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Research Article

Infinite Dimensional Widths and Optimal Recovery of a Function Class in Orlicz Spaces in L(R) Metric

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In this paper, we study the infinite dimensional widths and optimal recovery of Wiener–Sobolev smooth function classes $W_{M,1}(P_r(D))$ determined by the r-th differential operator $P_r(D)$ in Orlicz spaces with L(R) metric. Using tools such as the Hölder inequality, we give the exact values of the infinite dimensional Kolmogorov width and linear width of $W_{M,1}(P_r(D))$ in L(R) metric. We also study the related optimal recovery problem.

1. Introduction

In [1], the infinite dimensional widths problem and the optimal recovery problem of the Wiener–Sobolev class determined by differential operators in L_p spaces in $L(R)$ metric are studied (where $L(R)$ metric means the L_1 metric on R). In this paper, we study the infinite dimensional widths problem and the optimal recovery problem of the Wiener–Sobolev class $W_{M,1}(P_r(D))$ determined by differential operators in Orlicz spaces.

It is well known that Orlicz spaces are extensions and refinements of L_p spaces. In particular, the Orlicz spaces generated by N-functions that do not satisfy the Δ_2-condition are a substantial generalization and promotion of the L_p spaces. Therefore, the study of approximation problems in Orlicz spaces has potential application value and development prospect, such as in references [2–4]. In recent years, the research on widths problem in Orlicz spaces has made some progress, such as references [5–8].

In this paper, let $M(u)$ and $N(v)$ be complementary N-functions, the definition and properties of N-function are as follows.

Definition 1. A real valued function $M(u)$ defined on R is called an N-function, if it has the following properties:

(1) $M(u)$ is an even continuous convex function and $M(0) = 0$
(2) $M(u) > 0$ for $u > 0$
(3) $\lim_{u\longrightarrow 0}(M(u)/u) = 0, \lim_{u\longrightarrow\infty}(M(u)/u) = \infty$

The complementary N-function is given by $N(v) = \int_0^v (M')^{-1}(u)\mathrm{d}u$. The properties of N-functions are discussed in [9]. The Orlicz norm is defined by the following expression:

$$\|u\|_M = \sup_{\rho(v;N)\leq 1}\left|\int_I u(x)v(x)\mathrm{d}x\right|. \tag{1}$$

All measurable functions $\{u(x)\}$ with finite Orlicz norms constitute the Orlicz space $L_M^*(I)$ associated with the N-function $M(u)$, where $\rho(v;N) = \int_I N(v(x))\mathrm{d}x$ expresses the modulus of $v(x)$ with respect to $N(v)$. According to the reference [9], the Orlicz norm can also be defined as follows:

$$\|u\|_M = \inf_{\beta>0}\frac{1}{\beta}\left(1 + \int_I M(\beta u(x))\mathrm{d}x\right). \tag{2}$$

In this paper, $\|\cdot\|_{M[a,b]}$ represents the Orlicz norm taken on the corresponding interval $[a,b]$, and $\|\cdot\|_M$ represents the Orlicz norm taken on the interval of the definition domain involved in the conditions of the relevant conclusions. C is used to represent a constant, and in different places, its value can be different.

Set

$$\|f\|_{M,1} := \sum_{j\in Z} \|(\bullet + j)\|_{M[0,1]}, \tag{3}$$

where Z is the set of integers. As in references [10, 11], define a function space

$$L^*_{M,1}(R) := \left\{f\colon f \text{ is measurable on } R \text{ and } \|f\|_{M,1} < +\infty\right\}, \tag{4}$$

by reference [1], $L^*_{M,1}(R)$ is a Banach space.

Given a natural number r, let $C_0^r(R)$ represent the set of smooth functions

$$C_0^r(R) := \left\{f\colon f, \ldots, f^{(r-1)} \text{are absolutely continuous functions in an arbitrary finite interval}\right\}. \tag{5}$$

Let

$$P_r(t) = \prod_{j=1}^{r}\left(t - t_j\right), \quad t_j \in R, j = 1, 2, \ldots, r, \tag{6}$$

be a polynomial with only real roots, and $P_r(D)(D = d/dt)$ is the induced differential operator of $P_r(t)$. Define the Wiener–Sobolev space and the Wiener–Sobolev class in the Orlicz spaces as follows:

$$\begin{aligned} L^*_{M,1}(P_r(D)) &:= \left\{f \in C_0^r(R)\colon P_r(D)f \in L^*_{M,1}(R)\right\}, \\ W_{M,1}(P_r(D)) &:= \left\{f \in L^*_{M,1}(P_r(D))\colon \left\|P_r(D)f\right\|_{M,1} \le 1\right\}. \end{aligned} \tag{7}$$

2. Preliminaries

For arbitrary $\lambda > 0$, let

$$\Phi_{r,\lambda}(x) := \frac{2}{\pi i}\sum_{\nu=-\infty}^{+\infty} \frac{e^{i(2\nu+1)\lambda\pi x}}{(2\nu+1)P_r((2\nu+1)\lambda\pi i)}, \tag{8}$$

be a standard function with period $(2/\lambda)$ defined by the differential operator $P_r(D)$. Specially, if $P_r(D) = D^r$, then the function $\Phi_{r,\lambda}(x)$ has the following form:

$$\begin{aligned} \varphi_{r,\lambda}(x) := \Phi_{r,\lambda}(x) &= \frac{4}{\pi(\lambda\pi)^r} \\ &\cdot \sum_{\nu=0}^{+\infty} \frac{\cos((2\nu+1)\lambda\pi x - (r+1)(\pi/2))}{(2\nu+1)^{r+1}}. \end{aligned} \tag{9}$$

Set

$$\|f\|_{\infty,\infty} = \sup_{j\in Z}\|f(\bullet + j)\|_{\infty[0,1]}, \tag{10}$$

where $\|f(\bullet + j)\|_{\infty[0,1]}$ represents $\|f(\bullet + j)\|_{L_\infty[0,1]}$. Then, according to reference [1], we know

$$L_{\infty,\infty}(R) = \left\{f\colon f \text{ is measurable on } R \text{ and } \|f\|_{\infty,\infty} < +\infty\right\}, \tag{11}$$

is a Banach space with metric $\|\cdot\|_{\infty,\infty}$. Let

$$\begin{aligned} L_\infty(P_r(D)) &= \left\{f \in C_0^r(R)\colon P_r(D)f \in L_{\infty,\infty}(R)\right\}, \\ W_\infty(P_r(D)) &= \left\{f \in L_\infty(P_r(D))\colon \left\|P_r(D)f\right\|_{\infty,\infty} \le 1\right\}, \end{aligned} \tag{12}$$

be the corresponding Wiener–Sobolev space and Wiener–Sobolev class.

