Research Article

The Complex-Type $k$-Pell Numbers and Their Applications

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In this study, a new sequence called the complex-type $k$-Pell number is defined. Also, we give properties of this sequence such as the generating matrix, the generating function, the combinatorial representations, the exponential representation, the sums, the permanental and determinantal representations, and the Binet formula. Then, we determine the periods of the recurrence sequence according to the modulo $\nu$ and produce cyclic groups with the help of the generating matrices of the sequence. We also get some findings about the ranks and periods of the complex-type $k$-Pell sequence. Additionally, we create relations between the orders of the cyclic groups produced and the periods of the sequence. Then, this sequence is moved to groups and examined in detail in finite groups. As an application, we get the periods of the complex-type 2-Pell numbers in the polyhedral groups $(\nu, 2, 2)$, $(2, \nu, 2)$, and $(2, 2, \nu)$ and the quaternion group $\mathbb{Q}_2$.

1. Introduction

Recurrence sequences are widely utilized to solve some problems in various scientific fields, or different problems in different scientific disciplines are directly created by taking the structural aspects of these sequences into account. As an example of such studies, the scientific outputs in [1–7] can be given. All recurrence sequences defined by recurrence relations and initial values are a special case of recurrence sequences defined by the $a_{nk} = y_0a_n + y_1a_{n+1} + \cdots + y_{k-1}a_{n+k-1}$ relation. In this sense, recurrence sequences, such as Fibonacci, $k$-step Fibonacci, Pell, generalized $k$-order Pell, Jacobsthal, and generalized $k$-order Jacobsthal, which were previously defined and associated with many scientific disciplines, were discussed and various properties were obtained [8–24]. Furthermore, the authors defined the new sequences in [25, 26] using quaternions and complex numbers, and then they gave various features. In Section 3, using the reduction relation of the generalized order-$k$ Pell numbers, a new sequence called the complex-type $k$-Pell number is defined. Also, properties of this sequence such as the generating matrix, the generating function, the combinatorial representations, the exponential representation, the sums, permanental and determinantal representations, and the Binet formula are given.

Determining the periods of the recurrence sequences according to the modulo $m$ and reducing the elements of the generating matrices of the recurrence sequences according to the modulo $m$, and producing cyclic and semigroups by choosing these matrices as generating are extremely interesting current study topics and are examples of recent studies in this sense [27–30] whose scientific outputs can be given. In addition, when determining the period of a sequence in the group, it is a common situation to use the period of this sequence according to the modulo $m$. The scientific outputs in [31–34] can be given as an example of the studies in which such situations arise. With regard to the $k$-step Fibonacci sequence, Lu and Wang studied the Wall number [35]. The $k$-step Fibonacci sequence modulo $m$ is simply periodic according to Lu and Wang’s proof. In Section 4, the periods of the recurrence sequence were determined according to the modulo $\nu$, and producing cyclic and semigroups were produced with the help of the generating matrices of the sequence. Additionally, we get some findings about the ranks and periods of complex-type $k$-Pell sequences for any $k$ and $\nu$. Then, relations are created between the orders of the cyclic groups produced and the periods of
the sequence, and special formulas are given for the orders of the cyclic groups and the periods of the sequence.

As the recurrence sequences are moved into groups, the elements of the group also appear as terms of the recurrence sequence moved to that group. Therefore, transferring recurrence sequences to groups is an extremely useful method in terms of examining the structures of groups. As an example of this situation, the study of [36] can be given and the theorem expressed as “G is a 2-generating group with a and b generating, and if a unit element occurs in either of the \( F_2 \) or \( F_2 \) Fibonacci sequences of \( G \), it is a commutative group.” Recurrence sequences were first transferred to algebraic structures by Wall’s work in [37]. In this study, Wall examined the standard Fibonacci sequences in cyclic groups. Later, Wilcox extended the theory to abelian groups with his work in [38]. In the next process, the concept was moved to different recurrence sequences and different algebraic structures. Some of the current ones of these studies are given in [39–44]. Therefore, in Section 5, the defined sequence is moved to groups and examined in detail in finite groups, both to test the usefulness of the defined reduction sequences and to better understand the structures of the groups. As an application, the periods of the complex-type 2-Pell numbers in the polyhedral groups \((v, 2, 2)\), \((2, v, 2)\), and \((2, 2, v)\) and the quaternion group \(Q_{2v}\) are obtained.

2. Preliminaries

The generalized order-\(k\) Pell numbers were defined by Kilic and Tasci [17] as follows:

\[
P_n^i = 2P_{n-1}^i + P_{n-2}^i + \cdots + P_{n-k}^i,
\]

for \(n > 0\) and \(1 \leq i \leq k\), with initial conditions

\[
P_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad 1 \leq k \leq n \leq 0,
\]

where \(P_n^i\) is the \(n\)th term of the \(i\)th sequence. When \(k = 2\), the generalized order-\(k\) Pell sequence, \([P_{n}^i]\), is reduced to the usual Pell sequence, \([P_n]\).

The complex Fibonacci sequence \([F_n^i]\) is given by [45] with the subsequent equation for \(n \geq 0\)

\[
F_n^i = F_n + iF_{n+1},
\]

such that \(i = \sqrt{-1}\) and the \(n\)th Fibonacci number is designated as \(F_n\) (cf. [46, 47]).

