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Research Article

Abundance of Exact Solutions of a Nonlinear Forced (2 + 1)-Dimensional Zakharov–Kuznetsov Equation for Rossby Waves

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In this paper, an improved tan $(\varphi/2)$ expansion method is used to solve the exact solution of the nonlinear forced $(2+1)$-dimensional Zakharov–Kuznetsov equation. Firstly, we analyse the research status of the improved tan $(\varphi/2)$ expansion method. Then, exact solutions of the nonlinear forced $(2+1)$-dimensional Zakharov–Kuznetsov equation are obtained by the perturbation expansion method and the multi-spatiotemporal scale method. It is shown that the improved tan $(\varphi/2)$ expansion method can obtain more exact solutions, including exact periodic travelling wave solutions, exact solitary wave solutions, and singular kink travelling wave solutions. Finally, the three-dimensional figure and the corresponding plane figure of the corresponding solution are given by using MATLAB to illustrate the influence of external source, dimension variable y, and dispersion coefficient on the propagation of the Rossby wave.

1. Introduction

In ocean and atmospheric motion, researchers have found that Rossby wave propagation is affected by nonlinear action, and sometimes linear theory cannot explain well the role of Rossby waves at energy conversion and climate change. They constructed mathematical models to reveal the dynamic characteristics of the Rossby wave [1–7]. Also, the exact solutions of these equations are crucial to studying the propagation of Rossby waves [8–11]. So, many methods have been proposed on how to solve the exact solutions to nonlinear equations, for instance, the Hirota method [12–14], the Jacobi elliptic function expansion method [15–18], the G'/G-expansion method [19–23], the Exp $(-\Phi(\xi))$-expansion method [24, 25], the generalised exponential rational function method [26–28], the negative power expansion method [29], the hyperbolic function expansion method [30–33], the extended sub-equation method [34], (ω/g)-expansion method [35], the improved sub-ODE method [36], the Riccati–Bernoulli sub-ODE method [37–40], the Lie symmetry technique [41–47], the fractional sub-equation [48] etc. These are valid methods and tools for computing nonlinear equations.

Manafian et al. proposed a new method to solve nonlinear partial differential equations, namely, the improved tan $(\varphi/2)$ expansion method [49]. With the help of this method, many classical nonlinear partial differential equations have been investigated and abundant exact solutions have been obtained [50–69]. Mohyud-Din and Irshad used this method to construct an exact solution for the generalised KP equation and explained that it can provide better help for the study of generalised KP equations [60]. Foroutan et al. utilised the improved tan $(\varphi/2)$ expansion method to study the soliton perturbation in inverse-cubic nonlinear optical metamaterials and confirmed that the soliton is in a superconducting presence in the material by obtaining bright soliton, dark soliton, and strange soliton solutions [64]. Sendi et al. solved nonlinear partial differential equations with the help of an improved tan $(\varphi/2)$ expansion method. Exact periodic travelling wave solution, exact singular kink travelling wave solution, soliton and so on are obtained, and three-dimensional graphs corresponding to some exact solutions are depicted [66]. Zkan et al. used the improved tan $(\varphi/2)$ expansion method to obtain the exact solution to the $(2+1)$-dimensional KdV equation and explained the influence of different parameters on wave propagation through three-dimensional graphs and tables [69]. Compared to the negative power expansion method [29], the extended subequation method [34], and the improved sub-ODE method [36], we can obtain more formal solutions using the improved tan $(\varphi/2)$ expansion method, which is one of the efficient mathematical methods and tools that are widely used and easy to implement.

In reference [11], Yin et al. deduced that the amplitude of the large-amplitude Rossby long waves satisfies the forced Zakharov–Kuznetsov (ZK) equation based on the potential vorticity equation by utilising the perturbation expansion method and the coordinate change method. In this study, we count the case that the external source of ZK equation is constant as follows:

$$u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_{xyy} = q. \tag{1}$$

Here, the third derivative term represents the dispersion effect, y denotes the longitude variable, and q represents the influence of the external source. The research shows that these three factors have a certain weight on wave propagation. Equation (1) reflects the characteristics of the large-amplitude Rossby long waves and describes two-dimensional Rossby waves, which can show the propagation of Rossby waves more comprehensively. At $\gamma = 0$, equation (1) can be simplified to the forced KdV equation, described by one-dimensional Rossby solitary waves.

The 1-soliton and 2-soliton solution of the equation is derived using the coupled Burgers equation without considering the solution of external source, and the solitary waves are obtained to explain the influence of external source on the propagation of Rossby waves with the help of the extended Jacobi expansion method of elliptic functions in [11]. Other solutions of equation (1) have not been reported.

To better study the characteristics of the Rossby long wave reflected by equation (1), we use the improved tan $(\varphi/2)$ expansion method with the help of mathematics to obtain exact solutions. More forms of exact solutions, including exact periodic solutions, exact solitary wave solutions, and singular kink travelling wave solutions, are derived. Then, with the help of MATLAB image processing software, we draw the three-dimensional figure and plane figures corresponding to the partial solution to equation (1). What is more, we analyse the effects of external sources, longitude variables, and dispersion coefficients on Rossby wave propagation in more detail with graphs, and give the corresponding conclusions.

