Hindawi
Journal of Mathematics
Volume 2023, Article ID 6677650, 9 pages
https://doi.org/10.1155/2023/6677650

Research Article

Iterative Construction of Fixed Points for Functional Equations and Fractional Differential Equations

Latif Ur Rahman,[1] Muhammad Arshad,[1] Sabri T. M. Thabet ,[2] and Imed Kedim[3]

[1]*Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan*
[2]*Department of Mathematics, Radfan University College, University of Lahej, Lahej, Yemen*
[3]*Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia*

Correspondence should be addressed to Sabri T. M. Thabet; th.sabri@yahoo.com

Received 14 January 2023; Revised 24 September 2023; Accepted 21 October 2023; Published 13 November 2023

Academic Editor: Xiaolong Qin

This paper proposes some iterative constructions of fixed points for showing the existence and uniqueness of solutions for functional equations and fractional differential equations (FDEs) in the framework of CAT (0) spaces. Our new approach is based on the M^*-iterative scheme and the class of mappings with the KSC condition. We first obtain some Δ and strong convergence theorems using M^*-iterative scheme. Using one of our main results, we solve a FDE from a broad class of fractional calculus. Eventually, we support our main results with a numerical example. A comparative numerical experiment shows that the M^*-iterative scheme produces high accurate numerical results corresponding to the other schemes in the literature. Our results are new and generalize several comparable results in fixed point theory and applications.

1. Introduction

Fixed point theory in recent years has suggested very useful techniques for solving nonlinear problems (for details, see the survey article by Karapinar [1, 2]). Iterative solutions for functional equations and FDEs are a busy field of research on their own [3]. It is known that a sought solution of a functional equation or a FDE can be expressed as a fixed point of a certain linear or nonlinear operator acting on a subset of a suitable distance space (see, e.g., [4] and others). The existence as well as the iterative construction of the fixed point of this operator is always desirable. We know that, the existence of a fixed point is possible but to construct a suitable algorithm to approximate the value of the fixed point is not an easy work (see, e.g., [5, 6] and others). For example, the Banach Contraction Principle (BCP) [7] suggests a unique fixed point for contractions and suggests the Picard iteration [8], that is, $\nu_i = \Psi\nu_i$, to approximate the values of this unique fixed point [9]. On the other hand, the Browder–Gohde–Kirk result (see Browder [10], Gohde [11], and Kirk [12]) proved that every nonexpansive mapping on a closed convex bounded subset of a uniformly convex Banach space (UCBS) has a fixed point. Notice that a self-map Ψ on a subset V of a metric space is essentially called a contraction if

$$d(\Psi\nu, \Psi\xi) \leq \alpha d(\nu, \xi), \quad \text{for all } \nu, \xi \in V, \tag{1}$$

where $\alpha \in [0, 1)$.

A fixed point of Ψ in this case is any element, namely, $z \in V$ with the property $z = \Psi z$. The set of all fixed points of the operator Ψ will be denoted simply by F_Ψ throughout in this research paper. If (1) holds when we put $\alpha = 1$ then Ψ is known as nonexpansive. An example of a nonexpansive mapping for which Picard iteration does not converge is the following:

Example 1. Let $V = [0, 1]$ and $\Psi\nu = 1 - \nu$ for all $\nu \in V$. It follows that Ψ is nonexpansive with $F_\Psi = \{0.5\}$ and the Picard iteration is not convergent to 0.5 for all the starting value which is different from 0.5.

Example 1 suggests other iterative schemes instead of Picard iteration [8] which are convergent in the setting of nonexpansive mappings (or even generalized nonexpansive mappings). In 2008, Suzuki [13] introduced a condition on mappings, called (C) condition.

Definition 1 (see [13]). The self-map Ψ of V is said to satisfy the (C) condition of Suzuki if

$$\frac{1}{2}d(\nu,\Psi\nu)\leq d(\nu,\xi)\Longrightarrow d(\Psi\nu,\Psi\xi)\leq\alpha d(\nu,\xi),\quad \text{for all } \nu,\xi\in V. \tag{2}$$

The (C) condition is essentially weaker than the nonepensiveness property of any operator Ψ. For instance, see an example in [13].

Strongly motivated by Suzuki [13], the author Karapinar and Taş [14] suggested another condition for mappings.

Definition 2 (see [14]). The self-map Ψ of V is said to satisfy KSC condition (or said to satisfy Kannan–Suzuki (C) condition) if

$$\frac{1}{2}d(\nu,\Psi\nu)\leq d(\nu,\xi)\Rightarrow d(\Psi\nu,\Psi\xi)\leq\frac{1}{2}d(\nu,\Psi\nu)+d(\xi,\Psi\xi),\quad \text{for all } \nu,\xi\in V. \tag{3}$$

In fact, there are many iterative schemes in the literature, that are extensively used for approximating fixed points in different settings of mappings, (see e.g., Mann [15], Ishikawa [16], S-iteration of Agarwal et al. [17], three-step iteration of Noor [18], Abbas [19], Thakur et al. [20], and others).

Ullah and Arshad [21] constructed a new iteration called M^*-iteration and proved that this iteration is stable and suggests highly accurate results corresponding to other iterations of the literature. This iteration generates a sequence $\{\nu_i\}$ as follows:

$$\begin{cases}\nu_1\in V,\\ \omega_i=(1-b_i)\nu_i+b_i\Psi\nu_i,\\ \xi_i=\Psi\left[(1-a_i)\nu_i+a_i\Psi\omega_i\right],\\ \nu_{i+1}=\Psi\xi_i.\end{cases} \tag{4}$$

In the scheme (4), the operator Ψ is a self-map of the set V and the sequences $\{a_i\}$ and $\{b_i\}$ are in the interval (0,1). Although Ullah and Arshad [21] proved the convergence of the scheme (4) in the case of contractions. We extend here their main outcome to the more general setting of mappings satisfying the KSC condition. Using the same techniques, convergence of the above mentioned iterations can be proved on the same line of proof. Using a nontrivial example, we show that the iteration scheme M^* suggests accurate results corresponding to the other iterations in this new setting of mappings.