Lemma 2 (see [1]). *Let $g \in W_\infty(P_r(D))$ satisfy*

$$\|g\|_\infty \le \left\|\Phi_{r,\lambda}\right\|_\infty, g(\xi_0) = \Phi_{r,\lambda}(\eta_0), \tag{13}$$

and do not have any other restrictions. If $[\alpha, \beta]$ is the interval that contains the point η_0 and $\Phi_{r,\lambda}(x)$ is monotonous on $[\alpha, \beta]$, then the following statements are true:

(1) If $\Phi_{r,\lambda}(x)$ increases monotonically on $[\alpha, \beta]$, the following inequalities are true:

$$\begin{aligned} g(\xi_0 + u) &\le \Phi_{r,\lambda}(\eta_0 + u), 0 \le u \le \beta - \eta_0, \\ g(\xi_0 - u) &\ge \Phi_{r,\lambda}(\eta_0 - u), 0 \le u \le \eta_0 - \alpha, \end{aligned} \tag{14}$$

(2) If $\Phi_{r,\lambda}(x)$ decreases monotonically on $[\alpha, \beta]$, the following inequalities are true:

$$\begin{aligned} g(\xi_0 + u) &\ge \Phi_{r,\lambda}(\eta_0 + u), 0 \le u \le \beta - \eta_0, \\ g(\xi_0 - u) &\le \Phi_{r,\lambda}(\eta_0 - u), 0 \le u \le \eta_0 - \alpha. \end{aligned} \tag{15}$$

Lemma 3. *If $g \in W_\infty(P_r(D))$ satisfies*

$$\|g\|_\infty \le \left\|\Phi_{r,\lambda}\right\|_\infty, \sup_{x,y}\left|\int_x^y g(t)dt\right| \le 2\left\|\Phi_{r+1,\lambda}\right\|_\infty, \tag{16}$$

and $[a, b]$ is an interval such that g has only two zeros a and b on $[a, b]$, then

$$\|g\|_{M[a,b]} \le \left\|\Phi_{r,\lambda}\right\|_{M[0,(1/\lambda)]}, \tag{17}$$

where $\Phi_{r+1,\lambda}(x)$ represents the standard function defined by $DP_r(D)$.

Proof. From Lemma 2, similar to the proof of Theorem 5.7.1 in reference [12], Lemma 3 is easy to be proved. □

According to Lemma 3, we have the following expression.

Lemma 4. *Let $\Phi_{r+1,\lambda}(x)$ represent the standard function defined by $DP_r(D)$, and $\Phi_{r,\lambda}(x)$ be defined by (8), if $G \in W_\infty(DP_r(D))$ satisfies*

$$\|G\|_\infty \le 2\|\Phi_{r+1,\lambda}\|_\infty. \tag{18}$$

Then, for arbitrary $c \in R$,

$$\|g(\bullet + c)\|_{M[0,(1/\lambda)]} \le \|\Phi_{r,\lambda}\|_{M[0,(1/\lambda)]}, \tag{19}$$

holds, where $G' = g$.

Proof. Without losing generality, suppose $c = 0$, and we just consider this case.

(1) If g has not any zero on $(0, (1/\lambda))$, by the proof of Theorem 1 in reference [1], we define a non-negative function $\chi_r(t)$:

$$\chi_r(t) = \begin{cases} 1, & -1 \le t \le 1, \\ (-1)^r (t-2)^r \sum_{j=0}^{r-1} C_{r+j-1}^j (t-1)^j, & 1 \le t \le 2, \\ (t+2)^r \sum_{j=0}^{r-1} C_{r+j-1}^j (t+1)^j, & -2 \le t \le -1, \\ 0, & |t| \ge 2. \end{cases} \tag{20}$$

For given $\alpha \in (0,1)$ and every natural number N, let

$$F_N(t) = \alpha G(t)\chi_r\left(\frac{1}{N}\right), \tag{21}$$

by reference [1], $F_N(t)$ satisfies the following properties:

(i) If N is sufficiently large, then $F_N(t) \in W_\infty(D(P_r(D)))$ holds.
(ii) $\|F_N\|_\infty \le \|\Phi_{r+1,\lambda}\|_\infty$, $\sup_{x,y}|\int_x^y F_N'(t)dt| \le 2\|\Phi_{r+1,\lambda}\|_\infty$.

For sufficiently large N, we have $[0, (1/\lambda)] \subset [-N, N]$, and for arbitrary $t \in [0, (1/\lambda)], F_N'(t) = \alpha g(t)$, now we need to prove

$$\left\|F_N'\right\|_{M[0,(1/\lambda)]} \le \|\Phi_{r,\lambda}\|_{M[0,(1/\lambda)]}. \tag{22}$$

Obviously, if g has not zeros in $(0, (1/\lambda))$, then there exist two points a, b in $[-2N, 2N]$ such that $F_N'(t)$ has only two zeros on $[a,b]$. Therefore, by Lemma 3, we can obtain

$$\begin{aligned} \alpha\|g\|_{M[0,(1/\lambda)]} = \left\|F_N'\right\|_{M[0,(1/\lambda)]} \le \left\|F_N'\right\|_{M[a,b]} \\ \le \|\Phi_{r,\lambda}\|_{M[0,(1/\lambda)]}, \end{aligned} \tag{23}$$

let $\alpha \longrightarrow 1$ for both sides of (23), the lemma is proved.

(2) If there exists zeros $a, b (a \le b)$ of $g(t)$ in interval $(0, (1/\lambda))$ such that for every $t \in (0,a) \cup (b, (1/\lambda))$, $g(t) \ne 0$, then

$$\|g\|_{M[0,a]} \le \|\Phi_{r,\lambda}(x_0 - \bullet)\|_{M[0,a]}, \tag{24}$$

$$\|g\|_{M[b,(1/\lambda)]} \le \|\Phi_{r,\lambda}(x_0 + \cdot)\|_{M[0,(1/\lambda)-b]}, \tag{25}$$

where x_0 is the zero of $\Phi_{r,\lambda}(t)$. If one of (24) and (25) is not valid, assume that (24) is not true, then

$$\|g\|_{M[0,a]} > \|\Phi_{r,\lambda}(x_0 - \bullet)\|_{M[0,a]}. \tag{26}$$

Then by (26), Lemma 2 and $g(a) = \Phi_{r,\lambda}(x_0) = 0$, there exists a point $x_1 \in (0,a)$ such that

$$\begin{aligned} |g(a - x_1)| = |\Phi_{r,\lambda}(x_0 - x_1)|, |g(a-u)| \\ \ge |\Phi_{r,\lambda}(x_0 - u)|, x_1 \le u \le \frac{1}{\lambda}. \end{aligned} \tag{27}$$

On one hand, by (26) and (27), we have the following expression:

$$\|g(a - \cdot)\|_{M[0,(1/\lambda)]} > \|\Phi_{r,\lambda}(x_0 - \bullet)\|_{M[0,(1/\lambda)]}. \tag{28}$$

On the other hand, according to Lemma 2 and (27), we have $g(t) \ne 0$ for any $t \in ((a - 1/\lambda), a)$. Moreover, according to case (1), we have the following expression:

$$\|g(a - \cdot)\|_{M[0,(1/\lambda)]} \le \|\Phi_{r,\lambda}(x_0 - \bullet)\|_{M[0,(1/\lambda)]}. \tag{29}$$

This contradicts inequation (28). Therefore, inequations (24) and (25) hold.

