla u(\tau))\mathrm{d}\tau \\ + \displaystyle\int_\delta^t e^{-(t-\tau)A_n}P(\theta(\tau)e_n)\mathrm{d}\tau.\end{cases} \tag{79}$$

Since $u\cdot\nabla\theta\in C([\delta,\infty),L^{2n/5})$, $P(u\cdot\nabla u)\in C([\delta,\infty),L^n_\sigma)$ and $P(\theta(\tau)e_n)\in C([\delta,\infty),L^n_\sigma)$, by employing the maximal L^p– regularity of B_p and A_p (refer to [16, 17]) and unique solvability of (79), we conclude that

$$\begin{aligned} &\theta \in C\left([\delta,\infty), W^{1,2n/5}\right) \cap W^{1,2}_{\mathrm{loc}}\left(\delta,\infty; L^{2n/5}\right) \cap L^2_{\mathrm{loc}}\left(\delta,\infty; W^{2,2n/5}\right),\\ &u \in C\left([\delta,\infty), W^{1,n}\right) \cap W^{1,2}_{\mathrm{loc}}\left(\delta,\infty; L^{n}\right) \cap L^2_{\mathrm{loc}}\left(\delta,\infty; W^{2,n}\right), \end{aligned} \tag{80}$$

and (θ, u) verifies (34) a.e. on (δ, ∞). Thus, by the arbitrariness of $\delta > 0$, we eventually obtain (32), (33), and (34) for a.e. $t \in (0,\infty)$. In other words, (θ, u) is a strong solution for the Boussinesq system (1.1).

The second part of the proof is devoted to the validation of (35)–(37) where $\nabla u \in C_\beta((0,\infty); L^q)$ and $\|\nabla u(t)\|_q \le C\varepsilon t^{-\beta}$ for $n \le q < 2n$ has been verified in the first part. From the continuity of $u(t)$ in $L^n \cap L^\infty$, together with the estimates (29) and (78), and the method of interpolation, one can easily derive that $u \in C_\alpha((0,\infty); L^q)$ and $\|u(t)\|_q \le C\varepsilon t^{-\alpha}$ for all $2n < q < \infty$. Besides, for any $n < p < \infty$, let $\omega = (3 - n/p)/2$, then analogous to the treatment of $\|\theta^{k+1}(t)\|_n$, we have

$$\begin{aligned} \left\|\theta^{k+1}(t)\right\|_p &\le \left\|\widehat{\theta}_0(t)\right\|_p + \int_0^{t/2} \left\|K_1(t-\tau)\right\|_b \left\|\theta^k(\tau)\right\|_{(2n/3)} \left\|u^k(\tau)\right\|_n \mathrm{d}\tau\\ &\quad + \int_{t/2}^{t} \left\|K_1(t-\tau)\right\|_{q'} \left\|\theta^k(\tau)\right\|_p \left\|u^k(\tau)\right\|_q \mathrm{d}\tau\\ &\le \left\|\widehat{\theta}_0(t)\right\|_p + C\int_0^{t/2} (t-\tau)^{-7/4+(n/2p)} \tau^{-3/4}\mathrm{d}\tau\\ &\quad \cdot \left\|\theta^k\right\|_{L^\infty_{3/4}\left(L^{(2n/3)}\right)} \left\|u^k\right\|_{L^\infty(L^n)}\\ &\quad + C_\alpha \int_{t/2}^{t} (t-\tau)^{\alpha-1}\tau^{-\omega-\alpha}\mathrm{d}\tau \cdot \left\|\theta^k\right\|_{L^\infty_\omega(L^p)} \left\|u^k\right\|_{L^\infty_\alpha(L^q)}\\ &\le \left\|\widehat{\theta}_0(t)\right\|_p + C\varepsilon t^{-\omega} + C_\alpha C\varepsilon t^{-\omega}\left\|\theta^k\right\|_{L^\infty_\omega(L^p)}, \end{aligned} \tag{81}$$

which in turn yields $\|\theta^{k+1}\|_{L^\infty_\omega(L^p)} \le 3C\varepsilon$, and consequently $\theta \in C_\omega((0,\infty); L^p)$ with the same estimate provided that $\varepsilon > 0$ is small enough, where $b = (1 + 1/p - 5/2n)^{-1}$. This result can also be checked for $n/2 < p < n$ by means of interpolation.