2. Preliminaries

Now, we need some basic results of CAT (0) spaces. For more details on CAT (0) spaces, please see the books [22, 23].

We now state a result from [24].

Lemma 3. *Suppose B is any complete CAT (0) space and $\varnothing\neq V\subseteq B$. Then,*

(a) *If we have $\nu,\xi\in B$ and there is a fixed element θ in the set [0,1], then one has a unique point $q\in[\nu,\xi]$, such that*

$$d(\nu,q)=\theta d(\nu,\xi)\text{ and } d(\xi,q)=(1-\theta)d(\nu,\xi). \tag{5}$$

Sometimes, we may write $(1-\theta)\nu\oplus\theta\xi$ as the unique point q that satisfies (5).

(b) *If we have $\nu,\xi,\omega\in B$ and $\theta\in[0,1]$ is fix, then one has*

$$d(\omega,\theta\nu\oplus(1-\theta)\xi)\leq\theta d(\omega,\nu)+(1-\theta)d(\omega,\xi). \tag{6}$$

Definition 4. Let $\{\nu_i\}$ be a bounded sequence in a metric space B and $\varnothing\neq V\subseteq B$ closed convex. We denote and set the asymptotic radius of the sequence $\{\nu_i\}$ in the set V as $R(V,\{\nu_i\})=\inf\{\limsup_{i\longrightarrow\infty}d(\nu_i,\nu)\colon \nu\in V\}$. We denote and set the asymptotic center of the sequence $\{\nu_i\}$ in the set V as $A(V,\{\nu_i\})=\{\nu\in V\colon \limsup_{i\longrightarrow\infty}(\nu_i,\nu)=R(V,\nu_i)\}$. If B is a complete CAT (0) space then $A(V,\{\nu_i\})$ contains one and only one point.

The following is the definition of a Δ convergence that can be considered as an analog of the weak convergence in a Banach space.

Definition 5. A bounded sequence, namely, $\{\nu_i\}$ in a complete CAT (0) space B is said to be Δ-convergent to a point, namely, $z\in B$ (and denote it as $\Delta-\lim_i\nu_i=z$) if it is the case that the point z is the unique asymptotic center for each choice of the subsequence $\{s_i\}$ of $\{\nu_i\}$.

The CAT (0) version of the Opial's [25] condition holds, that is, if $\{\nu_i\}$ is any Δ-convergent sequence in a complete CAT (0) space B with the Δ limit z, then for any $y\neq z\in B$, one has

$$\limsup_{i\longrightarrow\infty} d(\nu_i, z) < \limsup_{i\longrightarrow\infty} d(\nu_i, y). \tag{7}$$

Lemma 6 (see [26]). *Suppose we have complete CAT (0) space B. Then any bounded sequence $\{\nu_i\}\subseteq B$ admits a Δ-convergent subsequence.*

Lemma 7 (see [27]). *Suppose we have complete CAT (0) space B. If $\varnothing \neq V\subseteq B$ is convex and closed then the asymptotic center of any bounded sequence $\{\nu_i\}$ is contained in the space B.*

Definition 8 (see [28]). A self-map Ψ on a subset V of a CAT (0) space is said to satisfy condition (I) if one has a function γ with $\gamma(0) = 0$, $\gamma(u) > 0$ for each $u > 0$ and $d(\nu, \Psi\nu) \geq \gamma(\text{dist}(\nu, F_\Psi))$ for every point $\nu \in V$, where the notation $\text{dist}(\nu, F_\Psi)$ is the distance of the point ν to the set F_Ψ.

Lemma 9 (see [14]). *Suppose B is any CAT (0) space and $\varnothing \neq V\subseteq B$. Let Ψ be a self-map of V satisfying KSC condition with $F_\Psi \neq \varnothing$. Then for any $\nu \in V$ and $z \in F_\Psi$, one has the following property:*

$$d(\Psi\nu, z) \leq d(\nu, z). \tag{8}$$

Lemma 10 (see [14]). *Suppose B is any CAT (0) space and $\varnothing \neq V\subseteq B$. Let T be a self-map of V satisfying KSC condition. Then for any $\nu, \xi \in V$, one has the following property:*

$$d(\nu, \Psi\xi) \leq 5d(\nu, \Psi\nu) + d(\nu, \xi). \tag{9}$$

Lemma 11 (see [14]). *Suppose B is any complete CAT (0) space and $\varnothing \neq V\subseteq B$. Let Ψ be a self-map of V satisfying KSC condition. Then, the following property holds:*

$$\{\nu_i\}\subseteq V, \Delta-\lim_i \nu_i = z, d(\nu_i, \Psi\nu_i) \longrightarrow 0 \Rightarrow \Psi z = z. \tag{10}$$

Lemma 12 (see [29]). *Consider $0 < j \leq b_i \leq k < 1$, for every choice of $i \geq 1$. If $\{\nu_i\}$ and $\{\xi_i\}$ be two sequences in a complete CAT (0) space B with $\limsup_{i\longrightarrow\infty} d(\nu_i, \nu_0) \leq r$ and $\limsup_{i\longrightarrow\infty} d(\xi_i, \nu_0) \leq r$ and $\limsup_{i\longrightarrow\infty} d((1-b_i)\xi_i \oplus a_i \nu_i, \nu_0) = r$ for a real number $r \geq 0$, and some $\nu_0 \in B$ then $\lim_{i\longrightarrow\infty} d(\nu_i, \xi_i) = 0$.*

3. Main Results

First, we define the CAT (0) space version of M^* iterative scheme (4) as follows:

$$\begin{cases} \nu_1 \in V, \\ \omega_i = (1-b_i)\nu_i \oplus b_i \Psi\nu_i, \\ \xi_i = \Psi[(1-a_i)\nu_i \oplus a_i \Psi\omega_i], \\ \nu_{i+1} = \Psi\xi_i. \end{cases} \tag{11}$$

Now, using (11), we prove our man results. We first provide the following lemma that will play a key role.