From Lemma 2, we have the following expression:

$$|g(t)| \le |\Phi_{r,\lambda}(x_0 + t)|, t \in (a,b). \tag{30}$$

Hence, according to inequations (24) and (25) and the inequation above, the lemma can be proved. □

3. Infinite Dimensional Widths Problem

Let $T = \{t_j\}_{j\in Z}$ be a real sequence and satisfy

$$\begin{aligned} t_j \le t_{j+1}, \forall j \in Z, \\ \lim_{j \longrightarrow -\infty} t_j = -\infty, \lim_{j \longrightarrow +\infty} t_j = +\infty. \end{aligned} \tag{31}$$

For each natural number $m \geq r$, let $P_m(t)$ be an algebraic polynomial of degree m with only real roots, and $P_r(t)$ be its factor. Define

$$
\begin{aligned}
S_T(P_m(D)) &:= \left\{s(t) \in C^{m-2}(R)\colon P_m(D)s(t) = 0, \forall t \in \left(t_j, t_{j+1}\right), \forall j \in Z\right\}, (m \geq 2),\\
S_T(P_1(D)) &:= \left\{s(t)\colon P_1(D)s(t) = 0, \forall t \in \left(t_j, t_{j+1}\right), \forall j \in Z\right\}.
\end{aligned}
\tag{32}
$$

If $T = \{j/n\}_{j\in Z}$, we replace $S_T(P_m(D))$ with $S_n(P_m(D))$.

For $f \in L_M^*(R)$, let

$$
E(f, S_T(P_m(D)))_M = \inf\{\|f - g\|_M\colon g \in S_T(P_m(D))\}.
\tag{33}
$$

Define

$$
\begin{aligned}
L_N^r(R) &= \left\{f\colon f \in L_N^*(R) \cap C(R), f^{(r-1)} \text{ is locally absolutely continuous}, f^{(r)} \in L_N^*(R)\right\},\\
W_N^r(R) &= \left\{f\colon f \in L_N^r(R), \left\|f^{(r)}\right\|_N \leq 1\right\},
\end{aligned}
\tag{34}
$$

where $N(\cdot)$ is the complementary N-function of $M(\cdot)$.

Lemma 5. *(1) Let $g \in L_M^*(R)$, then*

$$
\inf_\alpha \left\|g - \sum \alpha_j N_{j,r}\right\|_M = \sup\left\{\int_R g(x) f^{(r)}(x)\mathrm{d}x\colon f \in W_N^r(R), f\left(t_j\right) = 0\,(j \in Z)\right\},
\tag{35}
$$

where $\alpha = \left\{\alpha_j\right\}_{j\in Z}$ is a real sequence, $N_{j,r}(x)$ is the standardized B-spline

$$
N_{j,r}(x) = \frac{\left(t_{j+r} - t_j\right) M_{j,r}(x)}{r},
\tag{36}
$$

while

$$
M_{j,r}(x) = r\left[t_j, \ldots, t_{j+r}\right](\cdot + x)_+^{r-1},
\tag{37}
$$

and $\sum \alpha_j N_{j,r} \in S_{m,T} \cap L_M^*(R)$,

$$
S_{m,T} = \left\{s(t)\colon s(t) \in C^{(m-2)}(R), D^m s(t)\Big|_{\left(t_j, t_{j+1}\right)} = 0, j = 0, \pm 1, \pm 2, \cdots\right\},
\tag{38}
$$

where $M_{j,r}(x)$ is B-spline, its detailed definition refer to reference [13], there is no need to go into details here.

(2) (see [1])

$$
\sup\left\{\left\|P_{r,m}(-D)g\right\|_\infty\colon g \in W_\infty(P_m(-D)), g\left(\frac{j}{n}\right) = 0, \forall j \in Z\right\} = \left\|\Phi_{r,n}\right\|_\infty,
\tag{39}
$$

where $P_{r,m}(t) = (P_m(t)/P_r(t))$ is an algebraic polynomial of degree $m - r$.

Proof

(1) According to reference [14], we know

$$
\begin{aligned}
&\sup\left\{\int_R g(x) f^{(r)}(x)\mathrm{d}x\colon f \in W_N^r(R), f\left(t_j\right) = 0\,(j \in Z)\right\}\\
&= \sup\left\{\int_R g(x) f^{(r)}(x)\mathrm{d}x\colon f \in W_N^r(R), \int_R N_{j,r}(x) f^{(r)}(x)\mathrm{d}x = 0\,(j \in Z)\right\}.
\end{aligned}
\tag{40}
$$

Therefore, to prove (35), we just need to prove

$$\inf_{\alpha}\left\|g-\sum \alpha_j N_{j,r}\right\|_M=\sup\left\{\int_R g(x)f^{(r)}(x)\mathrm{d}x\colon\ f\in W_N^r(R),\int_R N_{j,r}(x)f^{(r)}(x)\mathrm{d}x=0\,(j\in Z)\right\}. \tag{41}$$

Since $\int_R N_{j,r}(x)f^{(r)}(x)\mathrm{d}x=0$ and $\|f^{(r)}\|_N\le 1$, using the Hölder inequality in the Orlicz spaces, we have the following expression:

$$\begin{aligned}\int_R g(x)f^{(r)}(x)\mathrm{d}x&=\int_R\left(g(x)-\sum \alpha_j N_{j,r}(x)\right)f^{(r)}(x)\mathrm{d}x\\&\le\left\|g-\sum \alpha_j N_{j,r}\right\|_M\left\|f^{(r)}\right\|_N\\&\le\left\|g-\sum \alpha_j N_{j,r}\right\|_M,\end{aligned}\tag{42}$$

where the Hölder inequality in the Orlicz spaces is

$$\int_I u(x)v(x)\mathrm{d}x\le\|u\|_M\|v\|_N. \tag{43}$$

See the reference [9].