Moreover, for $r \le p < n/2$, we take $\max\{n, p'\} < s < \mathrm{pr}/(p-r)$, then analogous to (77), we can derive that

$$\begin{aligned} \left\|\nabla\theta^{k+1}(t)\right\|_p &\le \left\|\nabla\widehat{\theta}_0(t)\right\|_p + C\int_0^{t} (t-\tau)^{-((1+n/s)/2)} \left\|u^k(\tau)\right\|_s \left\|\nabla\theta^k(\tau)\right\|_p \mathrm{d}\tau\\ &\le \left\|\nabla\widehat{\theta}_0(t)\right\|_p + C\int_0^{t} (t-\tau)^{-((1+n/s)/2)} \tau^{-((1+n/s)/2)-\varpi}\mathrm{d}\tau\\ &\quad \cdot \left\|u^k\right\|_{L^\infty_{(1-n/s)/2}(L^s)} \left\|\nabla\theta^k\right\|_{L^\infty_\varpi(L^p)}\\ &\le Ct^{-\varpi}\left\|\theta_0\right\|_r + C\varepsilon t^{-\varpi}\left\|\nabla\theta^k\right\|_{L^\infty_\varpi(L^p)}, \end{aligned} \tag{82}$$

which results in $\|\nabla\theta^{k+1}\|_{L^\infty_\varpi(L^p)} \le 2C\|\theta_0\|_r = C_0$, and finally, $\nabla\theta \in C_\varpi((0,\infty); L^p)$ with the same estimate for a sufficiently small number of $\varepsilon > 0$.

Remark 10. Uniqueness of the strong solution for the Boussinesq system (1) comes from the unique solvability of the system (12).

Remark 11. Under the initial condition $\theta_0 \in L^1 \cap L^r$ in the case of $n = 3$, we can check the following estimate (see [4, 5] for references):

$$\left\|\theta^{k+1}(t)\right\|_r \le C(1+t)^{-3(1-1/r)/2}. \tag{83}$$

Remark 12. By reviewing the proof of Theorem 6, we should mention that associated with the estimate (30), the upper bounds of the small number $\varepsilon > 0$ can only be taken uniformly for p belonging to an bounded interval $[r, p_1]$ for any $r < p_1 < \infty$, but uniformly for q lying in the unbounded interval $[n, \infty]$.

4. Decay Rates of the Higher Order Norms

Proposition 13. *Suppose that (θ, u) is the global strong solution of (1) obtained in Theorem 6 under the initial hypotheses of (10) and (12). Then, for $0 \le \gamma < \delta < 1$ and $h > 0$, it holds that*

$$\left\|(-\Delta)^\gamma(\theta(t+h)-\theta(t))\right\|_p \le C\varepsilon\left[h^{\delta-\gamma}t^{-\delta} + h^{1-\gamma}t^{-1}\right]t^{1/2-\varpi}, \quad t > T_0, \tag{84}$$

and

$$\left\|(-\Delta)^\gamma(u(t+h)-u(t))\right\|_q \le C\varepsilon\left[h^{\delta-\gamma}t^{-\delta} + h^{1-\gamma}t^{-1}\right]t^{-\alpha}, \quad t > 0, \tag{85}$$

where $r \le p < n/2$, $n \le q < 2n$, and $\varpi = n(1/r - 1/p)/2 + 1/2$.

Proof. We only prove (84), and (85) can be dealt in the same way. For $t, h > 0$, we consider the following decomposition:

$$\begin{aligned}
&(-\Delta)^\gamma(\theta(t+h)-\theta(t)) \\
&\quad = (-\Delta)^\gamma\left(e^{h\Delta}-I\right)e^{t\Delta/2}\theta(t/2) - \int_{t/2}^{t}(-\Delta)^\gamma\left(e^{h\Delta}-I\right)e^{(t-\tau)\Delta}(u(\tau)\cdot\nabla)\theta(\tau)\mathrm{d}\tau \\
&\qquad - \int_t^{t+h}(-\Delta)^\gamma e^{(t+h-\tau)\Delta}(u(\tau)\cdot\nabla)\theta(\tau)\mathrm{d}\tau =: I_1 + I_2 + I_3.
\end{aligned} \tag{86}$$

By invoking (20) and (21), we can deduce that

$$\begin{aligned}
\left\|I_1\right\|_p &\le Ch^{\delta-\gamma}\left\|(-\Delta)^\delta e^{t\Delta/2}\theta(t/2)\right\|_p \le C\varepsilon h^{\delta-\gamma}t^{-\delta-\omega}, \\
\left\|I_2\right\|_p &\le Ch^{\delta-\gamma}\int_{t/2}^{t}(t-\tau)^{-\delta}\|(u(\tau)\cdot\nabla)\theta(\tau)\|_p\mathrm{d}\tau \\
&\le Ch^{\delta-\gamma}\int_{t/2}^{t}(t-\tau)^{-\delta}\|u(\tau)\|_\infty\|\nabla\theta(\tau)\|_p\mathrm{d}\tau \\
&\le C\varepsilon h^{\delta-\gamma}\int_{t/2}^{t}(t-\tau)^{-\delta}\tau^{-1/2-\varpi}d\tau \le C\varepsilon h^{\delta-\gamma}t^{-\delta+1/2-\varpi}, \\
\left\|I_3\right\|_p &\le C\int_t^{t+h}(t+h-\tau)^{-\gamma}\|u(\tau)\|_\infty\|\nabla\theta(\tau)\|_p\mathrm{d}\tau \\
&\le C\varepsilon\int_t^{t+h}(t+h-\tau)^{-\gamma}\tau^{-1/2-\varpi}\mathrm{d}\tau \le C\varepsilon h^{1-\gamma}t^{-1/2-\varpi}.
\end{aligned} \tag{87}$$