Lemma 13. *Suppose B is any complete CAT (0) space and $\varnothing \neq V \subseteq B$ is closed and convex. Let Ψ be a self-map of V satisfying the (KSC) condition with $F_\Psi \neq \varnothing$. Then the sequence generated by M^*-iteration (11) satisfies $\lim_{i\longrightarrow\infty} d(\nu_i, z)$ exists for each $z \in F_\Psi$.*

Proof. Consider any point $z \in F_\Psi$, then applying Lemma 9, one has

$$\begin{aligned} d(\nu_{i+1}, z) &= d(\Psi\xi_i, z) \\ &\leq d(\xi_i, z) \\ &= d(\Psi[(1-a_i)\nu_i \oplus a_i\Psi\omega_i], z) \\ &\leq d((1-a_i)\nu_i \oplus a_i\Psi\omega_i, z) \\ &\leq (1-a_i)d(\nu_i, z) + a_i d(\Psi\omega_i, z) \\ &\leq (1-a_i)d(\nu_i, z) + a_i d(\omega_i, z) \\ &= (1-a_i)d(\nu_i, z) + a_i(d(1-b_i)\nu_i + b_i d(\Psi\nu_i, z)) \\ &\leq (1-a_i)d(\nu_i, z) + a_i((1-b_i)d(\nu_i, z) + b_i d(\Psi\nu_i, z)) \\ &\leq (1-a_i)d(\nu_i, z) + a_i((1-b_i)d(\nu_i, z) + b_i d(\nu_i, z)) \\ &= (1-a_i)d(\nu_i, z) + a_i d(\nu_i, z) \\ &= d(\nu_i, z). \end{aligned} \tag{12}$$

Hence, we obtained for all $z \in F_\Psi$, $d(\nu_{i+1}, z) \leq d(\nu_i, z)$. This means that $\{d(\nu_i, z)\}$ is essentially bounded as well as nonincreasing and hence it follows that $\lim_{i\longrightarrow\infty} d(\nu_i, z)$ exists for all $z \in F_\Psi$.

Now, for the existence of a fixed point, we give the necessary and sufficient condition for mappings with (KSC) condition defined on nonempty closed convex subsets of a UCBS as follows. □

Theorem 14. *Suppose B is any complete CAT (0) space and $\varnothing \neq V\subseteq B$ is closed and convex. If Ψ is a self-map of V satisfying KSC condition and $\{\nu_i\}$ is the sequence of M^*-iteration (11). Then, $F_\Psi \neq \varnothing$, if and only if $\{\nu_i\}$ is bounded and satisfies $\lim_{i\longrightarrow\infty} d(\Psi\nu_i, \nu_i) = 0$.*

Proof. First, we assume the case that the set $F_\Psi \neq \varnothing$ and prove that $\{\nu_i\}$ is bounded with $\lim_{i\longrightarrow\infty} d(\nu_i, \Psi\nu_i) = 0$. For this, Lemma 13 suggests that $\{\nu_i\}$ is bounded and $\lim_{i\longrightarrow\infty} d(\nu_i, \Psi\nu_i)$ exists. Put

$$\lim_{i\longrightarrow\infty} d(\nu_i, \Psi\nu_i) = r, \tag{13}$$

for some $r \in \mathbb{R}^+$. We assume the nontrivial case, that is, when $r > 0$. Then in the view of the proof of Lemma 13, $d(\omega_i, z) \leq d(\nu_i, z)$. It follows that

$$\limsup_{i\longrightarrow\infty} d(\omega_i, z) \leq \limsup_{i\longrightarrow\infty} d(\nu_i, z) = r. \tag{14}$$

Now, $d(\Psi\nu_i, z) \leq d(\nu_i, z)$ from Lemma 9. So,

$$\limsup_{i\longrightarrow\infty} d(\Psi\nu_i, z) \leq \limsup_{i\longrightarrow\infty} d(\nu_i, z) = r. \tag{15}$$

Again, we see that, $d(\nu_{i+1}, z) \leq (1-a_i)d(\nu_i, z) + a_i d(\omega_i, z)$ form the proof of Lemma 13. It follows that $d(\nu_{i+1}, z) \leq d(\omega_i, z)$. So,

$$r=\liminf_{i\longrightarrow\infty} d(\nu_{i+1},z)\le\liminf_{i\longrightarrow\infty} d(\omega_i,z). \tag{16}$$

Thus from (14) and (16), we have

$$\lim_{i\longrightarrow\infty} d(\omega_i,z)=r. \tag{17}$$

From (17), we have

$$r=\lim_{i\longrightarrow\infty} d((1-b_i)\nu_i\oplus b_i\Psi\nu_i,z). \tag{18}$$

Now, applying Lemma 12 on (13), (15), and (18), we obtain

$$\lim_{i\longrightarrow\infty} d(\Psi\nu_i,\nu_i)=0. \tag{19}$$

Finally, we shall assume $\{\nu_i\}$ is bounded with the property $\lim_{i\longrightarrow\infty} d(\Psi\nu_i,\nu_i)=0$ and show that the set $F_\Psi\neq\varnothing$. For this, we may assume any point, namely, z in the set $A(V,\{\nu_i\})$. By Lemma 10, we have

$$\begin{aligned}R(\Psi z,\{\nu_i\})&=\lim_{i\longrightarrow\infty}\sup d(\nu_i,\Psi z)\\&\le\lim_{i\longrightarrow\infty}\sup(5d(\Psi\nu_i,\nu_i)+d(\nu_i,z))\\&\le 0+\lim_{i\longrightarrow\infty}\sup d(\nu_i,z)\\&=R(z,\{\nu_i\}).\end{aligned} \tag{20}$$

This implies that $\Psi z\in A(V,\{\nu_i\})$. Since $A(V,\{\nu_i\})$ is singleton, hence, we have $\Psi z=z$ and hence $F_\Psi\neq\varnothing$.