Therefore,

$$\sup\int_R g(x)f^{(r)}(x)\mathrm{d}x\le\inf_{\alpha}\left\|g-\sum \alpha_j N_{j,r}\right\|_M. \tag{44}$$

Define $I_N=[t_{-N},t_N]$,

$$E_N=\left\{h\colon\ \mathrm{supp}h\subset I_N,\|h\|_N\le 1,\int_R N_{j,r}(x)h(x)\mathrm{d}x=0\,(j\in Z)\right\},\ (N=1,2,\cdots). \tag{45}$$

Also, for $h\in E_N$, define

$$f_h(x)=\frac{1}{(r-1)!}\int_{I_N}(x-t)_+^{r-1}h(t)\mathrm{d}t. \tag{46}$$

According to reference [14], we have $f_h\in W_N^r(R)$, $\mathrm{supp}f_h\subset I_N$, moreover

$$\begin{aligned}&\sup\left\{\int_R g(x)f^{(r)}(x)\mathrm{d}x\colon\ f\in W_N^r(R),\int_R N_{j,r}(x)f^{(r)}(x)\mathrm{d}x=0\,(j\in Z)\right\}\\&\ge\sup\left\{\int_{I_N}g(x)h(x)\mathrm{d}x\colon\ \|h\|_{N(I_N)}\le 1,\int_{I_N}N_{j,r}(x)h(x)\mathrm{d}x=0\,(j\in Z)\right\}.\end{aligned}\tag{47}$$

We notice that $N_{j,r}(x)$ has compact support $[t_j,t_{j+r}]$ and the orthogonal condition $\int_{I_N}N_{j,r}(x)h(x)\mathrm{d}x=0$ satisfies for $j\le -N-r$ and $j\ge N$. So, the orthogonal condition is necessary only if $-N-r<j<N$.

According to the duality theorem of the best approximation of a function in a finite dimensional subspace, we have

$$\begin{aligned}&\sup\left\{\int_{I_N}g(x)h(x)\mathrm{d}x\colon\ \|h\|_{N(I_N)}\le 1,\int_{I_N}N_{j,r}(x)h(x)\mathrm{d}x=0\,(j\in Z)\right\}\\&=\inf\left\{\left\|g-\sum \alpha_j N_{j,r}\right\|_{M(I_N)}\colon\ \alpha_j\in R,-N-r<j<N\right\}\\&=\left\|g-\sum \alpha_{j,N}^{*}N_{j,r}\right\|_{M(I_N)},\end{aligned}\tag{48}$$

where $\alpha_{j,N}^{*}=0$ if $j\le -N-r$ and $j\ge N$. The right-hand side of (47) defines a monotonically increasing bounded sequence, and we need to prove

$$\lim_{N\longrightarrow\infty}\left\|g-\sum \alpha_{j,N}^{*}N_{j,r}\right\|_{M(I_N)}\ge\inf_{\alpha}\left\|g-\sum \alpha_j N_{j,r}\right\|_M\coloneqq d>0. \tag{49}$$

Assume (49) is not true, then there exists $\varepsilon > 0$, such that when N is sufficiently large, we have

$$\left\| g - \sum \alpha^*_{j,N} N_{j,r} \right\|_{M(I_N)} < d - \varepsilon. \quad (50)$$

According to [14], there exists a constant C such that for every $j \in Z$, such that

$$\begin{aligned} \left|\alpha^*_{j,N}\right|\left(t_{j+r} - t_j\right) &\le C \int_{t_j}^{t_{j+r}} \sum \alpha^*_{k,N} N_{k,r}(x)\mathrm{d}x \\ &\le C \left\| \sum \alpha^*_{k,N} N_{k,r} \right\|_{M[t_j,t_{j+r}]} \|1\|_{N[t_j,t_{j+r}]}. \end{aligned} \quad (51)$$

Let $N \ge \max\{-j, j+r\}$, then $[t_j, t_{j+r}] \subset [t_{-N}, t_N] = I_N$, and

$$\begin{aligned} \left|\alpha^*_{j,N}\right| &\le C\left(t_{j+r} - t_j\right)^{(-1)} \left\| \sum \alpha^*_{k,N} N_{k,r} \right\|_{M[t_{-N},t_N]} \|1\|_{N[t_{-N},t_N]} \\ &\le 2C\left(t_{j+r} - t_j\right)^{(-1)} \|g\|_{M[t_{-N},t_N]} \|1\|_{N[t_{-N},t_N]} \\ &\le 2C\left(t_{j+r} - t_j\right)^{(-1)} \|g\|_{M[t_{-N},t_N]} < \infty. \end{aligned} \quad (52)$$

Thus, using the diagonal rule we can find a sequence of positive integers $\{N_n\}_{n=1}^{+\infty}$ that satisfies for every $j \in Z$, we have the following expression:

$$\lim_{n \longrightarrow \infty} \alpha^*_{j,N_n} = \beta_j \in R. \quad (53)$$

Set

$$f_n(x) = \begin{cases} \left|g(x) - \sum \alpha^*_{j,N_n} N_{j,r}(x)\right| v(x), & x \in I_{N_n}, \\ 0, & x \in \left(R \backslash I_{N_n}\right). \end{cases} \quad (54)$$

For $\forall x \in R$, we have $x \in I_{N_n}$ if n is sufficiently large. Since the support of $N_{j,r}(x)$ is compact, $\sum \alpha^*_{j,N_n} N_{j,r}(x)$ contains a finite number of terms which is not zero. According to (53) and (54), we have

$$\lim_{n \longrightarrow \infty} f_n(x) = \left|g(x) - \sum \beta_j N_{j,r}(x)\right| v(x) \, (x \in R), \quad (55)$$

where $v(x)$ satisfies $\rho(v; N) = \int_{I_{N_n}} N(v(x))\mathrm{d}x \le 1$. By Fatou Lemma, we have the following expression:

$$\begin{aligned} \left\| g - \sum \beta_j N_{j,r} \right\|_M &= \sup_{\rho(v;N)\le 1} \left| \int_R \left|g(x) - \sum \beta_j N_{j,r}(x)\right| v(x)\mathrm{d}x \right| \\ &\le \sup_{\rho(v;N)\le 1} \lim_{n \longrightarrow \infty} \inf \left| \int_R f_n(x)\mathrm{d}x \right| \\ &= \sup_{\rho(v;N)\le 1} \lim_{n \longrightarrow \infty} \left| \int_{I_{N_n}} \left|g(x) - \sum \alpha^*_{j,N_n} N_{j,r}(x)\right| v(x)\mathrm{d}x \right| \\ &= \lim_{n \longrightarrow \infty} \left\| g - \sum \alpha^*_{j,N_n} N_{j,r} \right\|_{M\left(I_{N_n}\right)} \le d - \varepsilon. \end{aligned} \quad (56)$$

It indicates $\alpha^*_{j,N_n} \in S_{m,T} \cap L^*_M(R)$. However, in this case,

$$\left\| g - \sum \beta_j N_{j,r} \right\|_{M(I_N)} \ge d. \quad (57)$$

So, we draw a contradiction. Therefore, inequation (49) holds. When the limit of (47) on the right side is taken, we can get the opposite inequation of (44). (1) is proved. □

As seen in the proof of Lemma 5(1), for $f \in L^*_M(R)$, we have the following expression:

$$E\left(f, S_T\left(P_m(D)\right)\right)_M = \sup\left\{ \left| \int_R f(t) P_m(-D) g(t)\mathrm{d}t \right| : g \in W^r_N\left(P_m(-D)\right), g\left(t_j\right) = 0, \forall j \in Z \right\}, \quad (58)$$

where $W^r_N(P_m(-D)) = \{f\colon f \in L^r_N(R), \|P_m(-D)f\|_N \le 1\}$.