2. Use of an Improved tan $(\varphi/2)$ Expansion Method

Using the transform $u(\xi) = u(\lambda x + \omega y + \sigma t)$, we substitute into equation (1), which can be simplified to the following ordinary differential equation:

$$\sigma u' + \lambda \alpha u u' + \lambda^3 \beta u''' + \lambda \omega^2 \gamma u''' = q. \tag{2}$$

We integrate once on both sides of the equation (2) with respect to ξ, getting

$$\sigma u + \frac{\lambda\alpha}{2} u^2 + \left(\lambda^3 \beta + \lambda \omega^2 \gamma\right) u'' = Q. \tag{3}$$

We assume $u(\xi)$ that can be expanded into the form of a power series with respect to $p + \tan(\varphi(\xi)/2)$, namely,

$$u(\xi) = \sum_{k=-m}^{m} A_k \left[p + \tan(\varphi(\xi)/2)\right]^k, \tag{4}$$

where $A_{-k} = B_k\,(1 \le k \le m)$, $\varphi(\xi)$ satisfies the following ordinary differential equation:

$$\varphi'(\xi) = a \sin(\varphi(\xi)) + b \cos(\varphi(\xi)) + c. \tag{5}$$

For details of $\varphi(\xi)$, we refer to [49].

Using the homogeneous equilibrium method for equation (3), considering u'' and u^2, we obtain $n + 2 = 2n$, thus $n = 2$.

For formula (4), we consider $p = 0$, then it can be transformed into

$$u(\xi) = A_0 + A_1 \tan\left(\frac{\varphi(\xi)}{2}\right) + A_2 \tan^2\left(\frac{\varphi(\xi)}{2}\right) + B_1 \cot\left(\frac{\varphi(\xi)}{2}\right) + B_2 \cot^2\left(\frac{\varphi(\xi)}{2}\right). \tag{6}$$

Substituting equations (5) and (6) into (3), and using the numerical calculation software Mathematics, we can get the following conclusions.

Case 1. $a = a, b = b, c = c, \Delta = a^2 + b^2 - c^2, \mu = \lambda^2\beta + \omega^2\gamma$, $\overline{\xi} = \xi + C$,

$$\begin{aligned} A_0 &= -\frac{\lambda\alpha\mu\left(a^2 - 2b^2 + 2c^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\Delta^2\mu^2\right)}}{\lambda\alpha^2}, A_1 = \frac{6a\mu(b-c)}{\alpha}, \\ A_2 &= -\frac{3\mu(b-c)^2}{\alpha}, B_1 = 0, B_2 = 0, \sigma = \pm\frac{\sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\Delta^2\mu^2\right)}}{\alpha}, \\ u(\xi) &= A_0 + A_1\tan\left(\frac{\varphi(\xi)}{2}\right) + A_2\tan^2\left(\frac{\varphi(\xi)}{2}\right). \end{aligned} \tag{7}$$

Referring to the 19 kinds of results in [66], we obtain the trigonometric functional solutions, hyperbolic functional solutions, exponential functional solutions, and rational functional solutions of equation (3).

$$\begin{aligned} u_1(\xi) = &-\frac{\lambda\alpha\mu\left(a^2 - 2b^2 + 2c^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\Delta^2\mu^2\right)}}{\lambda\alpha^2} \\ &+ \frac{3\mu}{\alpha}\left\{2a\left[a - \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\overline{\xi}\right)\right] - \left[a - \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\overline{\xi}\right)\right]^2\right\}, \end{aligned} \tag{8}$$

$$\begin{aligned} u_2(\xi) = &-\frac{\lambda\alpha\mu\left(a^2 - 2b^2 + 2c^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\Delta^2\mu^2\right)}}{\lambda\alpha^2} \\ &+ \frac{3\mu}{\alpha}\left\{2a\left[a + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\overline{\xi}\right)\right] - \left[a + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\overline{\xi}\right)\right]^2\right\}, \end{aligned} \tag{9}$$

$$\begin{aligned} u_3(\xi) = &-\frac{\lambda\alpha\mu\left(a^2 - 2b^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\mu^2\left(b^2 + a^2\right)^2\right)}}{\lambda\alpha^2} \\ &+ \frac{3\mu}{\alpha}\left\{2a\left[a + \sqrt{b^2 + a^2}\tanh\left(\frac{\sqrt{b^2 + a^2}}{2}\overline{\xi}\right)\right] - \left[a + \sqrt{b^2 + a^2}\tanh\left(\frac{\sqrt{b^2 + a^2}}{2}\overline{\xi}\right)\right]^2\right\}, \end{aligned} \tag{10}$$

$$\begin{aligned} u_4(\xi) = &-\frac{\lambda\alpha\mu\left(a^2 + 2c^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\mu^2\left(c^2 - a^2\right)^2\right)}}{\lambda\alpha^2} \\ &- \frac{3\mu}{\alpha}\left\{2a\left[-a + \sqrt{c^2 - a^2}\tan\left(\frac{\sqrt{c^2 - a^2}}{2}\overline{\xi}\right)\right] + \left[-a + \sqrt{c^2 - a^2}\tan\left(\frac{\sqrt{c^2 - a^2}}{2}\overline{\xi}\right)\right]^2\right\}, \end{aligned} \tag{11}$$

