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Research Article

Averaged Control Problems Governed by a Semilinear Distributed Systems

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In this work, we consider the regional averaged controllability (RAC) problem governed by a class of semilinear hyperbolic systems. We start by giving the definitions of the exact and approximate RAC systems. After that, we state the problem of RAC for semilinear systems. We propose two methods of solution: using a condition of the analytical operator to the nonlinear part of the system to characterize the optimal control via the fixed point theorem and the Hilbert Uniqueness Method (HUM) with an asymptotic condition on the nonlinear part to find the optimal control of the considered problem. Finally, we present a numerical example to show the effectiveness of the main results.

1. Introduction

The study of hyperbolic partial differential equations (PDE) is a historically important subject whose first steps go back to d'Alembert with the equation of waves and to Euler, with equations of the same name describing the evolution of a fluid [1]. An important example of hyperbolic PDEs is provided by the conservation laws of the first order [2], which appear quite naturally in physics, as soon as a balance of energy, mass, quantity of movement, matter... is carried out and that the phenomena of diffusion (thermal or viscosity) are neglected.

Solutions of this kind of problem have undulating characterizations. If a localized perturbation is occurred on the initial input, then points in space far from the support of the perturbation will not instantaneous feel the effects. With respect to a fixed spatiotemporal point, the disturbances have a finite propagation speed and move according to the characteristics of the equation. This property makes it possible to distinguish hyperbolic problems from elliptical or parabolic problems, where the perturbations of the initial conditions will have immediate effects on all points of the domain. Although there are specific requirements that depend on the family of PDEs being investigated, the concept of hyperbolicity is fundamentally qualitative, see [3, 4] for instance.

Hyperbolic systems, it is a part of distributed systems modeling many real-life problems in various areas [5]. Several problems of mechanics are hyperbolic, and therefore the study of hyperbolic problems is of substantial contemporary interest. Furthermore, we primarily use light, sound, and wave phenomena to perceive the outside world through sight and hearing. They are also utilized in metal smelting, medicine, in laser printers and communications technologies, etc. In particular, hyperbolic systems describe optimal control problems with constraints. In the last few years a substantial literature is focused on the study of the control of distributed nonlinear hyperbolic systems and especially bilinear and semilinear hyperbolic systems [6].

The idea of regional controllability is one of controllability's most significant practical applications, control problems when the objective is not fully characterized as a position have been discussed using this state, but refers only to a subregion ω of Ω. Exactly, it finds a control which directs the considered system, at the moment T, towards a prescribed function defined on a subregion ω of Ω.

For controllability problems, one considers a control system in the time interval $[0, T]$ and specifically inquires as to the best way to reach the space of executed instructions (exact controllability) or a dense set in the space of instructions (approximate controllability).

In real-life problems, parameter-dependent system modeling seems to be difficult, since the issues encountered, is that we work with space-temporal systems considered as a nonlinear problem. Others' difficulty comes from the existence and uniqueness of their solutions. Furthermore, to solve the average control problem associated with a semilinear system a complex formulation of the fixed point method is introduced and applied in infinite dimension.

Averaged control problems for semilinear distributed systems involve the design of control laws for a class of systems described by partial differential equations (PDEs). These systems are characterized by a linear spatial operator and a nonlinear temporal operator. The goal of an averaged control problem is to stabilize the system around a desired equilibrium state or to drive it to a target state while taking into account the effects of averaging.

In this case of an unknown value parameter, it is not possible to control each realization of the system by a single control using an independent control of the parameter. Which motivates this work, is the first time that we consider the averaged time of semilinear distributed systems. Such type of systems is important in theory as in applications and looked as a compromise between linear and nonlinear systems. The average controllability allows us to consider many types of nonautonomous systems. The average controllability introduced by Zuazua [7], purpose is to check the averaged state of a parameterized system instead of the state against the unknown parameter. Moreover, the problem of average controllability has recently been introduced in some papers [8–12].

The paper is organized as follows. In the second section, we begin by stating the problem and giving the definition of the RAC. The third section will be our main result, when we will present two methods to solve the considered problem. First, using a condition of the analytical operator on the nonlinear part of the system to characterize the optimal control via the fixed point theorem. Second, using the HUM with an asymptotic condition on the nonlinear part to find the optimal control. The fourth and last section, we propose a numerical example to show the effectiveness of the main results.