We assume that the \((n + k)\)th term in a sequence is defined recursively as the linear combination of the \(k\) terms that came before it

\[
a_{n+k} = y_0a_n + y_1a_{n+1} + \cdots + y_{k-1}a_{n+k-1},
\]

where \(y_0, y_1, \ldots, y_{k-1}\) are constants.

Number sequences can be derived from a matrix representation, as demonstrated by Kalman [15]. By using the companion matrix method, he arrived at the following closed-form formulas for the generalized sequence:

\[
A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ y_0 & y_1 & y_2 & \cdots & y_{k-2} & y_{k-1} \end{bmatrix}
\]

and he demonstrated that

\[
\left(A_k^n\right)^a = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n+k-1} \\ a_1 & a_2 & \cdots & a_{n+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+k-1} & a_{n+k-2} & \cdots & a_1 \\ a_{n+k} & a_{n+k-1} & \cdots & a_0 \end{bmatrix}.
\]

Definition 1. If after a particular point, a group of group elements only repeats a specified subsequence, then the sequence is periodic. The period of the sequence is determined by how many elements there are in the repeating subsequence. After the first element \(x_0\), for instance, the sequence \(x_0, x_1, x_2, x_3, x_4, \ldots\) is periodic and has period 3. If the first \(k\) elements in a group element sequence form a repeating subsequence, then the group element sequence is simply periodic with period \(k\). For instance, the sequence \(x_0, x_1, x_2, x_3, x_4, x_0, x_1, x_2, x_3, x_4, \ldots\) is simply periodic with period 5.

Definition 2. The polyhedral group \((l, m, n)\) for \(l, m, n > 1\) has the presentation as follows:

\[
\langle x, y, z: x^l = y^m = z^n = x y z = e \rangle,
\]

and the polyhedral group \((l, m, n)\) has the presentation for the generating pair \((x, y)\), as follows:

\[
\langle x, y: x^l = y^m = (x y)^n = e \rangle,
\]

where \(l, m, n > 1\) (Coxeter and Moser) [48].

Definition 3. The generalized quaternion group \(Q_{2v}\) is defined by the presentation

\[
Q_{2v} = \langle x, y \mid x^{v^{-1}} = e, y^2 = x^{v^{-2}}, y^{-1} x y = x^{-1} \rangle,
\]

for every \(v \geq 3\).

3. The Complex-Type \(k\)-Pell Numbers

We next consider a new \(k\)-step sequence to be called the complex-type \(k\)-Pell numbers. This sequence is defined for any given \(k \in \{2, 3, \ldots\}\) and \(n \geq 0\) by the following recurrence relation:

\[
P_{n+k}^{(s+k)} = 2iP_{n+k-1}^{(s+k)} + i^2 P_{n+k-2}^{(s+k)} + \cdots + i^{k-1} P_{n+1}^{(s+k)} + i^k P_n^{(s+k)},
\]

where \(P_0^{(s+k)} = \cdots = P_{k-2}^{(s+k)} = 0, P_{k-1}^{(s+k)} = 1,\) and \(i = \sqrt{-1}\).
Using equation (10), we get
\[
\begin{bmatrix}
p_{n-k-2}^{(2,k)} \\
p_{n-k-3}^{(2,k)} \\
\vdots \\
p_{n-1}^{(2,k)} \\
p_n^{(2,k)}
\end{bmatrix} = C_k^* 
\begin{bmatrix}
p_{n-k-2}^{(2,k)} \\
p_{n-k-3}^{(2,k)} \\
\vdots \\
p_{n-1}^{(2,k)} \\
p_n^{(2,k)}
\end{bmatrix}
\]
where \( C_k \) is a \((k)\)-square companion matrix as shown below:
\[
C_k = \begin{bmatrix}
2i & i^2 & \cdots & i^{k-1} & i^k \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]
\( (C_k)^n = \begin{bmatrix}
p_{n-k-2}^{(2,k)} & \cdots & t^{k-1}p_{n-k-2}^{(2,k)} & + & \cdots & + & t^k p_{n-k-2}^{(2,k)} \\
p_{n-k-3}^{(2,k)} & \cdots & t^{k-1}p_{n-k-3}^{(2,k)} & + & \cdots & + & t^k p_{n-k-3}^{(2,k)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
p_n^{(2,k)} & \cdots & t^{k-1}p_n^{(2,k)} & + & \cdots & + & t^k p_n^{(2,k)}
\end{bmatrix}
\]
where \( C_k^* \) is a \(k\times 2\) matrix such as
\[
\begin{bmatrix}
1 & 2i & \cdots & i^{2,k-1} & i^{2,k} \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]
Corollary 1. Suppose that \( p_n^{(2,k)} \) is the \(n\)th complex-type k-Pell number. Then,
\[
p_n^{(2,k)} = \sum_{t_1=1}^{t_k} \binom{t_1 + \cdots + t_k}{t_1, \ldots, t_k} (2i)^{t_1} \cdots (i^k)^{t_k},
\]
where \( t_1 + 2t_2 + \cdots + kt_k = n - k + 1 \) is the sum of non-negative numbers.