We first suggest a Δ convergence result. □

Theorem 15. *Suppose B is any complete CAT (0) space and $\varnothing\neq V\subseteq B$ is closed and convex. Let Ψ be a self-map of V satisfying the KSC condition with $F_\Psi\neq\varnothing$. Then the sequence of the M^*-iteration $\{\nu_i\}$ (11) Δ-converges to some fixed point of Ψ provided that the space B has Opial's property.*

Proof. Using Theorem 14, we have the sequence of iterates $\{\nu_i\}$ is bounded in the set V and satisfies the condition $\lim_{i\longrightarrow\infty} d(\nu_i,\Psi\nu_i)=0$. Set $\omega_\Delta(\{\nu_i\})=\cup A(\{s_i\})$, where $\{s_i\}$ denotes any subsequences $\{\nu_i\}$. We prove $\omega_\Delta(\{\nu_i\})\subseteq F_\Psi$. To achieve the objective, we let $s\in\omega_\Delta(\{\nu_i\})$, thus one can find a subsequence $\{s_i\}$ of $\{\nu_i\}$ such that $A(\{s_i\})=\{s\}$. Applying Lemmas 6 and 7, one has a subsequence $\{r_i\}$ of $\{s_i\}$ such that $\{r_i\}\Delta$ converges to a point r in B. Now, using Theorem 14, we have $\lim_{i\longrightarrow\infty} d(r_i,\Psi r_i)=0$. Also, Ψ is endowed with KSC condition, therefore

$$d(r_i,\Psi r)\le 5d(r_i,\Psi r_i)+d(r_i,r). \tag{21}$$

Applying limit on (21), it follows that $r\in F_\Psi$. Hence, using Lemma 13, one has $\lim_{i\longrightarrow\infty} d(r_i,r)$ exists. The next aim is to obtain that $s=r$. We prove this by contradiction, that is, we assume that $s\neq r$. Keeping the uniqueness of asymptotic centers in mind, one has

$$\begin{aligned}\limsup_{i\longrightarrow\infty} d(r_i,r)&<\limsup_{i\longrightarrow\infty} d(r_i,s)\le\limsup_{i\longrightarrow\infty} d(s_i,s)\\&<\limsup_{i\longrightarrow\infty} d(s_i,r)=\limsup_{i\longrightarrow\infty} d(\nu_i.r)\\&=\limsup_{i\longrightarrow\infty} d(r_i,r).\end{aligned} \tag{22}$$

Subsequently, we obtained $\limsup_{i\longrightarrow\infty} d(r_i,r)<\limsup_{i\longrightarrow\infty} d(r_i,r)$. Since this is a contradiction, we conclude that $s=r\in F_\Psi$ and hence $\omega_\Delta(\{\nu_i\})\subseteq F_\Psi$.

Eventually, we prove $\{\nu_i\}\Delta$-converges to a fixed point of Ψ, that is, we need to show $\omega_\Delta(\{\nu_i\})$ contains only one point. Suppose $\{s_i\}$ is a subsequence of $\{\nu_i\}$ and applying Lemmas 6 and 7, we have a Δ-convergent subsequence $\{r_i\}$ of $\{s_i\}$ that Δ-converges to a point r in B. Suppose $A(\{s_i\})=\{s\}$ and $A(\{\nu_i\})=\{q\}$. Then since we proved already that $s=r$ and $r\in F_\Psi$, we claim $q=r$. Because if $q\neq r$, then $\lim_{i\longrightarrow\infty} d(\nu_i,r)$ exists and keeping the uniqueness of asymptotic centers in mind, one has

$$\begin{aligned}\limsup_{i\longrightarrow\infty} d(r_i,r)&<\limsup_{i\longrightarrow\infty} d(r_i,q)\le\limsup_{i\longrightarrow\infty} d(b_i,q)\\&<\limsup_{i\longrightarrow\infty} d(\nu_i,r)=\limsup_{i\longrightarrow\infty} d(r_i.r),\end{aligned} \tag{23}$$

which is clearly a contradiction. Hence, we conclude that $q=r\in F_\Psi$. It follows that $\omega_\Delta(\{\nu_i\})=\{q\}$. So $\{\nu_i\}\Delta$-converges to a fixed point of Ψ.

The following theorem is based on compactness. □

Theorem 16. *Suppose B is any complete CAT (0) space and $\varnothing\neq V\subseteq B$ is compact and convex. Let Ψ be a self-map of V satisfying the KSC condition with $F_\Psi\neq\varnothing$. Then the sequence of the M^*-iteration (11) converges strongly to some fixed point of Ψ.*

Proof. As assumed, the set V is convex and compact, so the sequence of iterates $\{\nu_i\}$ contained in the set V and has a subsequence $\{\nu_{i_k}\}$ of $\{\nu_i\}$ that converges strongly to $\nu\in V$. Moreover, in the view of Theorem 14, we obtain $\lim_{i\longrightarrow\infty} d(\Psi\nu_{i_k},\nu_{i_k})=0$. Hence, using these facts together with Lemma 10, we have

$$d(\nu_{i_k},\Psi\nu)\le 5d(\Psi\nu_{i_k},\nu_{i_k})+d(\nu_{i_k},\nu)\longrightarrow 0 \text{ as } k\longrightarrow\infty. \tag{24}$$

It follows that $\Psi\nu=\nu$. By Lemma 13, $\lim_{i\longrightarrow\infty} d(\nu_i,\nu)$ exists and hence $\{\nu_i\}$ is strongly convergent to ν.