Define

$$\|f\|_{N,\infty} = \sup_{j \in Z} \|f(\cdot + j)\|_{N[0,1]}. \quad (59)$$

Also, a function space

$$L^*_{N,\infty}(R) = \left\{ f\colon f \text{ is measurable on } R \text{ and } \|f\|_{N,\infty} < \infty. \right\}, \quad (60)$$

according to reference [1], we know $L^*_{N,\infty}(R)$ is also a Banach space.

Lemma 6 (see [10, 15]). *Let $f \in L^*_{M,1}(R)$, $g \in L^*_{N,\infty}(R)$, then $f(t)g(t) \in L(R)$, and*

$$\|fg\|_1 \le \|f\|_{M,1} \|g\|_{N,\infty}. \quad (61)$$

For $g \in L^*_{M,1}(R)$, quantity

$$E(g, S_n(P_m(D)))_1 = \inf\{\|g - f\|_1 : f \in S_n(P_m(D))\}, \quad (62)$$

is called the best approximation of g by $S_n(P_m(D))$, and

$$E\big(W_{M,1}(P_r(D)), S_n(P_m(D))\big)_1 \coloneqq \sup\big\{E(g, S_n(P_m(D)))_1 : g \in W_{M,1}(P_r(D))\big\}, \quad (63)$$

is called the best approximation of $W_{M,1}(P_r(D))$ by $S_n(P_m(D))$ in $L(R)$ metric.

Let $n \geq 0$ be a fixed number (not necessarily an integer), $\mathfrak{F}_n$ represent the set of all linear subspaces F on $L(R)$, such that for every $F \in \mathfrak{F}_n$, we have the following expression:

$$\lim_{a \longrightarrow +\infty} \frac{\dim\big(F\,|_{[-a,a]}\big)}{2a} \leq n, \quad (64)$$

where $F|_{[-a,a]}$ indicates the limit of F on $[-a, a]$ and $\dim(F\,|_{[-a,a]})$ is the dimension of the linear space $F|_{[-a,a]}$.

Quantity

$$d_n\big(W_{M,1}(P_r(D)), L(R)\big) \coloneqq \inf_{F \in \mathfrak{F}_n} \sup_{f \in W_{M,1}(P_r(D))} \inf_{g \in F} \|f - g\|_1, \quad (65)$$

represents the infinite dimensional $n-K$ width of $W_{M,1}(P_r(D))$ in $L(R)$ metric. If there is a subspace $F^* \in \mathfrak{F}_n$ that satisfies

$$d_n\big(W_{M,1}(P_r(D)), L(R)\big) = \sup_{f \in W_{M,1}(P_r(D))} \inf_{g \in F^*} \|f - g\|_1. \quad (66)$$

Then, F^* is the optimal subspace that reaches d_n.

Theorem 7. *Let $P_r(t), P_m(t)$ be defined as above. If n is a natural number, then*

$$E\big(W_{M,1}(P_r(D)), S_n(P_m(D))\big)_1 \leq \|\Phi_{r,n}\|_{N([0,1])}, \quad (67)$$

where $m \geq r$, and $P_m(0) = 0, \lim_{t \longrightarrow 0}(P_m(t)/P_r(t)) = 0$ for $m > r$.

Proof. From (58) and integrating by parts, we can obtain the following expression:

$$\begin{aligned} &E\big(W_{M,1}(P_r(D)), S_n(P_m(D))\big)_1 \\ &= \sup\left\{\int_R P_r(D) f(t) P_{r,m}(-D) g(t) \mathrm{d}t : f \in W_{M,1}(P_r(D)), g \in W_{N,\infty}(P_m(-D)), g\left(\frac{j}{n}\right) = 0, \forall j \in Z\right\}. \end{aligned} \quad (68)$$

From Lemma 6, we have the following expression:

$$\begin{aligned} \|P_r(D) f \cdot P_{r,m}(-D) g\|_1 &\leq \|P_r(D) f\|_{M,1} \|P_{r,m}(-D) g\|_{N,\infty} \\ &\leq \|P_{r,m}(-D) g\|_{N,\infty}. \end{aligned} \quad (69)$$

Therefore, from Lemma 2, Lemma 4, and Lemma 6, we have the following expression:

$$\begin{aligned} &\|P_{r,m}(-D) g(\cdot + i)\|_{N[0,1]} \\ &= \sup_{\rho(u;M) \leq 1} \left|\int_0^1 P_{r,m}(-D) g(x + i) u(x) \mathrm{d}x\right| \\ &\leq \sup_{\rho(u;M) \leq 1} \sum_{j=1}^{n} \left|\int_{(j-1/n)}^{(j/n)} P_{r,m}(-D) g(x + i) u(x) \mathrm{d}x\right| \\ &\leq \sum_{j=1}^{n} \sup_{\rho(u;M) \leq 1} \left|\int_{(j-1/n)}^{(j/n)} P_{r,m}(-D) g(x + i) u(x) \mathrm{d}x\right| \\ &= \sum_{j=1}^{n} \|P_{r,m}(-D) g(\cdot + i)\|_{N[(j-1/n),(j/n)]} \\ &\leq \sum_{j=1}^{n} \big\|\|\Phi_{r,n}\|_{N[(j-1/n),(j/n)]} = \|\Phi_{r,n}\|_{N[0,1]}, \end{aligned} \quad (70)$$

where $u(x)$ satisfies $\int_{(j-1)/n}^{(j/n)} M(u(x))dx \le 1$ for any $j = 1, \ldots, n$, and $\int_0^1 M(u(x))dx \le 1$.

Note that

$$\|P_{r,m}(-D)g\|_{N,\infty} = \sup_{i\in Z}\|P_{r,m}(-D)g(\cdot + i)\|_{N[0,1]}. \quad (71)$$

So, the theorem is proved. □

Theorem 8. *Let n be a natural number, $m \ge r$, then*

$$d_n\big(W_{M,1}(P_r(D)), L(R)\big) = E\big(W_{M,1}(P_r(D)), S_n(P_m(D))\big)_1 = \|\Phi_{r,n}\|_{N[0,1]}, \quad (72)$$

where $P_m(0) = 0$, $\lim_{t\longrightarrow 0}(P_m(t)/P_r(t)) = 0$ for $m > r$.

Proof. First of all, we have

$$d_n\big(W_{M,1}(P_r(D)), L(R)\big) \le E\big(W_{M,1}(P_r(D)), S_n(P_m(D))\big)_1. \quad (73)$$

To prove the opposite inequality, let $d_n(A, X)$ represent the $n-K$ width of A in the usual sense of X (X is the space of functions defined on a finite interval). For every finite interval $I = [a, b]$, let

$$\widetilde{W}_M(P_r(D), I) := \left\{f \in C_0^r(R)\colon f \text{ has period b} - \text{a and } \|P_r(D)f\|_{M(I)} \le 1\right\}, \quad (74)$$

$$\widetilde{W}_M(P_r(D), I)_0 := \left\{f \in \widetilde{W}_M(P_r(D), I)\colon f^{(j)}(a) = 0, j = 0, 1, \ldots, r-1\right\}, \quad (75)$$

$$E(A, B)_X := \sup_{x\in A}\inf_{y\in B}\|x - y\|, A \subset X, B \subset X, \quad (76)$$

where $\|\cdot\|$ represents the norm on X.