Proof. In Theorem 1, if we choose \( r = k, j = 1 \) for the case i. and \( r = k - 1, j = k \) for the case ii. we can immediately view the outcomes from the matrix \((C_k)^n\).
We will now deal with the exponential representation of
the complex-type \( k \)-Pell numbers. By calculating directly, we
get the generating function of \( P^{(k)}_n \) as shown below.

\[
g^{(i)}_k(x) = \frac{x^{k-1}}{1 - 2ix - (i)^2x^2 - (i)^3x^3 - \cdots - (i)^kx^k}.
\]  

(19)

Theorem 2. The exponential representation for the complex-type \( k \)-Pell numbers is as follows:

\[
g^{(i)}_k(x) = x^{k-1} \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n} \left( 2i + (i)^2x + (i)^3x^2 + \cdots + (i)^kx^{k-1} \right)^n \right).
\]  

(20)

Proof. It is obvious that

\[
\ln g^{(i)}_k(x) = -\ln \left( 1 - 2ix - (i)^2x^2 - (i)^3x^3 - \cdots - (i)^kx^k \right).
\]  

(21)

Thus, we obtain

\[
\ln g^{(i)}_k(x) = \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n} \left( 2i + (i)^2x + (i)^3x^2 + \cdots + (i)^kx^{k-1} \right)^n \right).
\]  

(23)

As such, it complements the proof.

The sums of complex-type \( k \)-Pell numbers are now being
considered.

Let

\[
T_n = \sum_{t=1}^{n} P^{(k)}_t,
\]  

(24)

for \( n \geq 1 \) and let \( S_k \) be the \((k+1) \times (k+1)\) matrix as in-
dicated below.

\[
S_k = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & C_k \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]  

(25)

Then, it can be shown by induction that

\[
(S_k)^n = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
T_{n+k-2} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
T_{n-1} & T_{n-k+3} & (C_k)^n & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}.
\]  

(26)

Definition 4. If exactly two nonzero entries can be
found in the \( k \)th column or row of a \( u \times v \) real matrix
\( M = [m_{ij}] \), the matrix is said to be contractible in that
column or row.

According to Brualdi and Gibson’s findings in citation
[50], if \( M \) is a real matrix of order \( \alpha > 1 \) and \( N \) is contraction
of \( M \), \( \text{per}(M) = \text{per}(N) \).
Let \( k \geq 3 \) be a positive integer and let \( F^k_l = [f^k_{r,j}] \) be the \( l \times l \) superdiagonal matrix, defined by

\[
\begin{bmatrix}
2i & i^2 & \cdots & i^{k-1} & i^k & 0 & \cdots & 0 & 0 & 0 \\
1 & 2i & i^2 & \cdots & i^{k-1} & i^k & 0 & \cdots & 0 & 0 \\
0 & 1 & 2i & i^2 & \cdots & i^{k-1} & i^k & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 2i & i^2 & \cdots & i^{k-1} & i^k & 0 \\
0 & 0 & \cdots & 0 & 1 & 2i & i^2 & \cdots & i^{k-1} & i^k \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2i & i^2 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2i
\end{bmatrix}
\]  
(27)

for \( l > k \).

Then, we give the following theorem:

**Theorem 3.** For \( l > k \) and \( k \geq 3 \),

\[
\text{per} F^k_l = P_{l-1}^{(s,k)}.
\]  
(28)

**Proof.** An inductive method can be applied to \( l \) for proof. Suppose the equation for \( l > k \) is satisfied. Now let us prove that the equation satisfies \( k + 1 \). Then, expanding the \( \text{per} F^k_l \) with the Laplace expansion relative to the first row, we get

\[
\text{per} F^k_{l+1} = 2i \cdot F^k_l + i^2 \cdot F^k_{l-1} + \cdots + i^{k-1} \cdot F^k_{l-k+2} + i^k \cdot F^k_{l-k+1}.
\]  
(29)

Since \( \text{per} F^k_1 = P_{l-1}^{(s,k)} \), \( \text{per} F^k_2 = P_{l-2}^{(s,k)} \), \( \text{per} F^k_3 = P_{l-3}^{(s,k)} \), and \( \text{per} F^k_{l-k+1} = P_{l-k}^{(s,k)} \) from definition of the complex-type \( k \)-Pell numbers \( P_{n}^{(s,k)} \), the equation is easily obtainable as follows:

\[
\text{per} F^k_{l+1} = P_{l-k}^{(s,k)}.
\]  
(30)

This means that the proof is complete.

Let \( l > k \) such that \( k \geq 3 \). Define the \( l \times l \) matrix \( Y^k_l = [y^k_{r,j}] \) as shown below.

\[
\begin{bmatrix}
2i & i^2 & \cdots & i^{k-1} & i^k & 0 & \cdots & 0 & 0 & 0 \\
1 & 2i & i^2 & \cdots & i^{k-1} & i^k & 0 & \cdots & 0 & 0 \\
0 & 1 & 2i & i^2 & \cdots & i^{k-1} & i^k & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 2i & i^2 & \cdots & i^{k-1} & i^k & 0 \\
0 & 0 & \cdots & 0 & 1 & 2i & i^2 & \cdots & i^{k-1} & i^k \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2i & -1 & -i \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & i \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -i
\end{bmatrix}
\]  
(31)
Suppose that the $l \times l$ matrix $Z_i^k = [z_{ij}^k]$ is defined by

\[
Z_i^k = \begin{bmatrix}
1 & \cdots & 1 & 0 \\
1 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 1
\end{bmatrix}
\]  

where $l > k$. Then, we can yield the next theorem using \textit{permanent representations}.