2. Problem Statement

Let an open bounded subset $\Omega \subset IR^n$ and we denote $Q = \Omega \times]0, T[$. Let the semilinear hyperbolic state-space system

$$\begin{cases} \dfrac{\partial^2 X}{\partial t^2} + \mathscr{A}(\sigma)X + \mathscr{N}X = \mathrm{Pro}j_D f u(t), & Q, \\ X(x,0) = X_0(x), \dfrac{\partial X}{\partial t}(x,0) = X_1(x), & \Omega. \end{cases} \tag{1}$$

The operator $\mathscr{A}(\sigma)$ is linear elliptic depending on the uncertainty parameter $\sigma \in [a, b]$, the nonlinear operator $\mathscr{N}$ is independent of σ, $\mathrm{Pro}j_D$ is the indicator of D such that D is a zone of the system domain Ω, f is a function in $L^2(D)$, u in $\mathscr{C} = L^2(0, T)$ is the control and the initial conditions $(X_0, X_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Let $Y_u = (X_u, \partial X_u/\partial t)$ represent the solution of (1) and suppose that $Y_u(T, \sigma) \in (L^2(\Omega))^2 = \mathscr{E}$. The following definitions give the exact and the approximate averaged controllability of hyperbolic system. First, let ω a subregion of Ω

Definition 1. We say that (1) is ω-exactly RAC, if there is $u \in \mathscr{C}$ independent of σ such that

$$Y_1^d = \frac{1}{b-a}\int_a^b X_u(T,\sigma)d\sigma \text{ in } \omega \text{ and } Y_2^d = \frac{1}{b-a}\int_a^b \frac{\partial X_u}{\partial t}(T,\sigma)d\sigma \text{ in } \omega, \tag{2}$$

where (Y_1^d, Y_2^d) is the final target in $H_0^1(\omega) \times L^2(\omega)$.

Definition 2. We say that (1) is ω-approximately RAC, if there is $u \in \mathscr{C}$ independent of σ such that

$$\left\| \frac{1}{b-a}\int_a^b X_u(T,\sigma)d\sigma - Y_1^d \right\|_{L^2(\omega)} + \left\| \frac{1}{b-a}\int_a^b \frac{\partial X_u}{\partial t}(T,\sigma)d\sigma - Y_2^d \right\|_{L^2(\omega)} \le \varepsilon, \forall \varepsilon > 0. \tag{3}$$

Now, let the RAC problem for (1) and the actuator type zone internal (f, D) stated:

$$\begin{cases} \text{Determinate}\, u \in \mathscr{C}, \\ \dfrac{1}{b-a}\displaystyle\int_a^b X_u(T,\sigma)d\sigma = Y_1^d \text{ and } \dfrac{1}{b-a}\displaystyle\int_a^b \dfrac{\partial X_u}{\partial t}(T,\sigma)d\sigma = Y_2^d \quad \text{in} \quad \omega. \end{cases} \tag{4}$$

Let $A(\sigma) = \begin{pmatrix} 0 & I \\ \mathscr{A}(\sigma) & 0 \end{pmatrix}$, $Y = (X, \partial X/\partial t)$ $NY = (0, -\mathscr{N}X)^t$, $Y_0 = (X_0, X_1)$ and $\text{Bu} = (0, \text{Pro}j_D f u)^t$.

So, we rewrite (1) as

$$\begin{cases} \dfrac{\partial Y}{\partial t} + A(\sigma)Y = \text{NY} + \text{Bu}(t), & Q, \\ Y(0) = Y_0, & \Omega, \end{cases} \tag{5}$$

and associated linear system.

$$\begin{cases} \dfrac{\partial Y}{\partial t} + A(\sigma)Y = \text{Bu}(t), & Q, \\ Y(0) = Y_{0,} & \Omega, \end{cases} \tag{6}$$

Consider the operator $A(\sigma)$ generating the semigroup $S(t,\sigma)_{(t\geq 0)}$ on $\mathscr{E}$, we define the two operators $L(.,\sigma)$ and G_ω

$$\begin{aligned} L(t,\sigma)Y(.) &= \frac{1}{b-a}\int_a^b \int_0^t S(t-s,\sigma)Y(s)\text{d}s\text{d}\sigma, \\ G_\omega u &= \text{Pro}j_\omega L(T,\sigma)\text{Bu}(t). \end{aligned} \tag{7}$$

Now, we introduce the function.

$$\Phi(Y)(.) = S(.)Y_0 + L(.)\text{NY}(.) + L(.)\text{BG}_\omega^\dagger\left[y^d - \text{Pro}j_\omega S(T,\sigma)y_0 - \text{Pro}j_\omega L(T,\sigma)\text{NY}(.)\right], \tag{8}$$

and we denote the inverse of G_ω by $G_\omega^\dagger = (G_\omega^* G_\omega)^{-1} G_\omega^*$. The fixed point $Y^*(.)$ of (8) where $[Y^d - \text{Pro}j_\omega S(T,\sigma)Y_0 - \text{Pro}j_\omega L(T,\sigma)NY^*(.)] \in \text{Im}G_\omega$, directly If (6) is ω-approximately RAC, then

$$u^* = G_\omega^\dagger\left[Y^d - \text{Pro}j_\omega S(T,\sigma)Y_0 - \text{Pro}j_\omega L(T,\sigma)\text{NY}^*(.)\right]. \tag{9}$$

Conduct (1) to Y^d at time T. In the next section, an important situation will be analyzed in the analytical case.

