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Research Article

Numerical Solution of Fractional Order Integro-Differential Equations via Müntz Orthogonal Functions

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In this paper, we derive a spectral collocation method for solving fractional-order integro-differential equations by using a kind of Müntz orthogonal functions that are defined on [0, 1] and have simple and real roots in this interval. To this end, we first construct the operator of Riemann–Liouville fractional integral corresponding to this kind of Müntz functions. Then, using the Gauss–Legendre quadrature rule and by employing the roots of Müntz functions as the collocation points, we arrive at a system of algebraic equations. By solving this system, an approximate solution for the fractional-order integro-differential equation is obtained. We also construct an upper bound for the truncation error of Müntz orthogonal functions, and we analyze the error of the proposed collocation method. Numerical examples are included to demonstrate the validity and accuracy of the method.

1. Introduction

We consider the fractional-order integro-differential equation:

$$F_1\left(t, f(t), D^{q_0} f(t), D^{q_1} f(t), \ldots, D^{q_u} f(t)\right) = \lambda F_2\left(t, f(t), \int_0^t \kappa(t,s) G(s, f(s)) ds\right), \tag{1}$$

with initial conditions

$$f^{(k)}(0) = d_k, \quad k = 0, 1, \ldots, m_0 - 1, \tag{2}$$

where

$$q_0 \geq q_1 \geq \ldots \geq q_u \geq 0, m_k - 1 < q_k \leq m_k, 0 \leq t \leq 1, \quad \lambda \in \mathbb{R}. \tag{3}$$

In recent years, several numerical techniques have been proposed in the literature for solving fractional-order integro-differential equations, such as the composite functions [1], collocation method [2], Chebyshev series [3], fractional Legendre functions [4], modified hat functions [5], Bernstein polynomials [6], modified Adomian decomposition [7], Sumudu transformation and Hermite spectral collocation method [8], stable least residue method [9], discrete Galerkin method [10], Haar wavelet [11], Euler functions [12], Jacobi spectral method [13], Legendre wavelet [14], operational matrices [15], Runge–Kutta convolution quadrature methods [16], Dhage iteration principle [17], perturbation-iteration algorithms [18], and composite collocation method [19].

In several research papers, the topic of existence and uniqueness of the solution of fractional-order integro-differential equations have been discussed [20–24]. In [25], fractional-order integro-differential equations are utilized in the modeling of some phenomena in fluid dynamic. In [26], a numerical scheme is proposed based on basis functions. Moreover, in [27–29], the solvability of fractional-order integro-differential equations is assessed using the Krasnoselskii fixed-point theorem. In [30], nonlinear fractional-order integro-differential equations have been solved by using Riemann–Liouville integral and Caputo fractional derivative operators. The asymptotic stability, the boundedness of nonzero solutions, the stability of Mittag-Leffler zero solution, and the monotonic stability of solutions of fractional-order integro-differential equations are studied in [31]. Furthermore, existence and uniqueness of the solution of fractional-order integro-differential equations in Banach spaces are investigated in [32, 33], and the authors of [34] obtained similar results using the Schauder fixed-point theorem and the contraction map principle. In [35], by using Legendre wavelet collocation and definite and stochastic operational matrices, the uncertainty quantity in solving fractional-order integro-differential equations has been assessed. In [36], using the concept of extended distances with piecewise constant functions, the necessary criteria for existence and uniqueness are constructed via the Schauder and Banach fixed-point theorem. In [37], several classes of fractional-order integro-differential equations have been solved by using the Jacobi–Gauss collocation algorithm.

In the past few decades, fractional calculus has been of considerable importance due to its several applications in various fields of science and engineering. Hence, the theory of fractional differential and integral calculus is of interest to many mathematicians, and currently, different definitions of fractional derivatives and integrals are used. Moreover, researchers and engineers in different scientific fields have made efforts to construct fractional models for different problems in fields such as viscoelastic systems, electrode-electrolyte polarization, electrochemistry, processor and the process of publication, processing, and control, which have specified and defined a general framework for the issue of fractional calculus. The subject of fractional differential and integral calculus is actually the generalization of integral and derivative calculations from integer orders to arbitrary real order. In fact, the subject of fractional calculus is the generalization of derivation from the integer order and the ordinary multiple integration. Fractional derivatives are able to describe memory and inherited properties of materials and methods. In 1976, Caputo defined a fractional derivation method that has several good properties in modeling natural phenomena. Among the most important features and superiority of the Caputo definition compared to other existing definitions is that the fractional Caputo derivative of a constant function is equal to zero. Indeed, it can be said that the Caputo definition of fractional derivatives is a generalization of the ordinary derivative. Also, the most important features of the Riemann–Liouville integral operator are its commutative and semigroup properties [38, 39].

Many problems in physics and the real world lead to equations where zero is the singular point such as fractional-order equations. On the other hand, one of the most important features of Müntz orthogonal functions is that zero is their singular point, and therefore, these functions can provide suitable approximate solutions for such equations (40). In this paper, we present a numerical method based on Müntz orthogonal functions and collocation to approximate the solution of the fractional integro-differential (1) and (2).

The reminder of this paper is organized as follows: In Section 2, basic definitions which are required for our subsequent development are presented. In Section 3, Müntz orthogonal functions are defined and the best approximation of an arbitrary function via Müntz orthogonal functions is given. Moreover, the Riemann–Liouville fractional integral operator is constructed, which reduces the computational complexity and speeds up the solution process. Section 4 is devoted to the numerical solution of fractional-order integro-differential equations using the Müntz functions and collocation method. An error analysis of the method is also carried out. In Section 5, numerical examples are given to demonstrate the applicability and high accuracy of the proposed method. Finally, some conclusions are given in Section 6.