By combining the abovementioned three estimates, we obtained (84). ☐

Remark 14. It is observed that for $n = 3$, estimate (84) also holds on (0, 1) but only with the exponent $1/2 - \varpi$ in the power substituted by $-\omega$.

Proof of Theorem 8: Here, we only deal with (40), the derivation of (39) is much similar. The standard potential theory shows that $\|\nabla^2 f\|_r \le C\|(-\Delta)f\|_r$ for all $f \in W^{2,r}$ and $1 < r < \infty$ with $C = C(n, r)$. So, we first use (84) and (85) to derive the estimates for $\|(-\Delta)u(t)\|_q$. For this purpose, we divide $(-\Delta)u(t)$ into four parts

$$\begin{aligned}
(-\Delta)u(t) &= (-\Delta)e^{t\Delta/2}u(t/2) + \left(I - e^{t\Delta/2}\right)[P\theta(t) - P(u(t)\cdot\nabla u(t))] \\
&\quad - \int_{t/2}^{t}(-\Delta)e^{(t-\tau)\Delta}[P(u(\tau)\cdot\nabla)u(\tau) - P(u(t)\cdot\nabla u(t))]\mathrm{d}\tau \\
&\quad + \int_{t/2}^{t}(-\Delta)e^{(t-\tau)\Delta}[P\theta(\tau) - P\theta(t)]\mathrm{d}\tau \\
&:= J_1 + J_2 + J_3 + J_4, \quad t > 0.
\end{aligned} \tag{88}$$

It is easily deduced from (20), (30), and (37) that

$$\begin{aligned} \|J_1\|_q &\le Ct^{-1}\left\|u\left(\frac{t}{2}\right)\right\|_q \le C\varepsilon t^{-1-\alpha}, \\ \|J_2\|_q &\le C\|\theta(t)\|_q + \|u(t)\|_\infty\|\nabla u(t)\|_q\Big) \le C\varepsilon t^{-1-\alpha} \end{aligned} \tag{89}$$

for $t > 0$. By employing (84) and (85) with $\gamma = 0$ or $\gamma = 1/2$, we obtain

$$\begin{aligned} \|J_3\|_q \le& \int_{t/2}^{t}\left\|(-\Delta)e^{(t-\tau)\Delta}P[u(\tau)\cdot\nabla(u(\tau)-u(t))]\right\|_q \mathrm{d}\tau \\ &+ \int_{t/2}^{t}\left\|(-\Delta)e^{(t-\tau)\Delta}P[(u(\tau)-u(t))\cdot\nabla u(t)]\right\|_q \mathrm{d}\tau \\ \le& C\int_{t/2}^{t}(t-\tau)^{-1}\|u(\tau)\|_\infty\|\nabla(u(t)-u(\tau))\|_q \mathrm{d}\tau \\ &+ C\int_{t/2}^{t}(t-\tau)^{-1}\|u(t)-u(\tau)\|_\infty\|\nabla u(t)\|_q \mathrm{d}\tau \\ \le& C\int_{t/2}^{t}(t-\tau)^{-1}\|u(\tau)\|_\infty\left\|(-\Delta)^{1/2}(u(t)-u(\tau))\right\|_q \mathrm{d}\tau \\ &+ C\int_{t/2}^{t}(t-\tau)^{-1}\|u(t)-u(\tau)\|_{q_1}^{1-n/q_1}\|\nabla(u(t)-u(\tau))\|_{q_1}^{n/q_1}\|\nabla u(t)\|_q \mathrm{d}\tau \\ \le& C\varepsilon\int_{t/2}^{t}(t-\tau)^{-1}\tau^{-1/2}\left[(t-\tau)^{\delta_1-1/2}\tau^{-\delta_1} + (t-\tau)^{1/2}\tau^{-1}\right]\tau^{-\alpha}\mathrm{d}\tau \\ &+ C\int_{t/2}^{t}(t-\tau)^{-1}\|u(t)-u(\tau)\|_{q_1}^{1-n/q_1}\left\|(-\Delta)^{1/2}(u(t)-u(\tau))\right\|_{q_1}^{n/q_1} \\ &\bullet \|\nabla u(t)\|_q \mathrm{d}\tau \\ \le& C\varepsilon t^{-1-\alpha} + C\varepsilon t^{-\beta}\int_{t/2}^{t}(t-\tau)^{-1}\Big\{\left[(t-\tau)^{\delta_2}\tau^{-\delta_2} + (t-\tau)\tau^{-1}\right]^{1-n/q_1} \\ &\cdot \tau^{-(1-n/q_1)^2/2}\cdot\left[(t-\tau)^{\delta_1-1/2}\tau^{-\delta_1} + (t-\tau)^{1/2}\tau^{-1}\right]^{n/q_1}\tau^{-(1-n/q_1)n/2q_1}\Big\}\mathrm{d}\tau \\ \le& C\varepsilon t^{-1-\alpha}, \end{aligned} \tag{90}$$