3. Main Results

3.1. First Case Using Analytical Operator. Consider the previous problem (2) of the equation (3). By choosing $Y_0 = 0$ and let $(-A(\sigma))$ generates the analytic semigroup $S(t,\sigma)_{(t\geq 0)}$ on $\mathscr{E}$.

We consider $A_1(\sigma) = A(\sigma) + aI$ and we define $\text{Re}(A_1)$ the real part of the spectrum of $A_1(\sigma)$ (a is a real number providing $\text{Re}(A_1) > \delta > 0$). On the dance Banach space $\mathscr{E}^\alpha = D(A_1^\alpha(\sigma))\mathscr{E}$, we define for $0 \leq \alpha < 1$ the norm.

$$\left\|.\right\|_{\mathscr{E}^\alpha} = \left\|A_1^\alpha(.,\sigma)\right\|_{\mathscr{E}}, \tag{10}$$

and $\|S(t,\sigma)\|_{\mathscr{L}(\mathscr{E},\mathscr{E}^\alpha)} = c.t^{-\alpha}e^{(a-\delta)t} = g_1(t)$ ([13]).

Suppose that $g_1 \in L^q(0,T)$, $q \geq 1$, and let $1/q = 1 + 1/r - 1/s$, with $r, s \geq 1$ and that N is the map $L^r(0,T;\mathscr{E}^\alpha) \longrightarrow L^s(0,T;\mathscr{E})$ verifying.

$$\begin{cases} N(0) = 0, \\ \left\|\text{Na} - \text{Nb}\right\|_{L^s(0,T;\mathscr{E})} \leq k(\|a\|, \|b\|)\left\|a - b\right\|_{L^r(0,T;\mathscr{E}^\alpha)}; \forall a, b \in L^r(0,T;\mathscr{E}^\alpha), \\ \text{with}\, k\colon \text{IR}^+ \times \text{IR}^+ \longrightarrow \text{IR}^+;\ \lim\limits_{\mu_1,\mu_2 \longrightarrow 0} k(\mu_1,\mu_2) = 0. \end{cases} \tag{11}$$

This hypothesis is verified by several classes of semilinear hyperbolic systems.

Now, let the functions.

$$\widetilde{\Phi}(Y, u(t)) = L(.,\sigma)\text{NY} + L(.,\sigma)\text{Bu}(t), \tag{12}$$

$$\widetilde{\psi}_\omega\left(Y^d, u(t)\right) = G_\omega^\dagger\left(Y^d - \text{Pro}j_\omega L(T,\sigma)\text{NY}_u\right), \tag{13}$$

In the next, $\mathrm{Im}G_\omega$ is equipped with the seminorm:

$$\left\|Y^d\right\|_{\mathrm{Im}G_\omega} = \left\|G_\omega^\dagger Y^d\right\|_{L^2(0,T)}, \tag{14}$$

which give us the next theorem.

Theorem 1. *We suppose that (6) is ω-approximately RAC, (11) verified and*

$$\left\|L(.,\sigma)\mathrm{Bu}(t)\right\|_{L^r(0,T,\mathscr{E}^\alpha)} \leq \theta\left\|u(t)\right\|_{L^2(0,T)}, \tag{15}$$

θ is a positive constant.

$$\left\|\mathrm{Pro}j_\omega S(.)\right\|_{\mathscr{L}(\mathscr{E},\mathrm{Im}G_\omega)} = g_2(t) \in L^p(0,T) \text{ such that } \frac{1}{p}+\frac{1}{s} = 1, p, s \geq 1, \tag{16}$$

$g_2(t)$ is a positive function that is supposed belong $L^p(0,T)$. Then

(1) The solution $u^* \in B(0,m)$ of (2) exists and it is unique, for $m>0$ and $\rho>0$ such that $\forall Y^d \in B(0,\rho) \subset \mathrm{Im}G_\omega$.

(2) The map $Y^d \longrightarrow u^*(Y^d)$ from $B(0,\rho) \longrightarrow L^2(0,T)$ is Lipschitz.