\textbf{Theorem 4} \\
\textit{(i)} For $l > k$,

\[ \text{per} Y_{i+1}^k = 2i \cdot Y_1^k + i^2 \cdot Y_{l-1}^k + \cdots + i^{k-1} \cdot Y_{l-k+1}^k + i^k \cdot Y_{l-k}^k = 2i \cdot P_{i+k-3}^{(s,k)} + i^2 \cdot P_{i+k-4}^{(s,k)} + \cdots + i^{k-1} \cdot P_{i-1}^{(s,k)} + i^k \cdot P_{l-2}^{(s,k)}. \]  

\[ \text{per} Z_i^k = \sum_{a=0}^{l+k-4} p_i^{(s,k)}. \]  

\textit{(ii)} For $l > k$,

\[ \text{per} Y_{i+1}^k = \text{per} (F_i^k \ast J), \]  

\[ \text{per} Z_i^k = \text{per} (Z_i^k \ast J) \]  

\textit{Proof} \\
\textit{(i)} Assuming that the equation is valid for $l > k$, we now demonstrate that it is also valid for $l + 1$. When we expand the $\text{per} Y_{i+1}^k$ by the Laplace expansion of permanent with regard to the first row, we reach

\[ \text{per} Y_{i+1}^k = \text{per} Y_i^k. \]  

So, we achieve the conclusion.

\textit{(ii)} As we expand the $\text{per} Z_i^k$ with the Laplace expansion relative to the first row, we reach

\[ \text{per} Z_i^k = \text{per} Z_{i-1}^k + \text{per} Y_i^k. \]  

The conclusion is reached by considering the result of part \textit{(i)} in Theorem 4 and the inductive argument.

If there is a $n \times n$ $(1, -1)$-matrix $K$ such that $\text{per} M = \det (M \ast K)$, where $M \ast K$ stands for the Hadamard product of $M$ and $K$, and then the matrix $M$ is said to be convertible.

We will now deal with the determinantal representations for the complex-type $k$-Pell numbers. Let $J$ be the $l \times l$ matrix, defined by

\[
J = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & 1 & -1 & 1 \\
1 & \cdots & 1 & 1 & -1 \\
-1 & 1 & 1 & \cdots & 1
\end{bmatrix}
\]  

\textbf{Corollary 2.} For $l > k + 1$,

\[ \det (F_i^k \ast J) = P_{i+k-1}^{(s,k)}, \]  

\[ \det (Y_i^k \ast J) = P_{i+k-3}^{(s,k)}, \]  

and

\[ \det (Z_i^k \ast J) = \sum_{a=0}^{l+k-4} p_i^{(s,k)}. \]  

For the complex-type $k$-Pell numbers, we now derive a generalized Binet formula. For this reason, we start by making the following lemma.

\textbf{Lemma 1.} The equation $x^k - 2i \cdot x^{k-1} - i^2 \cdot x^{k-2} - \cdots - i^k \cdot x - i^k = 0$ does not have multiple roots for $k \geq 3$.

\textit{Proof.} Let $f(x) = x^k - 2i \cdot x^{k-1} - i^2 \cdot x^{k-2} - \cdots - i^k \cdot x - i^k$, then $x^k - 2i \cdot x^{k-1} - i^2 \cdot (x^{k-1} - i^{k-1}) = 0$. It is obvious that $f(0) \neq 0$ and $f(1) \neq 0$ for all $k \geq 3$. Let us assume that $t(x) = (x - i) \cdot f(x) = x^{k+1} - 3i \cdot x^k + i^2 \cdot x^{k+1} + i^{k+1}$. Let $\gamma$ be a multiple root of $f(x)$, then $\gamma \neq 0$ and $\gamma \neq 1$. Given that $\gamma$ is a multiple root, it is $f(\gamma) = 0$ and $f'(\gamma) = 0$. Considering the case $f'(\gamma)$ and $\neq 0$, we obtain $(k+1) \cdot \gamma^2 - (3ik) \cdot \gamma + i^2 \cdot (k+1) = 0$. Thus, we obtain

\[ \gamma_1 = \frac{3ik + \sqrt{5k^2 - 4}}{2k + 2}, \]  

and

\[ \gamma_2 = \frac{-3ik + \sqrt{5k^2 - 4}}{2k + 2}, \]  

for $k \geq 3$, $f(\gamma_1) \neq 0$, and $f(\gamma_2) \neq 0$, which is a contradiction and with this contradiction, it is concluded.

If $\sigma_1, \sigma_2, \ldots, \sigma_k$ be the eigenvalues of the matrix $C_k$, then by Lemma 1, $\sigma_1, \sigma_2, \ldots, \sigma_k$'s are known to be distinct. Define the Vandermonde matrix as shown below.
\[ Q^k = \begin{bmatrix} (\sigma_1)^{k-1} & (\sigma_2)^{k-1} & \cdots & (\sigma_k)^{k-1} \\ (\sigma_1)^{k-2} & (\sigma_2)^{k-2} & \cdots & (\sigma_k)^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1 & \sigma_2 & \cdots & \sigma_k \\ 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{(42)} \]

We assume that
\[ U^k = \begin{bmatrix} (\sigma_1)^{nk+r} \\ (\sigma_2)^{nk+r} \\ \vdots \\ (\sigma_k)^{nk+r} \end{bmatrix}, \quad \text{(43)} \]
and \( Q^k (r, j) \) is derived from \( Q^k \) by replacing the \( j \)th column of \( Q^k \) by the matrix \( U^k \).

**Theorem 5.** Let \((C_k)^n = [c_{rk}^n]\), then
\[ c_{rk}^n = \frac{\det Q^k (r, j)}{\det Q^k}, \quad \text{(44)} \]
for \( k \geq 3 \).