2. Preliminaries and Notations

2.1. Convolution

Definition 1. Convolution is a mathematical operator that has two functions as input and a third function as output. The convolution of two functions f and g has the notation $f * g$ and is given by [41]:

$$(f * g)(t) = \int_0^t f(t-y)g(y)dy. \tag{4}$$

Lemma 2. *Let* $\in L^1(\mathbb{R})$ *and* $g \in L^p(\mathbb{R})$*; then, for some* $p \in [1,\infty]$*, the following inequality holds [41]:*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p. \tag{5}$$

Definition 3. A function $f\colon S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $t, y \in S$ is Lipschitz with the Lipschitz constant $\lambda \geq 0$ if [42]

$$|f(t) - f(y)| \leq \lambda |t-y|. \tag{6}$$

2.2. Fractional-Order Integral and Derivative

Definition 4. The fractional Riemann–Liouville integral I^α of order $\alpha \in \mathbb{R}^+$ on the interval $[a,b]$ is defined as follows [43]:

$$I^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \dfrac{f(s)}{(t-s)^{1-\alpha}} ds = \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{7}$$

Definition 5. The Caputo fractional derivative D^{α} of order $\alpha \in \mathbb{R}^{+}$ is defined as follows [43]:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}, n-1 < \alpha \le n, \quad n \in \mathbb{N}. \tag{8}$$

The following relations between fractional Riemann–Liouville integral and the Caputo fractional derivative also hold [4, 43]:

$$\begin{aligned} &1. I^{\alpha}(\lambda_1 f(t) + \lambda_2 g(t)) = \lambda_1 I^{\alpha} f(t) + \lambda_2 I^{\alpha} g(t), \quad \alpha \in \mathbb{R}^{+}, \\ &2. I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\alpha+\beta}, \quad \beta > -1, \\ &3. D^{\alpha} I^{\alpha} f(t) = f(t), \\ &4. I^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}, \quad n \in \mathbb{N}, \\ &5. D^{\alpha} f(t) = I^{n-\alpha} D^{n} f(t), \\ &6. D^{\alpha}(\lambda_1 f(t) + \lambda_2 g(t)) = \lambda_1 D^{\alpha} f(t) + \lambda_2 D^{\alpha} g(t), \\ &7. D^{\alpha} c = 0, \end{aligned} \tag{9}$$

where λ_1, λ_2, and c are real constants.

3. Müntz Orthogonal Functions and Their Properties

3.1. Müntz Spaces. Let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ be an increasing sequence of distinct real numbers and let $M_n := \text{span}\{t^{\lambda_k}: k = 0, 1, \ldots, n\}$. A class of Müntz spaces is defined by $M := \bigcup_{n=0}^{\infty} M_n$. If $\lambda_0 = 0$, then the classical Müntz theorem states that the Müntz space M is dense in $C(I)$ if and only if $\sum_{k=1}^{\infty} \lambda_k^{-1} = +\infty$ [40]. This Müntz space defines an orthogonal basis of polynomials with real powers on the interval $[0, 1]$ that are called Müntz–Legendre polynomials (see [40]). Another special space of Müntz orthogonal functions is given in the following definition.

Definition 6. The logarithmic family of Müntz orthogonal functions $\{P_n(t)\}_{n=0}^{\infty}$ defined on the interval $[0, 1]$ is given by [40]:

$$P_n(t) = R_n(t) + S_n(t) \ln(t), \quad n = 0, 1, 2, \ldots, \tag{10}$$

where

$$\begin{aligned} R_n(t) &= \sum_{\nu=0}^{[n/2]} a_{\nu}^{(n)} t^{\nu}, \\ S_n(t) &= \sum_{\nu=0}^{[n-1/2]} b_{\nu}^{(n)} t^{\nu}. \end{aligned} \tag{11}$$

Furthermore, for $n = 2m$ and for every $0 \le \nu \le m-1$, we have

$$\begin{aligned} a_{\nu}^{(2m)} &= -\binom{m+\nu}{m}^2 \binom{m}{\nu}^2 \left[\frac{2m+1}{2\nu+1} + 2(m-\nu) \sum_{j=0, j\ne\nu}^{m-1} \frac{2j+1}{(j-\nu)(j+\nu+1)} \right], \\ b_{\nu}^{(2m)} &= -(m-\nu) \binom{m+\nu}{m}^2 \binom{m}{\nu}^2, \end{aligned} \tag{12}$$

and if $v = m$,

$$a_m^{(2m)} = \binom{2m}{m}^2, \quad b_m^{(2m)} = 0. \tag{13}$$

In addition, for $n = 2m + 1$ and for every $0 \le v \le m$, we have

$$a_v^{(2m+1)} = \binom{m+v}{m}^2 \binom{m}{v}^2 \left[\frac{2m+1}{2v+1} + 2(m+v+1) \sum_{j=0, j\neq v}^{m} \frac{2j+1}{(j-v)(j+v+1)}\right], \quad b_v^{(2m+1)} = (m+v+1)\binom{m+v}{m}^2 \binom{m}{v}^2. \tag{14}$$

For more clarity of the definition, the first few logarithmic Müntz orthogonal functions are as follows:

$$\begin{aligned} P_0(t) &= 1, \\ P_1(t) &= 1 + \ln(t), \\ P_2(t) &= -3 + 4t - \ln(t), \\ P_3(t) &= 9 - 8t + 2(1+6t)\ln(t), \\ P_4(t) &= -11 - 24t + 36t^2 - 2(1+18t)\ln(t), \\ P_5(t) &= 19 + 276t - 294t^2 + 3\left(1 + 48t + 60t^2\right)\ln(t), \\ P_6(t) &= -21 - 768t + 390t^2 + 400t^3 - 3\left(1 + 96t + 300t^2\right)\ln(t). \end{aligned} \tag{15}$$

Moreover, the following theorem adopted from [40] states that this new class of Müntz functions is orthogonal and has real distinct roots. In Table 1, the roots of $P_n(t)$ for $n = 1, 2, \ldots, 5$ are listed.