Proof

(1) Knowing that $\lim_{\theta_1,\theta_2 \longrightarrow 0} k(\theta_1,\theta_2) = 0$, then there exists $\gamma > 0$ such that.

$$C_1 := \|g_2\|_{L^q(0,T)} \sup_{\theta_1,\theta_2<\gamma} k(\theta_1,\theta_2). \tag{17}$$

While (6) is ω-approximately RAC, then (14) is a norm. Using Schauder fixed-point theorem see [4], The function $\widetilde{\Phi}(.,u(t))$ is a contraction on the nonempty convex closed ball $B(0,\gamma)$, then admit a unique fixed point $Y \in B(0,\gamma) \subset L^r(0,T,\mathscr{E}^\alpha)$ solution of (3), for all $u(t) \in B(0,m)$ with

$$m := \frac{\gamma}{\theta}\left(1-\|g_1\|_{L^q(0,T)} \sup_{\mu\leq\gamma} k(\mu,0)\right). \tag{18}$$

Furthermore, for $Y^d \in \mathrm{Im}G_\omega$ we have

$$\begin{aligned}
&\left\|\widetilde{\psi}_\omega\left(Y^d,u(t)\right) - \widetilde{\psi}_\omega\left(Y^d,v\right)\right\|_{L^2(0,T)} \\
&= \left\|G_\omega^\dagger \mathrm{Pro}j_\omega L(T,\sigma)\left(\mathrm{NY}_v - \mathrm{NY}_u\right)\right\|_{L^2(0,T)} \\
&= \left\|\mathrm{Pro}j_\omega L(T,\sigma)\left(\mathrm{NY}_v - \mathrm{NY}_u\right)\right\|_{\mathrm{Im}G_\omega} \\
&\leq \left\|g_2\right\|_{L^p(0,T)}\left\|\mathrm{NY}_v - \mathrm{NY}_u\right\|_{L^s(0,T,\mathscr{E})} \\
&\leq \frac{\theta}{1-C_1}\|g_2\|_{L^p(0,T)} \sup_{\mu_1,\mu_2<\gamma} k(\mu_1,\mu_2)\|u-v\|.
\end{aligned} \tag{19}$$

We deduce that $\widetilde{\psi}_\omega$ is contraction while

$$\left\|\widetilde{\psi}_\omega\left(Z^d,u(t)\right) - \widetilde{\psi}_\omega\left(T^d,v\right)\right\|_{L^2(0,T)} \leq C_2\|u-v\|. \tag{20}$$

with

$$C_2 := \frac{\theta}{1-C_1}\|g_2\|_{L^p(0,T)} \sup_{\mu_1,\mu_2<\gamma} k(\mu_1,\mu_2) < 1. \tag{21}$$

From (18) we have

$$\left\|Y^d\right\| \leq \frac{\gamma}{\theta}\left(1-\left(\left\|g_1\right\|_{L^q(0,T)} + \theta\left\|g_2\right\|_{L^p(0,T)}\right)\sup_{\mu\leq\gamma} k(\mu,0)\right) =: \rho. \tag{22}$$

Therefore, if $Y^d \in B(0,\rho) \subset \mathrm{Im}G_\omega$, then $\widetilde{\psi}_\omega(z^d,.)$ admit a unique fixed point in $B(0,m)$ solution of (2).

(2) To prove that the map $Y^d \longrightarrow u^*(Y^d)$ is Lipshitz, let consider $Y^d, X^d \in B(0,\rho)$, such that

$$\begin{aligned}
u^*\left(Y^d\right) - u^*\left(X^d\right) &= \widetilde{\psi}_\omega\left(Y^d,u^*\left(Y^d\right)\right) - \widetilde{\psi}_\omega\left(y^d,u^*\left(y^d\right)\right) \\
&= \widetilde{\psi}_\omega\left(Y^d,u^*\left(Y^d\right)\right) - \widetilde{\psi}_\omega\left(Y^d,u^*\left(X^d\right)\right) + \widetilde{\psi}_\omega Y^d,u^*\left(X^d\right) - \widetilde{\psi}_\omega\left(X^d,u^*\left(X^d\right)\right),
\end{aligned} \tag{23}$$