**Proof.** The \( C_k \) matrix may be diagonalized since \( \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \) are distinct. Then, we easily see that \( C_k Q^k = Q^k C_k \) where \( Q^k = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k) \). Since the matrix \( Q^k \) is invertible, we obtain \((Q^k)^{-1} C_k Q^k = Q^k\). Then, the matrix \( C_k \) is similar to \( Q^k \), so, we obtain \((C_k)^n Q^k = Q^k (G^k)^n\). We therefore have the following linear system of equations:

\[
\begin{align*}
&c_{r1}^n (\sigma_1)^{k-1} + c_{r2}^n (\sigma_2)^{k-1} + \cdots + c_{rk}^n (\sigma_k)^{k-1} = (\sigma_1)^{nk-r}, \\
&c_{r1}^n (\sigma_2)^{k-1} + c_{r2}^n (\sigma_2)^{k-1} + \cdots + c_{rk}^n (\sigma_k)^{k-1} = (\sigma_2)^{nk-r}, \\
&\vdots \\
&c_{r1}^n (\sigma_k)^{k-1} + c_{r2}^n (\sigma_k)^{k-1} + \cdots + c_{rk}^n (\sigma_k)^{k-1} = (\sigma_k)^{nk-r}.
\end{align*}
\]

Then, for each \( r, j = 1, 2, \ldots, k \), we get \( c_{rj}^n \) as follows:
\[ c_{rj}^n = \frac{\det Q^k (r, j)}{\det Q^k}. \quad \text{(46)} \]

**Corollary 3.** Suppose that \( P^{(s, k)} \) is the \( n \)th element of complex-type \( k \)-Pell number, then
\[ P_n^{(s, k)} = \frac{\det Q^k (1, k)}{\det Q^k} = \frac{\det Q^k (k-1, k)}{i^k \cdot \det Q^k}. \quad \text{(47)} \]

**4. The Period and Rank of the Complex-Type \( k \)-Pell Numbers Modulo \( v \)**

By a modulus \( v \) reducing of the complex-type \( k \)-Pell numbers \( \{P_n^{(s, k)}\} \), the repeating sequence is obtained as follows:
\[ \{P_n^{(s, k)}(v)\} = \{P_0^{(s, k)}(v), P_1^{(s, k)}(v), \ldots, P_{l-1}^{(s, k)}(v)\}, \]

where \( P_n^{(s, k)}(v) = P_n^{(s, k)} \pmod{v} \). Here, it is important to note that the recurrence relations in the sequences \( \{P_n^{(s, k)}\} \) and \( \{P_n^{(s, k)}(v)\} \) are identical.

**Theorem 6.** For every value of \( k \geq 2 \), the sequence \( \{P_n^{(s, k)}(v)\} \) is a simply periodic sequence.

**Proof.** Let us think about the set
\[ B = \left\{ (b_1, b_2, \ldots, b_k) \mid b_{rj}\'s \text{ are complex numbers } d_w, \text{ and } e_w \text{ are integers with values of } 0 \leq d_w, e_w \leq v - 1 \text{ and } 1 \leq w \leq k \right\}. \quad \text{(49)} \]

\[ P_n^{(s, k)} = (-1)^k \cdot P_{n+k} - 2(-1)^k \cdot P_{n+k-1} \cdot P_{n+k-1} \]
\[ - (-1)^k \cdot P_{n+k-2} \cdot P_{n+k-2} - \cdots - (-1)^k \cdot P_{n+k-1}. \quad \text{(50)} \]

Therefore, it is simple to observe that \( P_n^{(s, k)}(v) \equiv P_{n+k}^{(s, k)}(v) \pmod{v} \) and \( P_n^{(s, k)}(v) \equiv P_{n+1}^{(s, k)}(v) \pmod{v} \), indicating that the sequence \( \{P_n^{(s, k)}(v)\} \) is simply periodic.

Let us use the symbol \( \sigma_k(v) \) to indicate the sequence \( \{P_n^{(s, k)}(v)\} \)'s period.

An integer matrix \( A \pmod{v} \) shows that all of \( A \)'s elements are modulo \( v \), that is, \( A \pmod{v} = (a_{ij} \pmod{v}) \) for
an integer matrix \( A = [a_{ij}] \). Consider the set 
\( \langle A \rangle_v = \{(A)^n \mod v \mid n \geq 0\} \) for a moment. The set 
\( \langle A \rangle_v \) is a semigroup if \( \det(A, v) \neq 1 \); if \( \det(A, v) = 1 \), then the set 
\( \langle A \rangle_v \) is a cyclic group.

From companion matrices, we can easily obtain 
\[ \det(c_k) = (-1)^{k+1} k^2. \]
Consequently, it follows that for each integer \( v \geq 2 \), \( \langle c_k \rangle \) is a cyclic group. It is simple to observe that 
\[ a_k(v) = |\langle c_k \rangle| \] from (13).

\( \square \)

Definition 5. The rank of the sequence \( \{P_n^{(s,k)}(v)\} \) is the smallest positive integer \( v \) such that 
\[ P_n^{(s,k)}(v) \equiv \cdots \equiv P_{n-k+1}^{(s,k)}(v) \equiv 0 \mod v. \]
We denote the rank of \( \{P_n^{(s,k)}(v)\} \) by \( \tau_k(v) \).

If \( P_{n-k}^{(s,k)}(v) \equiv 0 \mod v \), then the terms of \( \{P_n^{(s,k)}(v)\} \) begin with the index \( \tau_k(v) \), that is, 
\[ \{0, \ldots, \mu, \mu, \ldots \} \]
the initial terms of \( \{P_n^{(s,k)}(v)\} \) multiplied by a factor \( \mu \).