For every $F \in \mathfrak{F}_n$ and $N \ge 1$, let $N_n := \dim(F|_{[-N,N]})$, and $I_N = [-N, N]$. For every $f \in \widetilde{W}_M(P_r(D), I)_0$, it's easy to prove

$$(2(N+1))^{-1} f\chi_{I_N} \in W_{M,1}(P_r(D)), \quad (77)$$

where χ_I represents the characteristic function on the interval I.

Therefore, we have the following expression:

$$\begin{aligned} E\big(W_{M,1}(P_r(D)), F\big)_1 &\ge (2(N+1))^{-1}E\big(\widetilde{W}_M(P_r(D), I_N)_0, F|_{[-N,N]}\big)_{L(I_N)} \\ &\ge (2(N+1))^{-1}d_{N_n}(\widetilde{W}_M(P_r(D), I_N)_0, L(I_N)) \\ &\ge (2(N+1))^{-1}d_{N_n+2r}(\widetilde{W}_M(P_r(D), I_N), L(I_N)). \end{aligned} \quad (78)$$

For $f \in \widetilde{W}_M(P_r(D), I_N)$, let $F(t) = f(Nt/\pi), t \in I_\pi$, then we have the following expression:

$$\begin{aligned} d_{N_n+2r}(\widetilde{W}_M(P_r(D), I_N), L(I_N)) &= \frac{N}{\pi}d_{N_n+2r}\left(\widetilde{W}_M\left(P_r\left(\frac{\pi D}{N}\right), I_\pi\right), L(I_\pi)\right) \\ &\ge \frac{N}{\pi}(2\pi)\lambda_N\|\Phi_{r,\lambda_N}\|_{N[0,(1/\lambda_N)]}, \end{aligned} \quad (79)$$

where

$$\lambda_N = N^{-1}\left(\left[\frac{N_n}{2}\right] + r\right). \quad (80)$$

By (78) and (79), we have the following expression:

$$E\big(W_{M,1}(P_r(D)), F\big)_1 \ge \frac{N}{N+1}\|\Phi_{r,\lambda_N}\|_{N[0,(1/\lambda_N)]}\lambda_N. \quad (81)$$

Let $N \longrightarrow +\infty$ on both sides of above-given inequation, then

$$E\big(W_{M,1}(P_r(D)), F\big)_1 \ge \|\Phi_{r,\lambda_N}\|_{N[0,1]}. \quad (82)$$

Hence, we obtain

$$d_n\big(W_{M,1}(P_r(D)), L(R)\big) \ge \|\Phi_{r,\lambda_N}\|_{N[0,1]}. \quad (83)$$

The theorem is proved. □

Define

$$\delta_n\big(W_{M,1}(P_r(D)), L(R)\big) := \inf_{(A,\mathfrak{F}_n)} \sup_{f\in W_{M,1}(P_r(D))}\|f - A(f)\|_1, \tag{84}$$

as the infinite dimensional linear n-width of $W_{M,1}(P_r(D))$ in $L(R)$ metric, where A runs over the set of all linear operators such that $A(\mathfrak{D}) \subset F$ for some $F \in \mathfrak{F}_n$, where $\mathfrak{D}$ denotes the linear closure of $W_{M,1}(P_r(D))$ in $L(R)$. If there is a linear operator A^*: $\mathfrak{D} \longrightarrow A^*(\mathfrak{D}) \in \mathfrak{F}_n$ such that

$$\delta_n\big(W_{M,1}(P_r(D)), L(R)\big) = \sup_{f\in W_{M,1}(P_r(D))}\big\|f - A^*(f)\big\|_1. \tag{85}$$

Then, A^* is called the optimal linear operator.

Lemma 9 (See [1]). *If $f \in L^*_{M,1}(P_r(D))$, then there exists an unique $s_r(f) \in S_n(P_r(D))$ that satisfies*

$$s_r\Big(f, \alpha_n + \frac{j}{n}\Big) = f\Big(\alpha_n + \frac{j}{n}\Big), \quad \forall j \in Z, \tag{86}$$

$$f(x) - s_r(f,x) = \int_R G(x,t)P_r(D)f(t)\mathrm{d}t, \tag{87}$$

where $G(x,t)$ satisfies

$$\begin{aligned}\|G(x,\cdot)\|_1 &= \big|\Phi_{r,n}(x)\big|,\\ \|G(\cdot,t)\|_1 &= \big|\Phi_{r,n}(t+1-\alpha_n)\big|.\end{aligned} \tag{88}$$

Theorem 10. *Let n be a natural number, then*

$$\begin{aligned}\delta_n\big(W_{M,1}(P_r(D)), L(R)\big) &= \sup\Big\{\big\|f - s_r(f)\big\|_1 : f \in W_{M,1}(P_r(D))\Big\}\\ &= \big\|\Phi_{r,n}\big\|_{N[0,1]},\end{aligned} \tag{89}$$

where s_r: $L^*_{M,1}(P_r(D)) \longrightarrow S_n(P_r(D))$ is the interpolation operator satisfying

$$s_r\Big(f, \alpha_n + \frac{j}{n}\Big) = f\Big(\alpha_n + \frac{j}{n}\Big), \forall j \in Z, \tag{90}$$

and α_n is a fixed constant.

Proof. According to Lemma 9, we have the following expression:

$$\delta_n\big(W_{M,1}(P_r(D)), L(R)\big) = \sup\Big\{\big\|f - s_r(f)\big\|_1 : f \in W_{M,1}(P_r(D))\Big\}. \tag{91}$$

So, we just need to prove

$$\sup\Big\{\big\|f - s_r(f)\big\|_1 : f \in W_{M,1}(P_r(D))\Big\} = \big\|\Phi_{r,n}\big\|_{N[0,1]}. \tag{92}$$

By (66) and (84), we have the following expression:

$$d_n\big(W_{M,1}(P_r(D)), L(R)\big) \le \delta_n\big(W_{M,1}(P_r(D)), L(R)\big). \tag{93}$$

Now we prove

$$\sup\Big\{\big\|f - s_r(f)\big\|_1 : f \in W_{M,1}(P_r(D))\Big\} \le \big\|\Phi_{r,n}\big\|_{N[0,1]}. \tag{94}$$