but

$$\begin{aligned}\left\|\widetilde{\psi}_{\omega}\left(Y^{d},u^{*}\left(Y^{d}\right)\right)-\widetilde{\psi}_{\omega}\left(Y^{d},u^{*}\left(X^{d}\right)\right)\right\| &\leq C_{3}\left\|u^{*}\left(Y^{d}\right)-u^{*}\left(X^{d}\right)\right\|\\ \left\|\widetilde{\psi}_{\omega}\left(Y^{d},u^{*}\left(X^{d}\right)\right)-\widetilde{\psi}_{\omega}\left(X^{d},u^{*}\left(X^{d}\right)\right)\right\| &=\left\|Y^{d}-X^{d}\right\|,\end{aligned} \tag{24}$$

hence,

$$\left\|u^{*}\left(Y^{d}\right)-u^{*}\left(X^{d}\right)\right\|\leq\frac{1}{1-C_{3}}\left\|Y^{d}-X^{d}\right\|, \tag{25}$$

which concludes the result. □

Proposition 1. *Let Y_{u_n} is the solution of the system (3) associate to the control u_n. The solution of the problem (2) is represented by the control sequences*

$$\begin{cases}u_{n+1}=G_{\omega}^{\dagger}\left(Y^{d}-\mathrm{Pro}j_{\omega}L(T,\sigma)\mathrm{N}Y_{u_{n}}\right),\\ u_{0}=0.\end{cases} \tag{26}$$

Which converges to $u^{*}\in\mathscr{C}$.

Proof. The proof is obtained using (20) and (14). □

3.1.1. Second Case Using HUM. Here, we address the issue (4) that arises when it is anticipated that the system (1) would verify

$$\lim_{|r|\longrightarrow+\infty}\frac{\mathscr{N}(r)}{r}=\alpha\quad(\alpha\geq0)\text{ with }\mathscr{N}'\in L^{\infty}(\mathrm{IR})\quad(\text{see [14]}). \tag{27}$$

The method we will choose is an expansion of the HUM, which has been used to prove controllability in the linear situation (see [3]), as well as the semilinear situation (see [14]).

We consider the set G

$$G=\left\{(Z_{1},-Z_{0})\in\mathscr{C}^{\infty}(\Omega)\times\mathscr{C}^{\infty}(\Omega)\text{ such that }Z_{0}=Z_{1}=0\quad\text{on}\frac{\Omega}{\omega}\right\}, \tag{28}$$

and let

$$\widehat{G}=\left\{(Z_{1},-Z_{0})\in\mathscr{C}^{\infty}(\Omega)\times\mathscr{C}^{\infty}(\Omega)\quad\text{such that}\quad Z_{0}=Z_{1}=0\quad\text{on}\quad\omega\right\}. \tag{29}$$

For $(Z_1,-Z_0)\in G$, the following system admit a unique solution $Z\in C(0,T,H_0^1(\Omega))\cap C^1(0,T,L^2(\Omega))$ ([3]).

$$\begin{cases}\dfrac{\partial^{2}Z}{\partial t^{2}}+\mathscr{A}(\sigma)Z=0, & Q,\\ Z(x,T)=Z_{0}(x),\dfrac{\partial Z}{\partial t}(x,T)=Z_{1}(x), & \Omega,\\ Z(\xi,t)=0, & \Sigma.\end{cases} \tag{30}$$

And the solution of (1) can be expressed as

$$X=\phi_{0}+\phi_{1}+\phi_{2}, \tag{31}$$

where ϕ_0 and ϕ_1 are respectively solutions of systems

$$\begin{cases}\dfrac{\partial^{2}\phi_{0}}{\partial t^{2}}+\mathscr{A}(\sigma)\phi_{0}=0, & Q,\\ \phi_{0}(x,0)=X_{0}(x),\qquad\dfrac{\partial\phi_{0}}{\partial t}(x,0)=X_{1}(x), & \Omega,\\ \phi_{0}(\xi,t)=0, & \Sigma,\end{cases} \tag{32}$$

$$\begin{cases}\dfrac{\partial^{2}\phi_{1}}{\partial t^{2}}+\mathscr{A}(\sigma)\phi_{1}=-<Z,f>_{L^{2}(D)}\mathrm{Pro}j_{D}f, & Q,\\ \phi_{1}(x,0)=0,\qquad\dfrac{\partial\phi_{1}}{\partial t}(x,0)=0, & \Omega,\\ \phi_{1}(\xi,t)=0, & \Sigma,\end{cases} \tag{33}$$

that verify (see [3])

$$\phi_0, \phi_1 \in C\left(0, T, H_0^1(\Omega)\right) \cap C^1\left(0, T, L^2(\Omega)\right), \tag{34}$$