Theorem 7. *The Müntz functions $P_n(t)$ and $n \ge 0$ are orthogonal on the interval $[0,1]$, and $P_n(t)$ has exactly n distinct real and simple root in this interval.*

We now briefly explain the source of defining $\{P_n(t)\}$. Let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ be a complex sequence such that $\mathrm{Re}(\lambda_k) > -1/2$ and consider the rational function:

$$W_n(s) = \prod_{k=0}^{n-1} \frac{s + \overline{\lambda}_k + 1}{s - \lambda_k} \cdot \frac{1}{s - \lambda_n}, \quad n \in \mathbb{N}_0. \tag{16}$$

Then,

$$\begin{aligned} P_n(t) &= P_n(t; \Lambda_n) \\ &= \frac{1}{2\pi i} \oint_{\Gamma} W_n(s) t^s ds, \end{aligned} \tag{17}$$

where Γ is a simple contour surrounding all zero of the denominator of $W_n(s)$. In the special case, $\lambda_{2k} = \lambda_{2k+1} = k$ for $k = 0, 1, 2, \ldots$, i.e., taking $\lambda_{2k} = k$ and $\lambda_{2k+1} = k + \varepsilon$ where ε decreases to zero; then, by the limit process, (16) becomes

$$W_n(s) = \begin{cases} \prod_{v=0}^{m-1} \left(\frac{s+v+1}{s-v}\right)^2 \frac{1}{s-m} & n = 2m, \\ \prod_{v=0}^{m} \left(\frac{s+v+1}{s-v}\right)^2 \frac{1}{s+m+1} & n = 2m+1. \end{cases} \tag{18}$$

By applying the Cauchy residue theorem to the integral in (17) and by the above $W_n(s)$, the orthogonal Müntz polynomials with logarithmic terms given in (10) are obtained.

Remark 8. As in other types of orthogonal functions, by the change of variable $t = x - a/b - a$, we can convert the interval $[0,1]$ to the interval $[a,b]$ and define shifted Müntz orthogonal functions on $[a,b]$.

3.2. Function Approximation Using Orthogonal Müntz Basis.

Let $\{P_n(t)\}_{n=0}^{N}$ be the set of Müntz orthogonal functions and $Y = \mathrm{span}\{P_0(t), P_1(t), \ldots, P_N(t)\}$. Moreover, we suppose that $f \in L^2(0,1)$ and $I_N f \in Y$ are the best approximation to f in Y, i.e.,

$$\forall y \in Y: \ \|f - I_N f\| \le \|f - y\|. \tag{19}$$

As $I_N f \in Y$, there exist unique coefficients $c_0, c_1, \ldots, c_N$ such that

$$\begin{aligned} f \simeq I_N f &= \sum_{n=0}^{N} c_n P_n(t) \\ &= C^T \varphi(t), \end{aligned} \tag{20}$$

where

$$C = [c_0, c_1, \ldots, c_N]^T, \tag{21}$$

and

$$\varphi = [P_0(t), P_1(t), \ldots, P_N(t)]^T. \tag{22}$$

Table 1: Roots of $P_n(t)$ for $n = 1, 2, \ldots, 5$.

n	t_1	t_2	t_3	t_4	t_5
1	0.3678794412	—	—	—	—
2	0.06442096633	0.6374173264	—	—	—
3	0.01871588194	0.2651887508	0.7969679223	—	—
4	0.007047297639	0.1154772486	0.4569410332	0.8683835323	—
5	0.003221796109	0.05672067679	0.2565492462	0.5974812127	0.9100748739

3.3. Fractional-Order Riemann–Liouville Integral Operator for Müntz Functions. Applying the fractional-order Riemann–Liouville integral operator I^α defined in (7) to the Müntz vector $\varphi(t)$, we can write

$$I^\alpha \varphi(t) = \overline{\varphi}(t,\alpha), \tag{23}$$

where

$$\overline{\varphi}(t,\alpha) = \left[I^\alpha P_0(t), I^\alpha P_1(t), \ldots, I^\alpha P_N(t)\right]. \tag{24}$$

Now, taking the Laplace transform of both sides of (10), we obtain

$$\begin{aligned} L\left[P_n(t)\right] &= L\left(\sum_{\nu=0}^{[n/2]} a_\nu^{(n)} t^\nu + \sum_{\nu=0}^{[n-1/2]} b_\nu^{(n)} t^\nu \ln(t)\right) \\ &= \sum_{\nu=0}^{[n/2]} a_\nu^{(n)} \frac{\Gamma(\nu+1)}{s^{\nu+1}} + \sum_{\nu=0}^{[n-1/2]} b_\nu^{(n)} \frac{\Gamma(\nu+1)}{s^{\nu+1}} \left(\sum_{k=1}^{\nu} \frac{1}{k} - \ln(s)\right). \end{aligned} \tag{25}$$