An easy arithmetic progression is formed by the exponents \( \alpha \) for which \( (C_k)^\alpha \equiv I \mod v \). Then, we have
\begin{equation}
(C_k)^\alpha \equiv I \mod v \Leftrightarrow a_k(v)|\alpha. \tag{51}
\end{equation}

Similarly, an easy arithmetic progression is formed by the exponents \( \alpha \) for which \( (C_k)^\alpha \equiv \mu I \mod v \) for some \( \mu \in C \), and hence
\begin{equation}
(C_k)^\alpha \equiv \mu I \mod v \Leftrightarrow \tau_k(v)|\alpha. \tag{52}
\end{equation}

As a result, it is obvious that \( \tau_k(v) \) divides \( a_k(v) \).

The formula \( a_k(v)/\tau_k(v) \) yields the order of the sequence \( \{P_n^{(s,k)}(v)\} \), and this order is shown by \( \eta_k(v) \). Let \( (C_k)^{\tau_k(v)} \equiv \mu I \mod v \), then \( \text{ord}_v(\mu) \) is the least positive value of \( s \) such that \( (C_k)^{s \tau_k(v)} \equiv I \mod v \). Thus, it is established that \( \text{ord}_v(\mu) \) is the least positive integer \( s \), \(\eta_k(v)\) is always a positive integer and that 
\[ \eta_k(v) = \text{ord}_v(P_n^{(s,k)}(v) \equiv \text{ord}_v(P_{n-k}^{(s,k)}(v)). \]

The multiplicative order of \( P_n^{(s,k)}(v) \).

\begin{example}
\begin{equation}
\{P_n^{(s,k)}(3)\} = \{0, 0, 1, 2i, -2, 0, 0, -1, 2i, 2, i, 0, 0, 1, 2i, \ldots\}, \tag{53}
\end{equation}
we get \( a_3(3) = 12, \tau_3(3) = 6, \) and \( \eta_3(3) = 2. \)
\end{example}

\begin{theorem}
Let \( \rho \) be a prime number. Then,
\begin{enumerate}
\item If \( \beta = 1, 2 \), then \( a_\beta(\rho^{p+1}) \neq a_\beta(\rho^p) \) for each integer \( k \geq 2 \), \( a_\beta(\rho^{p+1}) = \rho \cdot a_\beta(\rho^p) \).
\item If \( \beta = 1, 2 \), then \( \tau_\beta(\rho^{p+1}) \neq \tau_\beta(\rho^p) \) for each integer \( k \geq 2 \), \( \tau_\beta(\rho^{p+1}) = \rho \cdot \tau_\beta(\rho^p) \).
\end{enumerate}
\begin{proof}
We assume that \( b \) is an element of \( \mathbb{Z}^* \) and the smallest value such that \( a_\beta(b^{p+1}) \neq a_\beta(b^p) \) for each integer \( k \geq 2 \), \( a_\beta(b^{p+1}) = \rho \cdot a_\beta(b^p) \).
\end{proof}
\end{theorem}

Theorem 8. If \( v_1 \) and \( v_2 \) are integers with \( v_1, v_2 \geq 2 \), then \( \tau_k(\text{lcm}[v_1, v_2]) = \text{lcm}[(\tau_k(v_1), \tau_k(v_2))] \) for any \( k \geq 2 \). Similarly, \( a_k(\text{lcm}[v_1, v_2]) = \text{lcm}[a_k(v_1), a_k(v_2)] \).

Proof. Let \( \text{lcm}[v_1, v_2] = v \). Then,
\begin{equation}
P_{(s,k)}^{(s,k)}(v) \equiv P_{(s,k)}^{(s,k)}(v) \equiv \cdots \equiv P_{(s,k)}^{(s,k)}(v) \equiv 0 \mod v, \tag{55}
\end{equation}
and
\begin{equation}
P_{(s,k)}^{(s,k)}(v) \equiv P_{(s,k)}^{(s,k)}(v) \equiv \cdots \equiv P_{(s,k)}^{(s,k)}(v) \equiv 0 \mod v, \tag{56}
\end{equation}
for \( k = 2 \). It is seen that \( P_{(s,k)}^{(s,k)}(v) \equiv P_{(s,k)}^{(s,k)}(v) \equiv \cdots \equiv P_{(s,k)}^{(s,k)}(v) \equiv 0 \mod v \) for \( k = 1 \) by using the least common multiple operation. Consequently, \( \tau_k(\text{lcm}[v_1, v_2]) \) and \( \tau_k(\text{lcm}[v_1, v_2]) \) are obtained, indicating that \( \text{lcm}[(\tau_k(v_1), \tau_k(v_2))] \) divides \( \tau_k(\text{lcm}[v_1, v_2]) \). We also know that the following equivalences are satisfied:
\begin{equation}
P_{(s,k)}^{(s,k)}(v) \equiv P_{(s,k)}^{(s,k)}(v) \equiv \cdots \equiv P_{(s,k)}^{(s,k)}(v) \equiv 0 \mod v, \tag{57}
\end{equation}
for \( \beta = 1, 2 \). Then, we can write

\[
P_{lcm\{t_1(v_1), t_2(v_2)\}}^{(\ast, \ast)} = P_{lcm\{t_1(v_1), t_3(v_3)\}}^{(\ast, \ast)} = \cdots = P_{lcm\{t_1(v_1), t_k(v_k)\}}^{(\ast, \ast)} = 0 \pmod{t-2}, \tag{58}
\]

and as a result of this, it follows that \( t_k(\text{lcm}\{v_1, v_2\}) \) divides \( \text{lcm}\{t_k(v_1), t_k(v_2)\} \). The proof is finished.