and the exist θ_1 a positive constant verifying

$$\begin{aligned} \|\phi_0\|_{L^\infty\left(0,T,H_0^1(\Omega)\right)} + \left\|\frac{\partial \phi_0}{\partial t}\right\|_{L^\infty(0,T,L^2(\Omega))} \le \theta_1 \left\|\left(X_0, X_1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}, \\ \|\phi_0\|_{L^\infty\left(0,T,H_0^1(\Omega)\right)} + \left\|\frac{\partial \phi_1}{\partial t}\right\|_{L^\infty(0,T,L^2(\Omega))} \le \theta_1 \left\|\left(Z_0, Z_1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}, \end{aligned} \tag{35}$$

and ϕ_2 is solution of the system

$$\begin{cases} \dfrac{\partial^2 \phi_2}{\partial t^2} + \mathscr{A}(\sigma)\phi_2 + \mathscr{N}\left(\phi_0 + \phi_1 + \phi_2\right) = \alpha\left(\phi_0 + \phi_1\right), & Q, \\ \phi_2(x, 0) = 0, \dfrac{\partial \phi_2}{\partial t}(x, 0) = 0, & \Omega, \\ \phi_2(\xi, t) = 0, & \Sigma. \end{cases} \tag{36}$$

The map $\phi \longrightarrow \mathscr{N}\left(\phi_0(t) + \phi_1(t) + \phi(t)\right)$ is Lipschitz continuous, since $\mathscr{N}' \in L^\infty(\mathrm{IR})$. So (36) admits a unique solution.

$$\phi_2 \in C\left(0, T, H_0^1(\Omega)\right) \cap C^1\left(0, T, L^2(\Omega)\right). \tag{37}$$

Now, we define the operator

$$\eta\left(Z_1, -Z_0\right) = \chi\left(\phi_1(T), \frac{\partial \phi_1}{\partial t}(T)\right) + \chi\left(\phi_2(T), \frac{\partial \phi_2}{\partial t}(T)\right), \tag{38}$$

where $\chi = \mathrm{Pro}j_\omega^* \mathrm{Pro}j_\omega$.

Then, the problem of RAC of (1) turns up to solve the equation

$$\eta\left(Z_1, -Z_0\right) = \mathrm{Pro}j_\omega^*\left(z_1^d, z_2^d\right) - \chi\left(\phi_0(T), \frac{\partial \phi_0}{\partial t}(T)\right). \tag{39}$$

The equation (39) is equivalent to the equation

$$\Lambda\left(Z_1, -Z_0\right) = \mathrm{Pro}j_\omega^*\left(z_1^d, z_2^d\right) - \chi\left(\phi_2(T), \frac{\partial \phi_2}{\partial t}(T)\right) - \chi\left(\phi_0(T), \frac{\partial \phi_0}{\partial t}(T)\right). \tag{40}$$

For a positive constant θ_2, let

$$\mathscr{G} = \left\{\left(Z_1, -Z_0\right) \in G;\ \left\|\left(Z_0, Z_1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} \le \theta_2 \left\|\left(Z_1, -Z_0\right)\right\|_G\right\}. \tag{41}$$

Solve the problem (39), became a fixed point of

$$\tilde{\eta}\left(Z_1, -Z_0\right) = \Lambda^{-1}\mathrm{Pro}j_\omega^*\left(z_1^d, z_2^d\right) - \Lambda^{-1}\mathscr{K}_\omega\left(Z_1, -Z_0\right) - \Lambda^{-1}\chi\left(\phi_0(T), \frac{\partial \phi_0}{\partial t}(T)\right). \tag{42}$$

We define the operator $\mathscr{K}_\omega$ by

$$\begin{aligned} &\mathscr{K}_\omega: \mathscr{G} \longrightarrow \widehat{G}^*, \\ &\left(Z_1, -Z_0\right) \longrightarrow \chi\left(\phi_2(T), \frac{\partial \phi_2}{\partial t}(T)\right), \end{aligned} \tag{43}$$

with $\widehat{G}^*$ is the dual of $\widehat{G}$.

Theorem 2. *If (6) is ω-approximately RAC, then (42) admit a unique fixed point* $\left(Z_1, -Z_0\right)$ *and* $u^*(t) = -\langle Z(t), f\rangle_{L^2(D)}$ *steer (1) to* Z^d*, where Z is the solution (30).*

Proof. Let $\phi_2 \in C\left(0, T, H_0^1(\Omega)\right) \cap C^1\left(0, T, L^2(\Omega)\right)$, for all $t > 0$ there is $\theta_3 > 0$ verifying

$$\left\|\chi\left(\frac{\partial \phi_2}{\partial t}(t), -\phi_2(t)\right)\right\|_{\widehat{G}^*} \leq \theta_3\left[\left\|\phi_2(t)\right\|_{H_0^1(\Omega)} + \left\|\frac{\partial \phi_2(t)}{\partial t}\right\|_{L^2(\Omega)}\right]. \tag{44}$$