Strong convergence without compactness of the domain is the following. □

Theorem 17. *Suppose B is any complete CAT (0) space and $\varnothing\neq V\subseteq B$ is closed and convex. Let Ψ be a self-map of V satisfying the KSC condition with $F_\Psi\neq\varnothing$. Then, the sequence of the M^*-iteration (11) converges strongly to some fixed point of Ψ provided that* $\liminf_{i\longrightarrow\infty} dist(\nu_i,F_\Psi)=0$.

Proof. The proof of this result is easy and hence we exclude the proof. □

Theorem 18. *Suppose B is any complete CAT (0) space and $\varnothing \neq V \subseteq B$ is closed and convex. Let Ψ be a self-map of V satisfying the KSC condition with $F_\Psi \neq \varnothing$. Then, the sequence of the M^*-iteration (11) converges strongly to some fixed point of Ψ provided that the Ψ satisfies condition (I).*

Proof. According to Theorem 14$\liminf_{i\longrightarrow\infty} d(\xi_i, \Psi\xi_i) = 0$. Now, condition (I) of Ψ gives $\liminf_{i\longrightarrow\infty} \mathrm{dist}(\nu_i, F_\Psi) = 0$. Thus by Theorem 17, $\{\nu_i\}$ is strongly convergent in F_Ψ. □

4. Numerical Example

In this section, first we give a numerical example of a mapping with (KSC) condition which does not satisfy (C) condition and then we show that the sequence $\{\nu_i\}$ generated by M^*-iteration process converges faster than some other well-known iteration schemes.

Example 2. Define a mapping Ψ on $[-1, 1]$ as follows:

$$\Psi\nu = \begin{cases} -\dfrac{\nu}{2}, & \text{if } \nu \in [-1,0)\left\{-\dfrac{1}{2}\right\}, \\ 0, & \text{if } \nu = \left\{-\dfrac{1}{2}\right\}, \\ -\dfrac{\nu}{4}, & \text{if } \nu \in [0,1]. \end{cases} \tag{25}$$

Now, we see that the above self-map Ψ is not enriched with (C) condition. For example, if one chose $\nu = -(1/2)$ and $\xi = -(4/5)$, then Ψ does not satisfy the (C) condition. Eventually, we shall establish that this map is enriched with KSC condition. To achieve the objective, some elementary cases have been omitted, while nontrivial cases are considered as follows:

C1: When $\nu, \xi \in [-1, 0)/\{-(1/2)\}$, we have

$$\begin{aligned} d(\Psi\nu, \Psi\xi) &= d\left(\frac{\nu}{2}, \frac{\xi}{2}\right) \le \frac{3}{4}[d(\nu,\xi)] \le \frac{1}{2}\left[\left|\frac{3\nu}{2}\right| + \left|\frac{3\xi}{2}\right|\right] \\ &= \frac{1}{2}\left[\left|\left(\frac{-\nu}{2}\right) - \nu\right| + \left|\xi - \left(\frac{-\xi}{2}\right)\right|\right] \\ &= \frac{1}{2}[|\nu - \Psi\nu| + |\xi - \Psi\xi|] \\ &= \frac{1}{2}[d(\nu, \Psi\nu) + d(\xi, \Psi\xi)]. \end{aligned} \tag{26}$$

C2: When $\nu, \xi \in [0, 1]$, we have

$$\begin{aligned} d(\Psi\nu, \Psi\xi) &= d\left(\frac{\nu}{4}, \frac{\xi}{4}\right) \le \frac{1}{4}[|\nu| + |\xi|] \le \frac{3}{8}[|\nu| + |\xi|] \\ &= \frac{1}{2}\left[\left|\frac{3\nu}{4}\right| + \left|\frac{3\xi}{4}\right|\right] = \frac{1}{2}\left[\left|\frac{\nu}{4} - \nu\right| + \left|\xi - \frac{\xi}{4}\right|\right] \\ &= \frac{1}{2}[|\nu - \Psi\nu| + |\xi - \Psi\xi|] \\ &= \frac{1}{2}[d(\nu, \Psi\nu) + d(\xi, \Psi\xi)]. \end{aligned} \tag{27}$$

C3: When $\nu \in [-1, 0)/\{-(1/2)\}$ and $\xi \in [0, 1]$ we have

$$\begin{aligned} d(\Psi\nu, \Psi\xi) &= d\left(\frac{\nu}{2}, \frac{\xi}{4}\right) \le \frac{1}{2}|\nu| + \frac{1}{4}|\xi| \le \frac{3}{4}|\nu| + \frac{3}{8}|\xi| \\ &= \frac{1}{2}\left[\left|\frac{3\nu}{2}\right| + \left|\frac{3\xi}{4}\right|\right] = \frac{1}{2}\left[\left|\left(\frac{-\nu}{2}\right) - \nu\right| + \left|\xi - \frac{\xi}{4}\right|\right] \\ &= \frac{1}{2}[|\nu - \Psi\nu| + |\xi - \Psi\xi|] \\ &= \frac{1}{2}[d(\nu, \Psi\nu) + d(\xi, \Psi\xi)]. \end{aligned} \tag{28}$$

C4: When $\nu \in [-1, 0)/\{-(1/2)\}$ and $\xi \in \{-(1/2)\}$, we have

$$\begin{aligned} d(\Psi\nu, \Psi\xi) &= d\left(\frac{\nu}{2}, 0\right) = \left|\frac{\nu}{2}\right| \le \left|\frac{3\nu}{4}\right| \le \left|\frac{3\nu}{4}\right| + \left|\frac{\xi}{2}\right| \\ &= \frac{1}{2}\left[\left|\frac{3\nu}{2}\right| + |\xi|\right] = \frac{1}{2}\left[\left|\left(\frac{-\nu}{2}\right) - \nu\right| + |\xi - 0|\right] \\ &= \frac{1}{2}[|\nu - \Psi\nu| + |\xi - \Psi\xi|] \\ &= \frac{1}{2}[d(\nu, \Psi\nu) + d(\xi, \Psi\xi)]. \end{aligned} \tag{29}$$