Using (7), we obtain

$$\begin{aligned} L\left[I^\alpha P_n(t)\right] &= L\left(\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * P_n(t)\right) \\ &= \sum_{\nu=0}^{[n/2]} a_\nu^{(n)} \frac{\Gamma(\nu+1)}{s^{\alpha+\nu+1}} + \sum_{\nu=0}^{[n-1/2]} b_\nu^{(n)} \frac{\Gamma(\nu+1)}{s^{\alpha+\nu+1}} \left(\sum_{k=1}^{\nu} \frac{1}{k} - \ln(s)\right). \end{aligned} \tag{26}$$

Computing the inverse Laplace transform of (26) gives $I^\alpha P_n(t)$, so that

$$I^\alpha P_n(t) = \sum_{\nu=0}^{[n/2]} a_\nu^{(n)} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} t^{\nu+\alpha} + \sum_{\nu=0}^{[n-1/2]} b_\nu^{(n)} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} t^{\nu+\alpha} \left(\sum_{k=1}^{\nu} \frac{1}{k} - \sum_{l=1}^{\nu+\alpha} \frac{1}{l} + \ln(t)\right). \tag{27}$$

It should be noticed that computing $I^\alpha P_n(t)$ via (27) reduces the computational complexity and speeds up the solution procedure.

4. Numerical Method and Error Estimations

In this section, we first derive a new Müntz collocation method for solving the fractional-order integro-differential (1) and (2), and then, we investigate some error estimations.

4.1. Müntz Collocation Method. In our new Müntz collocation method, we first approximate $D^{q_0} f(t)$ by using (20) to write

$$D^{q_0} f(t) = C^T \varphi(t) = c_0 P_0(t) + c_1 P_1(t) + \ldots + c_N P_N(t). \tag{28}$$

Next, from (23), assumption (2), and the property 4 in subsection 2.2, we arrive at

$$f(t) = C^T \overline{\varphi}(t, q_0) + \sum_{k=0}^{m_0-1} \frac{t^k}{k!} d_k, \quad m_0 - 1 < q_0 \leq m_0. \tag{29}$$

Taking the Caputo fractional derivative of order q_i of both sides of (29) and according to the properties 3, 5, and 6 in subsection 2.2, we obtain

$$\begin{aligned} D^{q_i} f(t) &= D^{q_i}\left(C^T \overline{\varphi}(t, q_0) + \sum_{k=0}^{m_0-1} \frac{t^k}{k!} d_k\right) = D^{q_i} C^T \overline{\varphi}(t, q_0) + \sum_{k=0}^{m_0-1} \frac{D^{q_i}\left(t^k\right)}{k!} d_k \\ &= C^T D^{q_i} I^{q_0} \varphi(t) + \sum_{k=0}^{m_0-1} \frac{D^{q_i}\left(t^k\right)}{k!} d_k \\ &= C^T I^{q_0 - q_i} D^{q_0} I^{q_0} \varphi(t) + \sum_{k=0}^{m_0-1} \frac{D^{q_i}\left(t^k\right)}{k!} d_k \\ &= C^T \overline{\varphi}(t, q_0 - q_i) + \sum_{k=0}^{m_0-1} \frac{D^{q_i}\left(t^k\right)}{k!} d_k. \end{aligned} \tag{30}$$

Substituting equations (28), (29), and (30) into (1), we get

$$\begin{aligned} &F_1\Bigg(t, C^T \overline{\varphi}(t, q_0) + \sum_{k=0}^{m_0-1} \frac{t^k}{k!} d_k, C^T \varphi(t), C^T \overline{\varphi}(t, q_0 - q_1) \\ &+ \sum_{k=0}^{m_0-1} \frac{D^{q_1}\left(t^k\right)}{k!} d_k, \ldots, C^T \overline{\varphi}(t)(t, q_0 - q_u) + \sum_{k=0}^{m_0-1} \frac{D^{q_u}\left(t^k\right)}{k!} d_k\Bigg) \\ &- \lambda F_2\left(t, C^T \overline{\varphi}(t, q_0) \sum_{k=0}^{m_0-1} \frac{t^k}{k!} d_k, \int_0^t k(t, s) G\left(s, C^T \overline{\varphi}(t, q_0) + \sum_{k=0}^{m_0-1} \frac{t^k}{k!} d_k\right)\right) = 0. \end{aligned} \tag{31}$$

To approximate the integral term in (31), we utilize the Gauss–Legendre quadrature rule. To this end, the interval $[0, t]$ is transformed to $[-1, 1]$ to obtain

$$\begin{aligned} &F_1\Bigg(t, C^T \overline{\varphi}(t, q_0) + \sum_{k=0}^{m_0-1} \frac{t^k}{k!} d_k, C^T \varphi(t), C^T \overline{\varphi}(t, q_0 - q_1) \\ &+ \sum_{k=0}^{m_0-1} \frac{D^{q_1}\left(t^k\right)}{k!} d_k, \ldots, C^T \overline{\varphi}(t)(t, q_0 - q_u) + \sum_{k=0}^{m_0-1} \frac{D^{q_u}\left(t^k\right)}{k!} d_k\Bigg) \\ &- \lambda F_2\Bigg(t, C^T \overline{\varphi}(t, q_0) + \sum_{k=0}^{m_0-1} \frac{t^k}{k!} d_k, \sum_{j=0}^{m} \frac{t}{2} \omega_j \kappa\left(t, \frac{t}{2} + \frac{t}{2} \gamma_j\right) \\ &G\left(\frac{t}{2} + \frac{t}{2} \gamma_j, C^T \overline{\varphi}\left(\frac{t}{2} + \frac{t}{2} \gamma_j, q_0\right) + \sum_{k=0}^{m_0-1} \frac{\left(t/2 + t/2 \gamma_j\right)^k}{k!} dk\right)\Bigg) = 0, \end{aligned} \tag{32}$$