Then $\chi(\partial\phi_2/\partial t, -\phi_2) \in C(0, T, \widehat{G}^*)$.
From [3], there is $\varepsilon > 0$ and $\theta_4 > 0$ such that

$$\begin{aligned}\left\|\phi_2\right\|_{L^\infty\left(0,T,H_0^1(\Omega)\right)} + \left\|\frac{\partial \phi_2}{\partial t}\right\|_{L^\infty(0,T,L^2(\Omega))} \leq \varepsilon\Bigg(\left\|(y_0, y_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} \\ + \left\|(Z_0, Z_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}\Bigg) + \theta_4.\end{aligned} \tag{45}$$

Hence, while $(Z_1, -Z_0) \in \mathscr{G}$, then for all $t > 0$

$$\left\|\chi\left(\frac{\partial \phi_2}{\partial t}(t), -\phi_2(t)\right)\right\|_{\widehat{G}^*} \leq \varepsilon\left[\left\|(y_0, y_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \theta_2\left\|(Z_1, -Z_0)\right\|_G + \frac{\theta_4}{\varepsilon}\right]. \tag{46}$$

Using (46) with $\varepsilon = [2\theta_2\|\Lambda^{-1}\|_{\mathscr{L}(G^*,G)}]^{-1}$ and for some constant $\theta_5 > 0$, we have

$$\begin{aligned}\left\|\widetilde{\eta}(Z_1, -Z_0)\right\|_G &\leq \left\|\Lambda^{-1}\mathscr{K}_\omega(Z_1, -Z_0)\right\|_G + \left\|\Lambda^{-1}Proj_\omega^*\left(z_1^d, z_2^d\right) - \Lambda^{-1}\chi\left(\phi_0(T), \frac{\partial\phi_0}{\partial t}(T)\right)\right\|_G \\ &\leq \frac{1}{2}\left\|(Z_1, -Z_0)\right\|_G + \theta_5.\end{aligned} \tag{47}$$

Furthermore, by (44) and (46) $\mathscr{K}_\omega$ is a compact operator, we deduce that $\widetilde{\eta}$ is compact and there is $M \geq 2\theta_5$ such that

$$\left\|\widetilde{\eta}(Z_1, -Z_0)\right\|_G \leq M \forall (Z_1, -Z_0) \in G \text{ such that } \left\|(Z_1, -Z_0)\right\|_G \leq M. \tag{48}$$

To complete the proof, (42) admit at least one fixed point by using the Schauder's fixed point theorem in [13]. □

4. Simulations

In this section, we present a numerical example which illustrates the previous results. It shows that there exists a link between the subregion area and the reached state Consider the one dimensional system excited by a zone actuator located in D.

$$\begin{cases}\dfrac{\partial^2 X(x,t)}{\partial t^2} - \sigma^2\dfrac{\partial^2 X(x,t)}{\partial x^2} + \displaystyle\sum_{i=1}^{m}|\langle X(t), w_i\rangle|\langle X(t), w_i\rangle w_i(x) + \text{Pro}j_D 11(x)u(t) = 0, &]0,1[, \\ X(x,0) = 0, \dfrac{\partial X}{\partial t}(x,0) = 0, &]0,1[, \\ X(0,t) = X(1,t) = 0, &]0,1[,\end{cases} \tag{49}$$

Step 1:
(i) We choose (Y_1^d, Y_2^d) and the region ω.
(ii) Define the precision ε and the location D.
Step 2: Repeat
(i) Solve the system (30) to find Z.
(ii) Compute the control $u^*(t)$ by the formula $u(t) = -\langle Z(t), f\rangle_{L^2(D)}$.
(iii) Solve (49) to obtain $X_u(T,\sigma)$ and $\partial X_u/\partial t(T,\sigma)$.
(iv) Until $\mathscr{E} = \|\int_0^1 X_u(T,\sigma)d\sigma - Y_1^d\|_\omega + \|\int_0^1 \partial X_u/\partial t(T,\sigma) - Y_2^d\|_\omega \le \varepsilon$, repeat step 1 and step 2.
Step 3: Then, $\int_0^1 X_u(T,\sigma)d\sigma = Y_1^d$ and $\int_0^1 \partial X_u/\partial t(T,\sigma)d\sigma = Y_2^d$ in ω.

Algorithm 1: Algorithm for solving the considered problem.

Table 1: Relation subregion-cost.