From Lemma 9, Lemma 6, and equation (88), we have the following expression:

$$\begin{aligned}\big\|f - s_r(f)\big\|_1 &\le \int_R\Big(\int_R |G(x,t)\mathrm{d}x\Big)\big|P_r(D)f(t)\big|\mathrm{d}t\\ &\le \big\|P_r(D)f\big\|_{M,1}\big\|\Phi_{r,n}(\bullet + 1 - \alpha_n)\big\|_{N,\infty}\\ &\le \big\|\Phi_{r,n}\big\|_{N[0,1]}.\end{aligned} \tag{95}$$

Therefore, the theorem can be proved by the above-given expressions and Lemma 4. □

4. Optimal Recovery Problem

Let Θ_n be the set of all sequence $\xi = \{\xi_j\}_{j\in Z}$ satisfying

$$\begin{gathered}\xi_j < \xi_{j+1}, \forall j \in Z,\\ \lim_{a\longrightarrow +\infty} \inf \frac{\operatorname{card}(\xi \cap [-a,a])}{2a} \le n,\end{gathered} \tag{96}$$

where $\operatorname{card}(\xi \cap [-a,a])$ is the number of elements in $\xi \cap [-a,a]$. For every $\xi \in \Theta_n$, let I_ξ represent an information operator defined by ξ, while $I_\xi(f) = \{f(\xi_j)\}_{j\in Z}$ is called the information of f defined by ξ.

$$D_n\big(W_{M,1}(P_r(D)), S, L\big) = 2\inf_{\xi\in\Theta_n} \sup_{f\in W_{M,1}(P_r(D)), I_\xi(f)=0} \|Sf\|_1, \tag{97}$$

is called the minimal information diameter of $W_{M,1}(P_r(D))$ with respect to the solution operator S: $\mathfrak{D} \longrightarrow L(R)$ in $L(R)$ metric. If there exists $\xi^* \in \Theta_n$ such that

$$D_n\big(W_{M,1}(P_r(D)), S, L\big) = 2\sup_{f\in W_{M,1}(P_r(D)), I_{\xi^*}(f)=0} \|Sf\|_1, \tag{98}$$

then ξ^* is the optimal sampling. For every $\xi \in \Theta_n$, let $I_\xi(W_{M,1}(P_r(D)))$ represent the image set of I_ξ on $W_{M,1}(P_r(D))$, and

$$A\colon I_\xi\big(W_{M,1}(P_r(D))\big) \longrightarrow L(R), \tag{99}$$

is the mapping from $I_\xi(W_{M,1}(P_r(D)))$ to $L(R)$. Sometimes A is called an algorithm.

Now, we discuss the following optimal recovery problem:

$$E_n\big(W_{M,1}(P_r(D)), S, L\big) = \inf_{\xi\in\Theta_n} \inf_A \sup_{f\in W_{M,1}(P_r(D))} \big\|Sf - A\big(I_\xi f\big)\big\|_1, \tag{100}$$

where A takes the mapping set from $I_\xi(W_{M,1}(P_r(D)))$ to $L(R)$. If A only traverses the set of linear maps, $E_n(W_{M,1}(P_r(D)), S, L)$ is replaced by $E_n^L(W_{M,1}(P_r(D)), S, L)$, and $E_n^L(W_{M,1}(P_r(D)), S, L)$ is called the n-th fundamental error.

If S is the identity operator, $D_n(W_{M,1}(P_r(D)), S, L)$ and $E_n(W_{M,1}(P_r(D)), S, L)$ are replaced by $D_n(W_{M,1}(P_r(D)), L)$ and $E_n(W_{M,1}(P_r(D)), L)$, respectively.

Let $I = [a, b], \xi := \{\xi_j\}_{j\in Z} \in \Theta_n, \Delta := \xi \cap I$,

$$\begin{aligned} S_{r-1}^*(\Delta) &:= \left\{s \in C^{r-2}(I)\colon P_r(-D)s(x) = 0, x \in \big(\xi_j, \xi_{j+1}\big), \forall j, \text{such that } \big(\xi_j, \xi_{j+1}\big) \cap I \neq \varnothing\right\}; \\ T_M^r(\Delta)_0 &:= \left\{\begin{array}{l} f\colon f^{(r-1)} \text{ is absolutely continuous on } I, f\big(\xi_j\big) = 0, \forall \xi_j \in \Delta, \\ f^{(i)}(a) = f^{(i)}(b) = 0, i = 0, 1, \ldots, r-1, \big\|P_r(D)f\big\|_{M(I)} \leq 1 \end{array}\right\}. \end{aligned} \tag{101}$$

According to reference [1], we know $S_{r-1}^*(\Delta)$ is the splines space corresponding to $P_r(-D)$ and with simple nodes on Δ.

Lemma 11. *(1)*

$$\left\{P_r(D)f\colon f \in T_M^r(\Delta)_0\right\} = \left\{\psi\colon \psi \perp S_{r-1}^*(\Delta) \text{ and } \|\psi\|_{M(I)} \leq 1\right\}, \tag{102}$$

where $\psi \perp S_{r-1}^*(\Delta)$ means

$$\int_I \psi(x)s(x)\mathrm{d}x = 0, \forall s \in S_{r-1}^*(\Delta); \tag{103}$$

(2)

$$E\big(T_\infty^{r,*}(I)\big)_{N(I)} = \sup\left\{\|f\|_{L(I)}\colon f \in T_M^r(\Delta)_0\right\}, \tag{104}$$

where

$$T_\infty^{r,*}(I) := \left\{f\colon f^{(r-1)} \text{ is absolutely continuous on } I \text{ and } \big\|P_r(-D)f\big\|_{\infty(I)} \leq 1\right\}. \tag{105}$$

Proof

(1) Let

$$B = \{P_r(D)f\colon f \in T_M^r(\Delta)_0\},\quad C = \{\psi\colon \psi \perp S_{r-1}^*(\Delta) \text{and} \|\psi\|_{M(I)} \le 1\}, \tag{106}$$

let $\psi \in B$, then there exists $f \in T_M^r(\Delta)_0$ such that $\psi = P_r(D)f$, and $\|\psi\|_{M(I)} = \|P_r(D)f\|_{M(I)} \le 1$. For every $s \in S_{r-1}^*(\Delta)$, according to Lemma 2 of reference [16], we have $s \in C^{r-2}(I), f^{(i)}(a) = f^{(i)}(b) = 0, i = 0, 1, \ldots, r-1$, and $\int_I \psi(x)s(x)\mathrm{d}x = 0, \psi \in C$. Hence $B \subset C$.

On the other hand, let $\psi \in C$, by related knowledge of ordinary differential equation, we can find a function f, which satisfies that $f^{(r-1)}$ is absolutely continuous on I, and $P_r(D)f = \psi, f^{(i)}(a) = 0, i = 0, 1, \ldots, r-1$. Therefore, $\|P_r(D)f\|_{M(I)} = \|\psi\|_{M(I)} \le 1$. Since $\psi \perp S_{r-1}^*(\Delta)$, that is, for every $s \in S_{r-1}^*(\Delta)$, $\int_I s(x)P_r(D)f(x)dx = 0$, according to [16], we have $f^{(i)}(b) = 0, i = 0, 1, \ldots, r-1$, and for every $\xi_k \in \Delta, f(\xi_k) = 0$. Therefore $f \in T_M^r(\Delta)_0$, and $\psi = P_r(D) f \in B$. Hence, $C \subset B$. (1) is proved.