where γ_j, ω_j, and $j = 0, 1, \ldots, m$ are the corresponding Gauss–Legendre points and weights. By collocating (32) at the roots t_i, $i = 0, 1, \ldots, N$ of the Müntz function $P_{N+1}(t)$, we obtain the following collocation equations:

$$
\begin{aligned}
&F_1\Bigg(t_i, C^T\overline{\varphi}(t_i, q_0) + \sum_{k=0}^{m_0-1}\frac{t_i^k}{k!}d_k, C^T\varphi(t), C^T\overline{\varphi}(t_i, q_0 - q_1)\\
&+\sum_{k=0}^{m_0-1}\frac{D^{q_1}\left(t_i^k\right)}{k!}d_k, \ldots, C^T\overline{\varphi}(t_i)(t_i, q_0 - q_u) + \sum_{k=0}^{m_0-1}\frac{D^{q_u}\left(t_i^k\right)}{k!}d_k\Bigg)\\
&-\lambda F_2\Bigg(t, C^T\overline{\varphi}(t_i, q_0) + \sum_{k=0}^{m_0-1}\frac{t_i^k}{k!}d_k, \sum_{j=0}^{m}\frac{t_i}{2}\omega_j\kappa\left(t_i, \frac{t_i}{2} + \frac{t_i}{2}\gamma_j\right)\\
&G\left(\frac{t}{2} + \frac{t}{2}\gamma_j, C^T\overline{\varphi}\left(\frac{t}{2} + \frac{t}{2}\gamma_j, q_0\right) + \sum_{k=0}^{m_0-1}\frac{\left(t/2 + t/2\gamma_j\right)^k}{k!}dk\right)\Bigg) = 0.
\end{aligned}
\tag{33}
$$

Equation (33) for $i = 0, 1, \ldots, N$ gives a system of algebraic equations with $N+1$ equations and $N+1$ unknowns of the vector C^T. By solving this system of equations using standard solvers, such as the Newton iterative method, and using f in (29), the approximate solution of problem (1) is obtained. In the Newton iterative method, the starting point is very important, and to get a suitable starting point, we first solve the problem for a small value of N; based on the obtained answer, we choose the starting point for larger values of N.

4.2. Error Estimation. We first consider the following lemma that will be used in deriving our main convergence results.

Lemma 9. *We suppose that $f \in H^r(0,1)$ with integers $r \geq 0$ where [44],*

$$H^r(a,b) = \left\{v \in \mathbb{C}^{r-1}([a,b]): \frac{d}{dx}v^{r-1} \in L^2(a,b)\right\}, \tag{34}$$

is the Sobolev space. Let $I_N f = \sum_{n=0}^{N} c_n P_n(t)$ be the best approximation of f in Y. Then, if $r \leq N+1$

$$\left\|f - I_N f\right\|_{L^2(0,1)} \leq c(N+1)^{-r}\left\|f^{(r)}\right\|_{L^2(0,1)}, \tag{35}$$

and for $1 \leq \mu \leq r$, we have

$$\left\|f - I_N f\right\|_{H^\mu(0,1)} \leq c(N+1)^{2\mu-1/2-r}\left\|f^{(r)}\right\|_{L^2(0,1)}, \tag{36}$$

where c is a constant depending only on r.

Proof. We suppose that $L_N f$ is the truncated Legendre series of the function f; then, according to Eq. (5.4.11) in [45] for $r \leq N+1$, we have

$$\left\|f - L_N f\right\|^2_{L^2(0,1)} \leq c(N+1)^{-2r}\left\|f^{(r)}\right\|^2_{L^2(0,1)}. \tag{37}$$

As $I_N f$ is the best approximation of f in L^2- norm, we can write

$$
\begin{aligned}
\left\|f - I_N f\right\|^2_{L^2(0,1)} &= \left\|f - L_N f\right\|^2_{L^2(0,1)}\\
&\leq c(N+1)^{-2r}\left\|f^{(r)}\right\|^2_{L^2(0,1)},
\end{aligned}
\tag{38}
$$

which proves (35). Equation (36) is proved using equation (5.5.11) in [45] in a similar manner.

The next theorem establishes the convergence result of the proposed Müntz orthogonal collocation method. □

Theorem 10. *Let $f \in H^r(0,1)$ with integers $r \geq 0$, and*

$$K = \max|\kappa(t,s)|, \quad (t,s) \in [0,1]\times[0,1]. \tag{39}$$

Moreover, we suppose that the functions F_1, F_2, and G satisfy the Lipschitz condition (6) with Lipschitz constants η_1, η_2, and η_3, respectively. Then, the residual error E_N of the proposed collocation method has he following bound:

$$
\begin{aligned}
\left\|E_N\right\|_{L^2(0,1)} &\leq \left(\eta_1 + \lambda\eta_2 + \lambda K\eta_2\eta_3\right)c(N+1)^{-r}\left\|f^{(r)}\right\|_{L^2(0,1)}\\
&+ \eta_1\sum_{k=0}^{u}\frac{c(N+1)^{2\mu-1/2-r}\left\|f^{(r)}\right\|_{L^2(0,1)}}{\Gamma(m_k - q_k + 1)}.
\end{aligned}
\tag{40}
$$