It also uses a similar proof method for the period \( o_k(v) \).

5. The Complex-Type \( k \)-Pell Numbers in Groups

Let \( G \) be a finite \( k \)-generator group and let \( E \) be the subset of \( G^{xG \cdots G_k} \) such that \( (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in E \) if and only if \( G \) is generated by \( \epsilon_1, \epsilon_2, \ldots, \epsilon_k \). We call \( (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \) a generating \( k \)-tuple for \( G \).

Definition 6. Let \( G \) be a \( k \)-generator group. For a generating \( k \)-tuple \( (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in E \), the complex-type \( k \)-Pell orbit is defined as follows:

\[
\alpha_0 = \epsilon_1, \alpha_1 = \epsilon_2, \ldots, \alpha_{k-1} = \epsilon_k, \alpha_{nk} = (\alpha_n)^{\beta} \epsilon_{n+1} \cdots \epsilon_{nk-1} \epsilon_{nk} \epsilon_{nk+1} \cdots = (\alpha_n)^{\beta} (\alpha_{n+1})^{\beta} \cdots (\alpha_{nk-2})^{\beta} (\alpha_{nk-1})^{\beta} (n \geq 0). \tag{59}
\]

\( (i) \) Given \( z_1 = a_1 + ib_1 \) and \( z_2 = a_2 + ib_2 \), where \( a_1, b_1, a_2, \) and \( b_2 \) are integers, \((x^2 - y^2) = y^2 - x^2\).

\( (ii) \) Let \( x^2 = \alpha \), then \( x^2 y^2 = y^2 x^2 \).

\( (v) \) \( x^2 y^2 = y^2 x \) and so \( y^2 x^l = x^l y^2 \) and \( y^2 x^l = x^l y^2 \).

It is important to note that we will obtain the terms of the complex-type \( k \)-Pell orbit according to the above rules.

Theorem 9. Let \( G \) be a \( k \)-generator group. The complex-type \( k \)-Pell orbit of \( G \) is periodic if \( G \) is finite.

Proof. We consider the following set:

\[
H = \left\{ \left( x_1 \right)^{d_1} (\text{mod} \left( h_1 \right)) \epsilon_{i_1}, (\text{mod} \left( h_1 \right)) \right\} \left( x_2 \right)^{d_2} (\text{mod} \left( h_2 \right)) \epsilon_{i_2}, (\text{mod} \left( h_2 \right)) \cdots \left( x_k \right)^{d_k} (\text{mod} \left( h_k \right)) \epsilon_{i_k}, (\text{mod} \left( h_k \right)) \right\} \quad : \ h_1, h_2, \ldots, h_k \in G \text{ and } d_n, e_n \in Z \text{ such that } 1 \leq n \leq k. \tag{60}
\]

Proof. Let us consider the group \((v, 2, 2)\). We consider the following sequence:

\[
\begin{align*}
w_0 &= 1, \\
w_1 &= 0, \\
w_{n+2} &= 2w_n - w_{n-1} (n \geq 0).
\end{align*} \tag{61}
\]

We can obtain a repeating sequence by reducing this sequence by a modulus \( \alpha \), which is indicated by:

\[
\left\{ w_n (\alpha) \right\} \left\{ w_0 (\alpha), w_1 (\alpha), \ldots, w_n (\alpha), \ldots \right\}. \tag{62}
\]

Let us use the notation \( o_k (v) \) to denote the period of the sequence \( \{ w_n (\alpha) \} \). Since \( o_k (v) = |(C_2)_k| \) is obvious that

\[
\begin{align*}
\text{Theorem 10. For } v \geq 3, \\
t^*_{(x,y)} ((v, 2, 2)) &= t^*_{(x,y)} ((2, v, 2)) = \text{lcm}[4, o_2 (v)].
\end{align*}
\]
\( o_2(a) = o_{w_2}(a) \) from equation (13). The sequence \( P^*_<(x, y) ((v, 2, 2)) \) is
\[
\begin{align*}
  x_0 &= x, \\
x_1 &= y, \\
x_2 &= x^{-1}, \\
x_3 &= yx^{-2i}, \\
x_4 &= x^5, \\
x_5 &= yx^{-2i}, \\
x_6 &= x^{-29}, \\
x_7 &= yx^{-70}, \ldots, \\
x_9 &= x^{w_{i+1}}, \\
x_{10} &= yx^{w_{i+1}}, \\
x_{11} &= x^{w_{i+2}}, \\
x_{12} &= yx^{w_{i+2}}, \\
x_{13} &= x^{w_{i+3}}, \\
x_{14} &= yx^{w_{i+3}}, \ldots.
\end{align*}
\] (63)

It is evident to say that \( o_2(2) = 2 \), and the complex-type 2-Pell orbit is in the form of four layers. Since \( |x| = v \) and \( |y| = 2 \),
\[
\begin{align*}
  x_4 &= x^{w_{i+1}}, \\
x_{4+1} &= yx^{w_{i+1}}, \\
x_{4+2} &= x^{w_{i+2}}, \\
x_{4+3} &= yx^{w_{i+3}}, \ldots.
\end{align*}
\] (64)

Then, we achieve
\[
\begin{align*}
x_{\text{lcm}[4, o_2(v)]} &= x, \\
x_{\text{lcm}[4, o_2(v)]+2} &= x^{-1}, \\
x_{\text{lcm}[4, o_2(v)]+3} &= yx^{-2i}, \ldots
\end{align*}
\] (65)

Thus, it is established that \( \text{lcm}[4, o_2(v)] \) is the length of the period of the sequence \( P^*_<(x, y) ((v, 2, 2)) \).