for $t > 0$. Here, $1/2 < \delta_1 < 1$, $0 < \delta_2 < 1$, $n < q_1 < 2n$, and $\beta = 1 - n/2q = \alpha + 1/2$.

As for $\|J_4\|_q$, we use (84) with $\gamma = 0$ to get

$$\begin{aligned} \|J_4\|_q \le& C\int_{t/2}^{t}(t-\tau)^{-1-n(1/p_1-1/q)/2}\|\theta(t)-\theta(\tau)\|_{p_1}\mathrm{d}\tau \\ \le& C\varepsilon\int_{t/2}^{t}(t-\tau)^{-1-n(1/p_1-1/q)/2}\left[(t-\tau)^{\delta_3}\tau^{-\delta_3} + (t-\tau)\tau^{-1}\right] \\ &\cdot\tau^{-n(1/r-1/p_1)/2}\mathrm{d}\tau \le C\varepsilon t^{-1-n(1/r-1/q)/2}, \end{aligned} \tag{91}$$

for $t > 2T_0$, where $r < p_1 < n/2$ is taken so close to $n/2$ such that $n(1/p_1 - 1/q)/2 < 1$ and $n(1/p_1 - 1/q)/2 < \delta_3 < 1$.

Putting all the estimates for $\|J_j\|_q$ and $j = 1, 2, 3, 4$ together, we obtain the desired estimates for $\|(-\Delta)u(t)\|_q$ and consequently for $\|\nabla^2 u(t)\|_q$. In addition, the continuity of $\nabla^2 u(t)$ in L^q with respect to $t > 0$ can be deduced from the decomposition (88), jointly with (84) and (85), in light of Lemma 2.

Finally, by performing the operator P on both sides of the second equation in (1), we can deduce that $\partial_t u \in C((0,\infty), L^q)$, and

$$\begin{aligned}\left\|\partial_t u(t)\right\|_q &\le \|\Delta u(t)\|_q + \|P(u(t)\cdot\nabla u(t))\|_q + \left\|P(\theta(t)e_n)\right\|_q\\ &\le C\varepsilon t^{-1-n(1/r-1/q)} + C\|u(t)\|_\infty \|\nabla u(t))\|_q + C\|\theta(t)\|_q\\ &\le C\varepsilon t^{-1-n(1/r-1/q)},\end{aligned} \tag{92}$$

which in turn infers the existence of the pressure π of the fluid verifying $\nabla\pi \in C((0,\infty), L^q)$ and the estimate

$$\|\nabla\pi(t)\|_q \le \left\|\partial_t u(t)\right\|_q + \|\Delta u(t)\|_q + \|u(t)\cdot\nabla u(t)\|_q + \left\|\theta(t)e_n\right\|_q \le C\varepsilon t^{-1-n(1/r-1/q)}, \tag{93}$$

for $t > 2T_0$. Thus, the desired estimate of (40) has been reached.

Remark 15. Under some extra reasonable hypothesis, the velocity of the flow exhibits rapid decaying behavior as $t \longrightarrow \infty$. Precisely, in addition to (10) and (11), if $\int_{\mathbb{R}^n}\theta_0 dx = 0$ in the case $n = 3$ or $\theta_0 \in L^r$ for some $1 \le r < n/3$ in the case $n > 3$, then for every $n/3 \le q < \infty$, it holds that $\|u(t)\|_q = o(t^{-(1-n/q)/2})$ as $t \longrightarrow \infty$ (cf. [4, 5]).

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The first part of the manuscript, including introduction and preliminaries, was written by Shuokai Yan and Qinghua Zhang. The second part, including the main results and proofs, was written by Lu Wang and Qinghua Zhang. All authors reviewed the manuscript.

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