Proof. According to (1), we have

$$\begin{aligned}\|E_N\|_{L^2(0,1)} = \Big\| & F_1\left(t, I_N f(t), D^{q_0} I_N f(t), D^{q_1} I_N f(t), \ldots, D^{q_u} I_N f(t)\right) \\ & - \lambda F_2\left(t, I_N f(t), \int_0^t \kappa(t,s) G(s, I_N f(s)) ds\right) \\ & - \left(F_1\left(t, f(t), D^{q_0} f(t), D^{q_1} f(t), \ldots, D^{q_u} f(t)\right)\right. \\ & \left. - \lambda F_2\left(t, f(t), \int_0^t \kappa(t,s) G(s, f(s)) ds\right)\right)\Big\|_{L^2(0,1)}.\end{aligned} \tag{41}$$

Since F_1, F_2, and G are Lipschitz functions, we deduce that

$$\|E_N\|_{L^2(0,1)} \leq \left(\eta_1 + \lambda\eta_2 + \lambda K \eta_2 \eta_3\right)\|f(t) - I_N f(t)\|_{L^2(0,1)} + \eta_1 \sum_{k=0}^{u} \|D^{q_k} f(t) - D^{q_k} I_N f(t)\|_{L^2(0,1)}. \tag{42}$$

Considering that $m_k - 1 < q_k \leq m_k$ and using equations (5), (7), and (36) and the property 5 in subsection 2.2, we obtain

$$\begin{aligned}\|D^{q_k} f(t) - D^{q_k} I_N f(t)\|^2_{L^2[0,1]} &= \|I^{m_k - q_k}\left(D^{m_k} f(t) - D^{m_k} I_N f(t)\right)\|^2_{L^2(0,1)} \\ &= \left\|\frac{1}{\Gamma(m_k - q_k)} t^{m_k - q_k - 1} * \left(D^{m_k} f(t) - D^{m_k} I_N f(t)\right)\right\|^2_{L^2(0,1)} \\ &\leq \left(\frac{1}{(m_k - q_k)\Gamma(m_k - q_k)}\right)^2 \|D^{m_k} f(t) - D^{m_k} I_N f(t)\|^2_{L^2(0,1)} \\ &\leq \left(\frac{1}{\Gamma(m_k - q_k + 1)}\right)^2 \|D^{m_k} f(t) - D^{m_k} I_N f(t)\|^2_{L^2(0,1)} \\ &\leq \left(\frac{1}{\Gamma(m_k - q_k + 1)}\right)^2 \|f(t) - I_N f(t)\|^2_{H^{\mu}(0,1)} \\ &\leq \left(\frac{1}{\Gamma(m_k - q_k + 1)}\right)^2 c(N+1)^{2\mu - 1/2 - r} \left\|f^{(r)}\right\|_{L^2(0,1)}.\end{aligned} \tag{43}$$

Combining equations (35), (42), and (43), the desired result in (40) is obtained. □

Remark 11. The upper bound (40) for the residual error shows that the residual error of the proposed Müntz collocation method converges exponentially to zeros as N increases, which demonstrates the spectral accuracy of the proposed method.

5. Numerical Examples

Example 1. We consider the fractional-order integro-differential equation [1, 4]:

$$\begin{aligned} & D^{0.5} f(t) = g(t) f(t) + h(t) + \sqrt{t} \int_0^t f^2(s) ds, \\ & f(0) = 0, \end{aligned} \tag{44}$$

with

$$g(t) = 2\sqrt{t}+2t^{3/2} - \left(\sqrt{t} + t^{3/2}\right)\ln(1+t),$$
$$h(t) = \frac{2Arc\sinh\sqrt{t}}{\sqrt{\pi}\sqrt{1+t}} - 2t^{3/2}. \quad (45)$$

The exact solution to this problem is $f(t) = \ln(1+t)$.

Taking $q_0 = 0.5$ and $m = 10$ and by solving an algebraic system of order $(N+1)\times(N+1)$ with various values of N, approximate solutions of (42) are obtained using the present Müntz collocation method. In Table 2, maximum absolute errors of the present method are compared with the composite functions method given in [1] (in which N and M are the orders of block-pulse and Bernoulli polynomials, respectively) and the fractional alternative Legendre function method given in [4] (in which m is the fixed nonnegative integer). In Figure 1, the logarithmic values of the absolute errors for different values of N are depicted, which show the exponential convergence.

Example 2. We consider the following system of fractional-order integro-differential equations [1, 46]:

$$\begin{cases} D^{\alpha} f_1(t) = g_1(t) - f_2(t) - \int_0^t f_1(s) + f_2(s)ds, \\ D^{\alpha} f_2(t) = g_2(t) + f_1(t) - \int_0^t f_1(s) - f_2(s)ds, \\ f_1(0) = 1, f_2(0) = -1, \end{cases} \quad (46)$$

with

$$g_1(t) = t^2 + t + 1,$$
$$g_2(t) = -t - 1. \quad (47)$$

For $\alpha = 1$, the exact solutions are $f_1(t) = t + e^t$ and $f_2(t) = t - e^t$.

By setting $q_0 = \alpha$, $m = 10$ and for various values of N algebraic systems of order $2(N+1)\times 2(N+1)$ with unknowns of the vectors C_1 and C_2 are obtained. In Tables 3 and 4, the absolute errors of approximations of f_1 and f_2 using the proposed method are compared with those given in [1, 46]. Furthermore, to show the exponential convergence, the logarithmic values of absolute errors are shown in Figure 2.