Subregion ω	Cost
$]0.43, 0.62[$	$1.02 * 10^{-4}$
$]0.38, 0.63[$	$3.41 * 10^{-3}$
$]0.32, 0.65[$	$1.32 * 10^{-3}$
$]0.12, 0.68[$	$4.36 * 10^{-2}$
$]0.21, 0.7[$	$1.31 * 10^{-2}$

Table 2: Relation subregion-error.

Subregion ω	$\mathscr{E}$
$]0.43, 0.62[$	$3.02 * 10^{-4}$
$]0.38, 0.63[$	$1.032 * 10^{-4}$
$]0.32, 0.65[$	$3.31 * 10^{-3}$
$]0.32, 0.68[$	$1.03 * 10^{-3}$
$]0.21, 0.7[$	$3.24 * 10^{-2}$

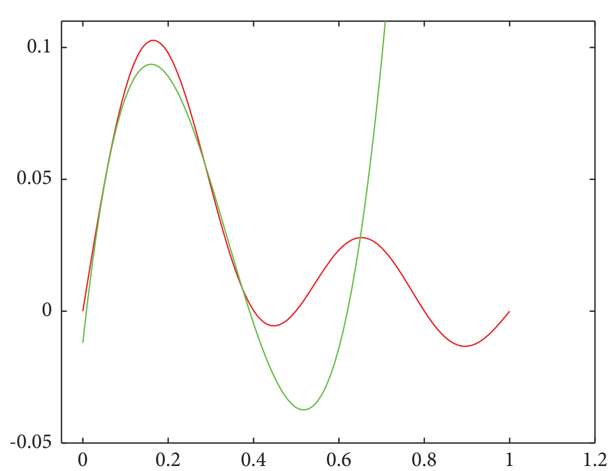

Figure 1: Desired position (green) and reached position (red) on ω.

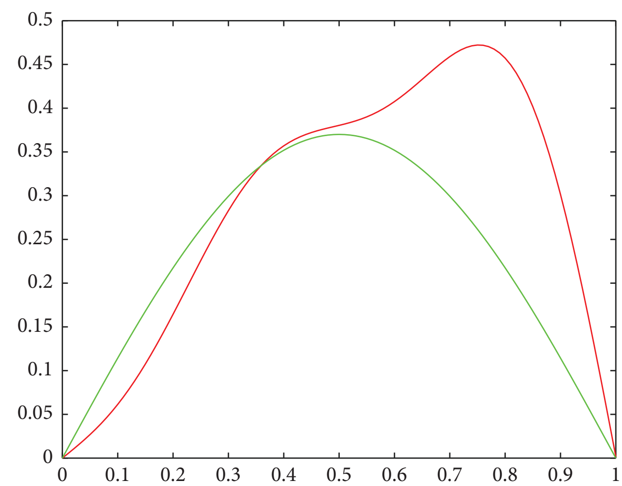

Figure 2: Desired speed (green) and reached speed (red) on ω.

where $w_i(x) = \sqrt{2}\sin(i\pi x)$, $i \in IN^*$, $\sigma \in [0,1]$ and $D = [0,52; 0,63]$. To solve the considered problem, we consider the following Algorithm 1.

Remark 1. In the next simulation, we will apply the previous algorithm to the second case where the optimal control is given by $u(t) = -\langle Z(t), f\rangle_{L^2(D)}$. The characterization established in the first case by Proposition 1 can be tested using the same algorithm.

Choosing the time optimal control $T = 3$, $\omega = [0.2.0.4]$ and applying the previous algorithm the system (49), we have the following results.

Tables 1 and 2 show numerically how the cost and the error respectively grow with respect to the subregion area.

Figures 1 and 2 represent the profile of the energy dissipated to command the system (49) from its initial states to the desired ones at the time $T = 3$ with the cost $\mathscr{E} = 2.031 \times 10^{-4}$.

From the reached state solution of the system (49) presented by Figure 1, we can remark that the desired position given by Figure 1 is very close to the reached position. Therefore, for the reached speed of the system (49) presented by Figure 2, we will have that the desired speed is very close to the reached one.

5. Conclusion

In this study, we describe the regional averaged controllability problem governed by a class of semilinear hyperbolic systems. The definitions of the precise and approximate regional averaged controllability systems are provided first. The issue of regional averaged controllability for semilinear systems is then raised. We suggest two approaches to the problem: the Hilbert Uniqueness Method with an asymptotic condition on the nonlinear part to find the optimal control of the considered problem, and using a condition of the analytical operator to the nonlinear part of the system to characterize the optimal control via the fixed point theorem. Finally, we give a numerical example to illustrate the effectiveness of our approach and to validate our results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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