C5: When $\nu \in [0, 1]$ and $\xi \in \{-(1/2)\}$, we have

$$\begin{aligned} d(\Psi\nu, \Psi\xi) &= d\left(\frac{\nu}{4}, 0\right) = \left|\frac{\nu}{4}\right| \le \left|\frac{3\nu}{8}\right| \le \left|\frac{3\nu}{8}\right| + \left|\frac{\xi}{2}\right| \\ &= \frac{1}{2}\left[\left|\frac{3\nu}{4}\right| + |\xi|\right] = \frac{1}{2}\left[\left|\frac{\nu}{4} - \nu\right| + |\xi - 0|\right] \\ &= \frac{1}{2}[|\nu - \Psi\nu| + |\xi - \Psi\xi|] \\ &= \frac{1}{2}[d(\nu, \Psi\nu) + d(\xi, \Psi\xi)]. \end{aligned} \tag{30}$$

Table 1: Some iterates of Ψ given in Example 2.

i	M^*	Thakur	Abbas	Noor	Ishikawa
1	0.99000000	0.99000000	0.99000000	0.99000000	0.99000000
2	0.07208437	0.07858125	0.13037100	0.13631062	0.28833800
3	0.00524864	0.00623738	0.01716820	0.01876826	0.08397830
4	0.00038216	0.00049509	0.00226083	0.002584156	0.02445800
5	0.00002782	0.00003929	0.00029772	0.00035580	0.00712359
6	0.00000202	0.00000311	0.00003920	0.00004899	0.00207475
7	0.00000014	0.00000024	0.00000516	0.00000674	0.00060427
8	**0**	0.00000001	0.00000067	0.00000092	0.00017599
9	0	**0**	0.00000008	0.00000012	0.00005125
10	0	0	0.00000001	0.00000001	0.00001492
11	0	0	**0**	**0**	0.00000434
12	0	0	0	0	0.00000126
13	0	0	0	0	0.00000036
14	0	0	0	0	0.00000010
15	0	0	0	0	0.0000003
16	0	0	0	0	**0**

The bold values indicate the first value tends to zero for every iteration scheme.

Now, we draw a graph and table which show that the sequence $\{\nu_i\}$ of the scheme M^*-iteration moving faster to the fixed point 0 of Ψ as compared to Thukar, Abbas, Noor, and Ishikawa iterative schemes. Assume that $a_i = 0.70$, $b_i = 0.65$, and $c_i = 0.90$. The iterative results are shown in Table 1 while the behavior of iterates are given in Figure 1. The effectiveness of M^* iterative scheme is clear in both the table and graph.

We finish the paper with a nontrivial example.

Example 3. Let $B_1 = \{(\nu, 0): \nu \in \mathbb{R}\}$ and $B_2 = \{(0, \xi): \xi \in \mathbb{R}\}$. Put $B = B_1 \cup B_2$. Clearly, $B \subset \mathbb{R}^2$. Define d on B as follows:

$$d((\nu_1, \nu_2), (\xi_1, \xi_2)) = \begin{cases} |\nu_1 - \xi_1|, & \text{if } \nu_2 = 0 = \xi_2, \\ |\nu_2 - \xi_2|, & \text{if } \nu_1 = 0 = \xi_1, \\ |\nu_1| + |\xi_2|, & \text{if } \nu_2 = 0 = \xi_1. \end{cases} \quad (31)$$

Here, the space (B, d) is only a CAT (0) space but not a Banach space [22]. Also, B is closed and convex. Now, let Ψ be the metric projection on B, then by a well-known result (see, p178 in [22]) that Ψ is nonexpansive and hence it satisfies KSC condition. By our main results, the sequence (11) converges to a fixed point of Ψ.

5. Application to Differential Equations

In this section, we study the solution of a FDE in our new setting of mappings. This problem has been considered by some authors in the class of nonexpansive mappings [30] and other types of spaces [31, 32]. It is important to note here that our approach is alternative and based on the class of mappings with KSC. The main difference between our approach and classical approaches to the problems are that mappings with KSC are not necessarily continuous throughout on the their domains. Moreover, our iterative method is more effective and suggests very high accurate numerical results in less step of iterates. To achieve the objective, we follow the idea given by [33].

We consider the following general class of boundary value problems from fractional calculus:

$$\left.\begin{array}{l} D^{\gamma} h(t) + \Omega(u, h(t)) = 0 \\ h(0) = h(1) = 0 \end{array}\right\}, \quad (32)$$

where $(0 \le t \le 1)$, $(1 < \gamma < 2)$, and D^{γ} stands for the Caputo fractional derivative with order γ and Ω: $[0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$.

Consider $B = C[0, 1]$ and Green's function associated with (32) that reads as follows:

$$G(t, s) = \begin{cases} \dfrac{1}{\Gamma(\xi)} t(1 - s)^{(\xi - 1)} - (t - s)^{(\xi - 1)}, & \text{if } s \le t \le 1, \\ \dfrac{t(1 - s)^{(\xi - 1)}}{\Gamma(\xi)}, & \text{if } t \le s \le 1. \end{cases} \quad (33)$$

The main result is provided in the following way.