Example 3. In this example, we consider the following fractional differential equation [1]:

$$D^{\alpha} f(t) + f^{3/2}(t) = \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)} t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1) + \left(t^8 - 3t^{4+\alpha/2} + \frac{9}{4}t^{\alpha}\right)^{3/2},$$
$$f(0) = f'(0) = 0. \quad (48)$$

Table 2: Comparison between absolute errors for Example 1.

Methods	Absolute error
Present method	
$N = 4$	$8.4e-6$
$N = 6$	$1.8e-6$
$N = 8$	$2.4e-6$
$N = 10$	$3.9e-8$
$N = 12$	$3.4e-7$
Method in [1]	
$M = 4, N = 1$	$6.7e-5$
$M = 6, N = 1$	$2.5e-6$
$M = 4, N = 2$	$7.4e-6$
$M = 6, N = 2$	$1.3e-7$
Method in [4]	
$m = 4$	$2.6e-2$
$m = 6$	$8.7e-4$
$m = 8$	$9.7e-5$
$m = 10$	$1.2e-5$
$m = 12$	$1.0e-7$

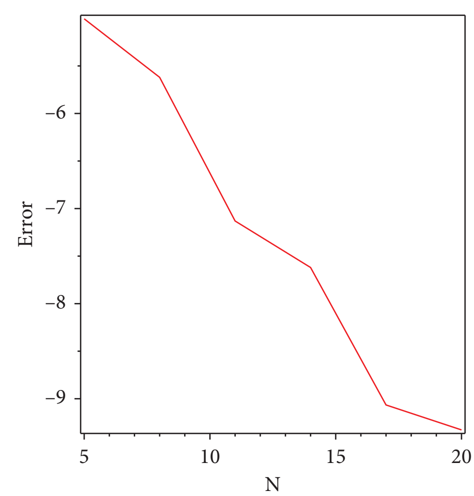

Figure 1: Logarithmic values of absolute errors for Example 1.

Table 3: Absolute errors of $f_1(t)$ for Example 2.

t	Present method ($N = 9$)	Method in [1] ($M = 5, N = 1$)	Method in [46] $M_2 = 5$
0.1	$4.2e-9$	$2.6e-7$	$1.0e-5$
0.2	$6.9e-9$	$2.0e-7$	$6.4e-5$
0.3	$7.3e-9$	$2.1e-7$	$9.2e-5$
0.4	$3.4e-9$	$2.3e-7$	$7.2e-5$
0.5	$9.7e-9$	$2.3e-7$	$1.5e-5$
0.6	$2.5e-9$	$2.2e-7$	$4.6e-5$
0.7	$1.0e-8$	$2.4e-7$	$7.6e-5$
0.8	$2.2e-10$	$2.4e-7$	$5.6e-5$
0.9	$7.1e-9$	$1.7e-7$	$5.1e-6$

Table 4: Absolute errors of $f_2(t)$ for Example 2.

t	Present method ($N = 9$)	Method in [1] ($M = 5, N = 1$)	Method in [46] $M_2 = 5$
0.1	$3.6e-9$	$1.9e-7$	$1.5e-5$
0.2	$6.1e-9$	$1.1e-7$	$6.3e-6$
0.3	$7.4e-9$	$9.9e-8$	$1.2e-5$
0.4	$2.2e-9$	$1.0e-7$	$4.9e-6$
0.5	$9.9e-9$	$9.0e-8$	$3.8e-5$
0.6	$1.1e-9$	$7.7e-8$	$6.7e-5$
0.7	$1.0e-8$	$9.5e-8$	$7.2e-5$
0.8	$1.5e-9$	$9.1e-8$	$4.5e-5$
0.9	$7.5e-9$	$2.1e-8$	$3.1e-6$