The proof of (2) is similar to the proof of Lemma 3 in reference [16]. □

Lemma 12 (see [15, 17]). *Let n be a natural number, then*

$$2^{-1}D_n\big(W_{M,1}(P_r(D)), L\big) \le E_n\big(W_{M,1}(P_r(D)), L\big) \le E_n^L\big(W_{M,1}(P_r(D)), L\big). \tag{107}$$

Theorem 13. *Let n be a natural number, then*

$$\begin{aligned} 2^{-1}D_n\big(W_{M,1}(P_r(D)), L\big) &= E_n\big(W_{M,1}(P_r(D)), L\big) \\ &= E_n^L\big(W_{M,1}(P_r(D)), L\big) \\ &= \|\Phi_{r,n}\|_{N[0,1]}. \end{aligned} \tag{108}$$

Furthermore, $\xi^* = \{(j + \alpha_n)/n\}_{j \in Z}$ is the set of optimal sampling points, and the basic interpolation operator s_r defined in Lemma 9 is the optimal algorithm.

Proof. First, we give the lower estimate of $D_n(W_{M,1}(P_r(D)), L)$. Let $\xi = \{\xi_j\}_{j \in Z} \in \Theta_n$, for every $N \ge 1$, set $I_N = [-N, N]$ and $N_n := \operatorname{card}(\xi \cap I_N)$.

By (74)–(77), we obtain the following expression:

$$\begin{aligned} e\big(W_{M,1}(P_r(D)), \xi, L\big) &:= \sup\big\{\|f\|_1\colon f \in W_{M,1}(P_r(D)), I_\xi(f) = 0\big\} \\ &\ge (2(N+1))^{-1}\sup\big\{\|f\|_{L(I_N)}\colon f \in T_M^r(\Delta_N)_0\big\}, \end{aligned} \tag{109}$$

where $\Delta_N := \xi \cap I_N$. Obviously

$$\dim S_{r-1}^*(\Delta_N) \le N_n + r. \tag{110}$$

From the properties of Kolmogorov n-width and Lemma 11(2), we obtain the following expression:

$$\begin{aligned} \sup\big\{\|f\|_{L(I_N)}\colon f \in T_M^r(\Delta_N)_0\big\} &= E(T_\infty^{r,*}(I_N), S_{r-1}^*(I_N))_{N(I_N)} \\ &\ge d_{N_n+r}(T_\infty^{r,*}(I_N), L_N^*(I_N)) \\ &\ge d_{N_n+r}\big(\widetilde{T}_\infty^{r,*}(I_N), L_N^*(I_N)\big). \end{aligned} \tag{111}$$

To make the distinction, we replace $\widetilde{T}_\infty^{r,*}(I_N)$ by $\widetilde{W}_\infty(P_r(-D), I_N)$, by appropriate variation, we obtain the following expression:

$$d_{N_n+r}(\widetilde{W}_\infty(P_r(-D), I_N), L_N^*(I_N)) = \frac{N}{\pi}d_{N_n+r}\Big(\widetilde{W}_\infty\Big(P_r\Big(-\frac{\pi D}{N}\Big), I_\pi\Big), L_N^*(I_\pi)\Big), \tag{112}$$

where

$$\widetilde{W}_{\infty}(P_r(-D), I_N) := \left\{ f \in C_0^r(R)\colon f \text{ is of period 1 and } \|P_r(-D)\|_{\infty(I_N)} \leq 1 \right\}. \tag{113}$$

According to reference [15] and the Theorem 7.2–4 in reference [17], and make the appropriate calculations, we obtain the following expression:

$$\frac{N}{\pi} d_{N_n+r}\left(\widetilde{W}_{\infty}\left(P_r\left(-\frac{\pi D}{N}\right), I_\pi\right), L_N^*(I_\pi)\right) \leq \frac{N}{\pi}(2\pi\lambda_N)\left\|\Phi_{r,\lambda_N}\right\|_{N[0,(1/\lambda_N)]}. \tag{114}$$

By (109)–(114), we get

$$e\left(W_{M,1}(P_r(D)), \xi, L\right) \geq \left(\frac{N\lambda_N}{(N+1)}\right)\left\|\Phi_{r,\lambda_N}\right\|_{N[0,(1/\lambda_N)]}, \tag{115}$$

where $\lambda_N = N^{-1}([N_n/2] + r)$.

Let $N \longrightarrow +\infty$ for inequation (115), we obtain the following expression:

$$e\left(W_{M,1}(P_r(D)), \xi, L\right) \geq \left\|\Phi_{r,n}\right\|_{N[0,1]}. \tag{116}$$

By (98), (109), and (116), we get the lower estimate of $D_n(W_{M,1}(P_r(D)), L)$.

According to Theorem 10 and Lemma 12, the theorem is proved. □

Theorem 14. *Let n be a natural number, then*

$$D_n\left(W_{\infty}(P_r(D)), L_{M,\infty}^*\right) = 2\left\|\Phi_{r,n}\right\|_{M[0,1]}. \tag{117}$$

Proof. For $T = \{t_j\}_{j\in Z} \in \Theta_n$, from Lemma 5(1), we obtain the following expression:

$$E\left(W_{N,1}(P_r(-D)), S_T(P_r(-D))\right)_1 \leq \sup\left\{\|g\|_{M,\infty}\colon g \in W_{\infty}(P_r(D)), g(t_j) = 0, \forall j \in Z\right\}. \tag{118}$$

On one hand, from Theorem 8, we obtain the following expression:

$$E\left(W_{N,1}(P_r(-D)), S_T(P_r(-D))\right)_1 \geq d_n\left(W_{N,1}(P_r(-D)), L\right) = \left\|\Phi_{r,n}^*\right\|_{M[0,1]}, \tag{119}$$

where $\Phi_{r,n}^*(x)$ is the standard function related to $P_r(-D)$.

On the other hand, from Lemma 2, we obtain the following expression:

$$\begin{aligned}
&\inf_{T\in\Theta_n} \sup\left\{\|g\|_{M,\infty}\colon g \in W_{\infty}(P_r(D)), g(t_j) = 0, \forall j \in Z\right\} \\
&\leq \sup\left\{\|g\|_{M,\infty}\colon g \in W_{\infty}(P_r(D)), g\left(\frac{j}{n}\right) = 0, \forall j \in Z\right\} \\
&= \left\|\Phi_{r,n}\right\|_{M[0,1]}.
\end{aligned} \tag{120}$$

Hence, according to (119) and (120), the theorem is proved. □

Data Availability

The data are not available as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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