A similar proof exists for \( t^*_<(x, y) ((2, v, 2)) \). \( \square \)

**Example 2.** The sequence \( P^*_<(x, y) ((7, 2, 2)) \) is
\[
\begin{align*}
  x, y, x^{-1}, yx^{-2i}, x^{-5}, yx^{3i}, x^{-1}, y, x, yx^{2i}, x^{-5}, yx^{3i}, x, y, x^{-1}, yx^{2i}, \ldots.
\end{align*}
\] (66)

This demonstrates that \( t^*_<(7, 2, 2) = 12 \). It is simply seen that \( \text{lcm}[4, o_2(7)] = \text{lcm}[4, 12] = 12 \).

**Theorem 11.** \( t^*_<(2, 2, v) = 2 \).

**Proof.** The direct calculation is used for proof. First, it should be noted that the present polyhedral group \( (2, 2, v) \) is as follows:
\[
(2, 2, v) = \bigl( x, y; x^2 = y^2 = (x, y)^v = e \bigr). \] (67)

Considering this group representation, the sequence \( P^*_<(x, y) ((2, 2, v)) \) is
\[
\begin{align*}
  x_0 &= x, \\
x_1 &= y, \\
x_2 &= x, \\
x_3 &= y, \ldots
\end{align*}
\] (68)

So, the length of the period of the sequence \( P^*_<(x, y) ((2, 2, v)) \) for all values of \( v \) is 2. \( \square \)

**Theorem 12.** For \( v \geq 3 \), \( t^*_<(x, y) (Q_{2v}) = 2^{v-2} \cdot o_2(2) \).

**Proof.** Using the period \( o_2(2) \), we consider the length of the period of the complex-type 2-Pell orbit in the quaternion group. Note that \( |x| = 2^{v-1} \), \( |y| = 4 \), and \( o_2(2) = 2 \). The complex-type 2-Pell orbit is
\[
\begin{align*}
  x, y, y^2 x^{-1}, y^{-1} x^{-2i}, x, y x^{12i}, y^{3i} x^{-29}, \\
y^{-1} x^{-70i}, x, y x^{408i}, y^{2i} x^{-985}, y^{-1} x^{-2378i}, \\
x^{574i}, y x^{13860i}, y^{2i} x^{-33461}, y^{-1} x^{-80782i}, \ldots,
\end{align*}
\] (69)

and so, using the above equation, the complex-type 2-Pell orbit can be represented in the form below.
\[
\begin{align*}
  x_0 &= x, \\
x_1 &= y, \\
x_2 &= y^2 x^{-1}, \\
x_3 &= y^{-1} x^{-2i}, \ldots
\end{align*}
\] (70)

where \( \gcd (k_1, k_2, k_3, k_4) = 1 \) and \( k_1, k_2, k_3, \) and \( k_4 \) are the positive integers. So, we require the smallest integer \( v \) such that \( 2^{v-1} \cdot b = 4v \). If we choose \( 2^{v-1} = 3 \), we get
\[
\begin{align*}
  x^{2v-2+1}, x x^{2v-2+1}, x x^{2v-2} = y, \ldots
\end{align*}
\] (71)

The cycle begins again with the element after \( x x^{2v-2} \), since elements \( x x^{2v-2} \) and \( x x^{2v-2} \) depend on \( x \) and \( y \) for their values.

Thus, \( t^*_<(Q_{2v}) = 2^{v-2} \cdot o_2(2) \). \( \square \)

**Example 3.** The sequence \( P^*_<(x, y) (Q_9) \) is
\[
\begin{align*}
  x, y, y^2 x^{-1}, y^{-1} x^{-2i}, x, y x^{12i}, y^{3i} x^{-29}, \\
y^{-1} x^{-6i}, x, y x^{408i}, y^{2i} x^{-985}, y^{-1} x^{-2378i}, \\
x^{574i}, y x^{13860i}, y^{2i} x^{-33461}, y^{-1} x^{-80782i}, \ldots
\end{align*}
\] (72)

which implies that \( t^*_<(Q_9) = 16 \).

**6. Conclusion**

As it is known, while a problem is being constructed in all fields of modern science, even a small change to any concept to be used for a solution causes significant differences in the result to be achieved. Therefore, such changes can lead to shorter and more original results. In this sense, in this study, we defined a sequence called the complex-type \( k \)-Pell number for the first time and gave its structural properties, such as the generating matrix, the generating function, the combinatorial representations, the exponential representation, the sums, the permanental and determinantal representations, and the Binet formula. Also, we determined the periods of the recurrence sequence according to the modulo \( v \) and produced cyclic groups with the help of the generating matrices of the sequence. Then, this sequence was moved to groups and examined in detail in finite groups. Finally, we obtained the periods of the complex-type 2-Pell numbers in
the polyhedral groups $(v,2,2), (2,v,2), \text{ and } (2,2,v)$ and the quaternion group $\mathbb{Q}_4$.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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