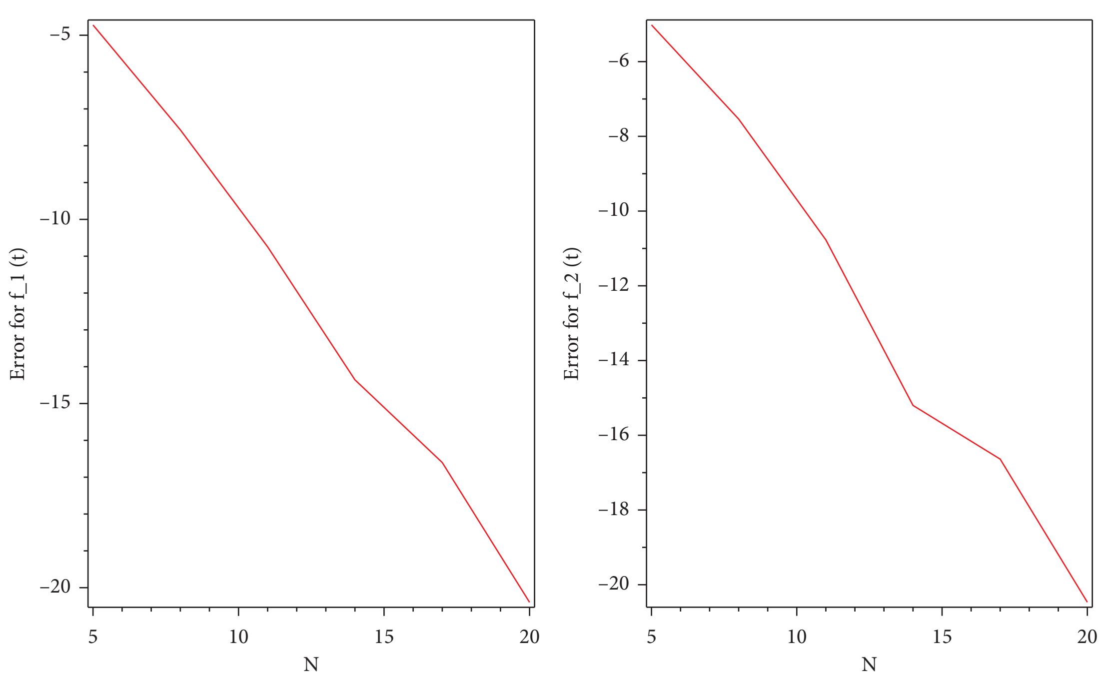

Figure 2: Logarithmic values of absolute errors for Example 2.

Table 5: Absolute errors of $f(t)$ for Example 3.

t	Present method ($N=14$) $\alpha=0.25$	Method in [1] ($M=8, N=1$) $\alpha=0.25$	Present method ($N=14$) $\alpha=1.25$	Method in [1] ($M=8, N=1$) $\alpha=1.25$
0.1	$1.4e-9$	$5.9e-8$	$1.4e-11$	$5.5e-6$
0.2	$7.5e-9$	$5.6e-8$	$8.0e-12$	$5.9e-6$
0.3	$7.4e-9$	$2.6e-8$	$1.0e-11$	$6.2e-6$
0.4	$5.6e-9$	$2.0e-8$	$2.0e-11$	$6.2e-6$
0.5	$9.5e-10$	$1.5e-8$	$2.7e-11$	$6.0e-6$
0.6	$5.0e-9$	$1.3e-8$	$1.9e-11$	$5.7e-6$
0.7	$8.7e-9$	$8.2e-9$	$1.6e-12$	$5.3e-6$
0.8	$9.2e-9$	$2.4e-8$	$1.1e-11$	$4.8e-6$
0.9	$1.0e-8$	$7.1e-8$	$9.4e-12$	$4.5e-6$

Table 6: Comparison between absolute errors for Example 4.

Method	Absolute error
Present method	
$N=2$	$6.8e-5$
$N=4$	$4.3e-7$
$N=5$	$2.7e-8$
$N=8$	$1.6e-10$
$N=10$	$7.2e-11$
$N=12$	$7.2e-13$
$N=14$	$2.0e-13$
Method in [1]	
$M=2, N=1$	$8.0e-4$
$M=2, N=2$	$8.0e-5$
$M=2, N=3$	$8.0e-6$
Method in [47]	
$M_1=2$	$6.7e-3$

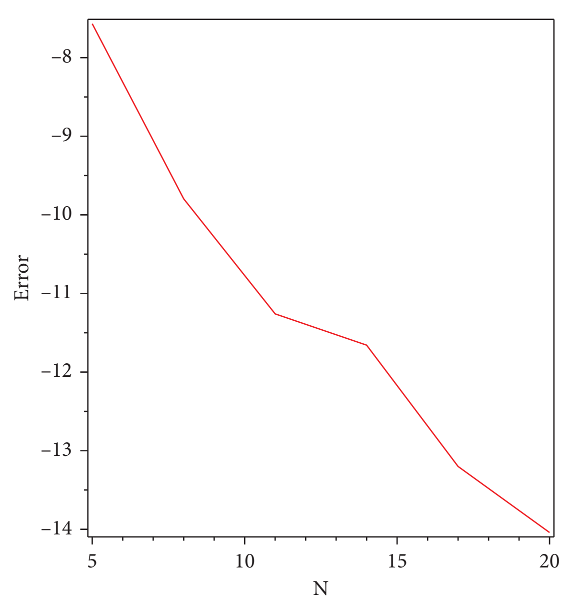

Figure 3: Logarithmic values of absolute errors for Example 4.

For $1 < \alpha \leq 2$, the exact solution is $f(t) = t^8 - 3t^{4+\alpha/2} + 9/4t^{\alpha}$. Again, we set $q_0 = \alpha$ and $m = 10$ and for $\alpha = 0.25, 1.25$; the errors are given in Table 5.

Example 4. We consider the following multiorder fractional integro-differential equation [1, 47]:

$$D^{5/3} f(t) + f^{'}(t) + t f(t) = g(t) + \frac{1}{2}\int_0^t (t-s)^2 f(s)ds,$$
$$f(0) = f^{'}(0) = 0, \tag{49}$$

with the exact solution $f(t) = t^{10/3}/\Gamma(13/3) + 2t^{11/3}/\Gamma(14/3) - t^4/12$.

Here, we set $q_0 = 5/3$, $q_1 = 1$, and $m = 5$. Table 6 reports the absolute errors, and Figure 3 depicts the logarithm values of absolute errors for different values of N.

6. Conclusion

A new Müntz orthogonal collocation method has been proposed for the numerical solution of fractional-order integro-differential equations with the Caputo derivative. The fractional integral operator associated with Müntz orthogonal functions was derived that assists in reducing the computational cost. An error bound for approximating function using Müntz series was obtained, and then, the behavior of the residual error of the proposed collocation method was analyzed. To demonstrate the applicability and high accuracy of the method, several numerical examples were solved. Comparisons between the present spectral collocation method and some other spectral methods in the literature show the superiority of the present method.

Data Availability

The data that support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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