Theorem 19. *Set a self-map Ψ: $B \longrightarrow B$ by the following formula:*

$$\Psi(\nu(t)) = \int_0^1 G(t, s)\Omega(s, \nu(s))ds, \quad \text{for each } \nu(t) \in B. \quad (34)$$

If

$$|\Omega(\nu, h(\nu)) - \Omega(\nu, g(\nu))| \le \frac{1}{2} (|h(\nu)\Psi(h(\nu))| + |g(\nu) - \Psi(g(\nu))|), \quad (35)$$

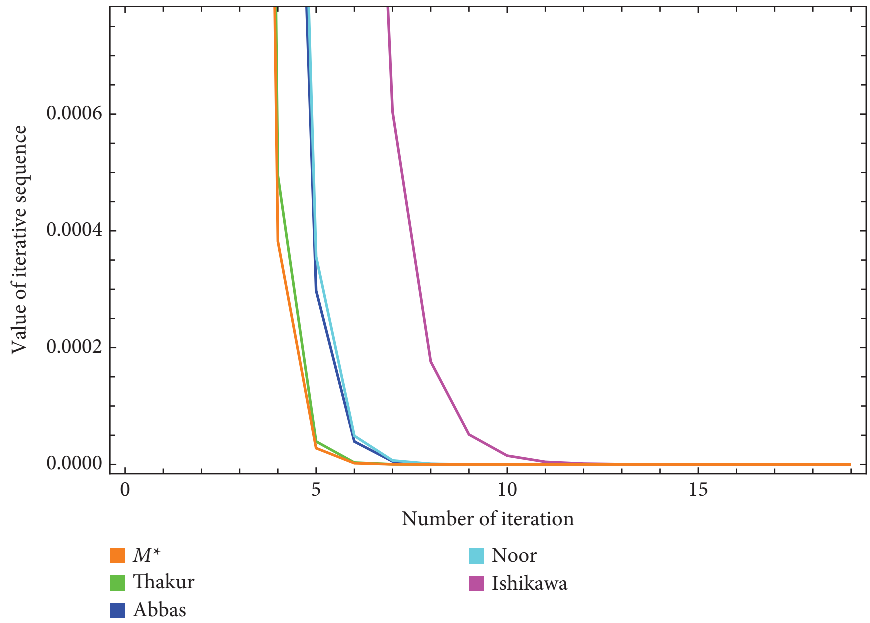

Figure 1: Graphical comparison of different iterations in the class of mappings with KSC condition.

then, the$M$ iterates (11) associated with the Ψ (as defiend above) converge to the point of S provided that $\liminf_{i\longrightarrow\infty}\text{dist}(v_i, S) = 0$, where S denotes the set of all solutions of (32).

Proof. Since G is a Green's function to our problem, so by [34], the sought solution can be expressed as an integral form as follows:

$$h(u) = \int_0^1 G(u,v)\Omega(v,h(v))\text{d}v. \tag{36}$$

Now, for every choice of $h, g \in B$ and $0 \le u \le 1$, it follows that

$$\begin{aligned}
d(\Psi(h(u)),\Psi(g(u))) &\le d\left(\int_0^1 G(u,v)\Omega(v,h(v))\right)\text{d}v, \int_0^1 G(u,v)\Upsilon(v,g(v))\text{d}v \\
&= \left|\int_0^1 G(u,v)[\Omega(v,h(v)) - \Omega(v,g(v))]\text{d}v\right| \\
&\le \int_0^1 G(u,v)|\Omega(v,h(v)) - \Omega(v,g(v))|\text{d}v \\
&\le \int_0^1 G(u,v)\left(\frac{1}{2}|h(v)) - \Psi(g(v))| + \frac{1}{2}|g(v) - \Psi(h(v))|\right)\text{d}v \\
&\le \left(\frac{1}{2}\|h(v)\right) - \Psi h(v)\| + \frac{1}{2}\|g(v) - \mathcal{J}(g(v))\| \\
&\quad \cdot \left(\sup_{t\in[0,1]}\int_0^1 G(u,v)\text{d}v\right) \\
&\le \frac{1}{2}d(h(v)),\Psi(h(v)) + \frac{1}{2}\text{d}(g(v)) - \Psi(g(v)) \\
&= \frac{1}{2}d(h(v)),\Psi(h(v)) + \frac{1}{2}\text{d}(g(v)) - \Psi(g(v)).
\end{aligned} \tag{37}$$

Consequently, we obtain

$$d(\Psi(h), \Psi(g)) \leq \frac{1}{2}(d(h, \Psi(h)) + d(g, \Psi(g))). \quad (38)$$

Hence, Ψ satisfies the (KSC) condition. In the view of Theorem 17, the sequence of the M^* iterates converges to a fixed point of Ψ and hence to the solution of the given equation. □

6. Conclusions

Existence as well as iterative constructional for the class of mappings satisfying the KSC condition is established under the iterative scheme M^* in a CAT (0) space setting. We proved Δ and strong convergence results for these mappings under certain mild conditions. It has been shown by providing an example that the class of mappings satisfying the KSC condition is different than the class of mappings satisfying (C) condition. Eventually, we performed a comparative numerical experiment and proved that the M^* iterative scheme in the class of KSC mappings is more effective than the many other iterative scheme. One application is also carried out. Our results refine and improve some main results due to Ullah and Arshad [21] from the case of the (C) condition to the more general case of KSC condition. Similarly, our results extend the results of Abbas and Nazir [19], Agarwal et al. [17], Noor [18], Thakur et al. [20], and others.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Every author contributed equally to each part of the paper.

Acknowledgments

This study is supported via funding from the Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1445).

References

[1] A. Das, B. Hazarika, V. Parvaneh, and M. Mursaleen, "Solvability of generalized fractional order integral equations via measures of noncompactness," *The Mathematical Scientist*, vol. 15, no. 3, pp. 241–251, 2021.
[2] C. Nuchpong, T. Khoployklang, S. K. Ntouyas, and J. Tariboon, "Boundary value problems for Hilfer-Hadamard fractional differential inclusions with nonlocal integro-multipoint boundary conditions," *J. Nonlinear Funct. Anal.*, vol. 37, 2022.
[3] B. Mohammadia, M. Paunović, V. Parvanahd, and M. Mursaleen, "Existence of solution for some ϕ-Caputo fractional differential inclusions via Wardowski-Mizoguchi-Takahashimulti-valued contractions," *Filomat*, vol. 37, no. 12, pp. 3777–3789, 2023.
[4] M. Paunovic, B. Mohammadi, and V. Parvaneh, "On weak wardowski contractions and solvability of p-caputo implicit fractional pantograph differential equation with generalized anti-periodic boundary conditions," *Journal of nonlinear and convex analysis*, vol. 23, no. 6, pp. 1261–1274, 2022.
[5] S. T. M. Thabet and I. Kedim, "Study of nonlocal multiorder implicit differential equation involving Hilfer fractional derivative on unbounded domains," *Journal of Mathematics*, vol. 2023, Article ID 8668325, 14 pages, 2023.
[6] M. Nazam, M. Arshad, C. Park, O. Acar, S. Yun, and G. A. Anastassiou, "On solution of a system of differential equations via fixed point theorem," *Journal of Computational Analysis and Applications*, vol. 27, no. 3, pp. 417–426, 2019.
[7] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations integrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.
[8] E. Picard, "Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives," *Journal de Mathematiques Pures et Appliquees*, vol. 6, pp. 145–210, 1880.
[9] H. M. Srivastava, A. Ali, A. Hussain, M. Arshad, and H. Al-Sulami, "A certain class of θL-type non-linear operators and some related fixed point results," *J. Nonlinear Var. Anal.*, vol. 6, pp. 69–87, 2022.
[10] F. E. Browder, "Nonexpansive nonlinear operators in a Banach space," *Proceedings of the National Academy of Sciences*, vol. 54, no. 4, pp. 1041–1044, 1965.
[11] D. Gohde, "Zum prinzip der Kontraktiven abbildung," *Mathematische Nachrichten*, vol. 30, no. 3-4, pp. 251–258, 1965.
[12] W. A. Kirk, "A fixed point theorem for mappings which do not increase distances," *The American Mathematical Monthly*, vol. 72, no. 9, pp. 1004–1006, 1965.
[13] T. Suzuki, "Fixed point theorems and convergence theorems for some generalized nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 2, pp. 1088–1095, 2008.
[14] E. Karapınar and K. Taş, "Generalized (C)–conditions and related fixed point theorems," *Computers & Mathematics with Applications*, vol. 61, no. 11, pp. 3370–3380, 2011.
[15] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, no. 3, pp. 506–510, 1953.
[16] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, no. 1, pp. 147–150, 1974.
[17] R. P. Agarwal, D. O'Regon, and D. R. Sahu, "Iterative construction of fixed points of nearly asymptotically nonexpansive mappings," *J. Nonlinear Convex Anal.*, vol. 8, pp. 61–79, 2007.
[18] M. A. Noor, "New approximation schemes for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 217–229, 2000.
[19] M. Abbas and T. Nazir, "A new faster iteration process applied to constrained minimization and feasibility problems," *Matematicki Vesnik*, vol. 66, pp. 223–234, 2014.
[20] B. S. Thakur, D. Thakur, and M. Postolache, "A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings," *Applied Mathematics and Computation*, vol. 275, pp. 147–155, 2016.
[21] K. Ullah and M. Arshad, "New iteration process and numerical reckoning fixed points in Banach spaces," *UPB*

Scientific Bulletin, Series A: Applied Mathematics and Physics, vol. 79, no. 4, pp. 113–122, 2017.

[22] M. Bridson and A. Haiger, *Metric Spaces of Non-positive Curvature*, Springer-Verlag, Berlin, Heidelberg, 1999.

[23] D. Burago, Y. Burago, and S. Ivanov, “A course in metric geometry,” *Graduate Studies in Math*, Amer Math Soc, Providence, RI, 2001.

[24] S. Dhompongsa and B. Panyanak, “On Δ-convergence theorems in CAT(0) spaces,” *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2572–2579, 2008.

[25] Z. Opial, “Weak convergence of the sequence of successive approximations for nonexpansive mappings,” *Bulletin of the American Mathematical Society*, vol. 73, no. 4, pp. 591–597, 1967.

[26] W. A. Kirk and B. Panyanak, “A concept of convergence in geodesic spaces,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 12, pp. 3689–3696, 2008.

[27] S. Dhompongsa, W. A. Kirk, and B. Panyanak, “Nonexpansive set-valued mappings in metric and Banach spaces,” *J. Nonlinear Convex Anal.*, vol. 8, pp. 35–45, 2007.

[28] H. F. Senter and W. G. Dotson, “Approximating fixed points of nonexpansive mappings,” *Proceedings of the American Mathematical Society*, vol. 44, no. 2, pp. 375–380, 1974.

[29] J. Schu, “Weak and strong convergence to fixed points of asymptotically nonexpansive mappings,” *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.

[30] J. Ali, M. Jubair, and F. Ali, “Stability and convergence of F iterative scheme with an application to the fractional differential equationFiterative scheme with an application tot the fractional differential equation,” *Engineering with Computers*, vol. 38, no. 1, pp. 693–702, 2020.

[31] S. T. M. Thabet, M. M. Matar, M. A. Salman, M. E. Samei, M. Vivas-Cortez, and I. Kedim, “On coupled snap system with integral boundary conditions in the $ \mathbb{G} $-Caputo sense,” *AIMS Mathematics*, vol. 8, no. 6, pp. 12576–12605, 2023.

[32] S. T M Thabet and M. B Dhakne, “On positive solutions of higher order nonlinear fractional integro-differential equations with boundary conditions,” *Malaya Journal of Matematik*, vol. 7, no. 1, pp. 20–26, 2019.

[33] M. I. Ayari and S. T. M. Thabet, “Qualitative properties and approximate solutions of thermostat fractional dynamics system via a nonsingular kernel operator,” *Arab Journal of Mathematical Sciences*, vol. 2023, 2023.

[34] S. Bayin, *Mathematical Methods in Science and Engineering*, WILEY, Hoboken, NY, USA, 2006.