$$u_5(\xi) = -\frac{2\lambda\alpha\mu\left(b^2 - c^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\mu^2\left(b^2 - c^2\right)^2\right)}}{\lambda\alpha^2} - \frac{3\mu\left(b^2 - c^2\right)}{\alpha}\tanh^2\left(\frac{\sqrt{b^2 - c^2}}{2}\overline{\xi}\right), \tag{12}$$

$$u_6(\xi) = -\frac{2\lambda\alpha b^2\mu \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda b^4\mu^2\right)}}{\lambda\alpha^2} - \frac{3b^2\mu}{\alpha}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\overline{\xi}} - 1}{e^{2b\overline{\xi}} + 1}, \frac{2e^{b\overline{\xi}}}{e^{2b\overline{\xi}} + 1}\right]\right), \tag{13}$$

$$u_7(\xi) = -\frac{3\lambda\alpha a^2\mu \pm \sqrt{-2\lambda\alpha^3 Q}}{\lambda\alpha^2} + \frac{3\mu}{\alpha}\left\{2a\left[\frac{2+a\bar{\xi}}{\bar{\xi}}\right] - \left[\frac{2+a\bar{\xi}}{\bar{\xi}}\right]^2\right\}, \tag{14}$$

$$u_8(\xi) = -\frac{\lambda\alpha a^2k^2\mu \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda a^4k^4\mu^2\right)}}{\lambda\alpha^2} + \frac{12a^2k^2\mu}{\alpha}\left\{\left[\frac{e^{ka\bar{\xi}}}{-1+e^{ka\bar{\xi}}}\right] - \left[\frac{e^{ka\bar{\xi}}}{-1+e^{ka\bar{\xi}}}\right]^2\right\}, \tag{15}$$

$$u_9(\xi) = -\frac{\lambda\alpha\mu\left(3a^2-2b^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda b^4\mu^2\right)}}{\lambda\alpha^2} + \frac{3\mu}{\alpha}\left\{2a(a-b)\left[\frac{(a+b)e^{b\bar{\xi}}-1}{(a-b)e^{b\bar{\xi}}-1}\right] - (a-b)^2\left[\frac{(a+b)e^{b\bar{\xi}}-1}{(a-b)e^{b\bar{\xi}}-1}\right]^2\right\}, \tag{16}$$

$$u_{10}(\bar{\xi}) = -\frac{\lambda\alpha\mu\left(3c^2-2b^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda b^4\mu^2\right)}}{\lambda\alpha^2} + \frac{3\mu}{\alpha}\left\{2(b-c)c\left[\frac{(b+c)e^{b\bar{\xi}}+1}{(b-c)e^{b\bar{\xi}}-1}\right] - (b-c)^2\left[\frac{(b+c)e^{b\bar{\xi}}+1}{(b-c)e^{b\bar{\xi}}-1}\right]^2\right\}, \tag{17}$$

$$u_{11}(\bar{\xi}) = -\frac{\lambda\alpha\mu\left(3a^2-2b^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda b^4\mu^2\right)}}{\lambda\alpha^2} - \frac{3\mu}{\alpha}\left\{2a(a+b)\left[\frac{e^{b\bar{\xi}}+b-a}{e^{b\bar{\xi}}-b-a}\right] + (a+b)^2\left[\frac{e^{b\bar{\xi}}+b-a}{e^{b\bar{\xi}}-b-a}\right]^2\right\}, \tag{18}$$

$$u_{12}(\bar{\xi}) = -\frac{\lambda\alpha\mu\left(3a^2-2b^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda b^4\mu^2\right)}}{\lambda\alpha^2} - \frac{12\mu}{\alpha}\left\{ac\left[\frac{ae^{a\bar{\xi}}}{1-ce^{a\bar{\xi}}}\right] + c^2\left[\frac{ae^{a\bar{\xi}}}{1-ce^{a\bar{\xi}}}\right]^2\right\}, \tag{19}$$

$$u_{13}(\bar{\xi}) = -\frac{3\lambda\alpha c^2\mu \pm \sqrt{-2\lambda\alpha^3 Q}}{\lambda\alpha^2} + \frac{3c^2\mu}{\alpha}\left\{2\left[\frac{c\bar{\xi}+2}{c\bar{\xi}}\right] - \left[\frac{c\bar{\xi}+2}{c\bar{\xi}}\right]^2\right\}, \tag{20}$$

$$u_{14}(\bar{\xi}) = \frac{\mp\sqrt{-2\lambda\alpha^3 Q}}{\lambda\alpha^2} - \frac{12c^2\mu}{\alpha}\tan^2\left[\frac{1}{c\bar{\xi}}\right], \tag{21}$$

$$\begin{aligned} u_{15}(\bar{\xi}) &= -\frac{2\lambda\alpha c^2\mu \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda c^4\mu^2\right)}}{\lambda\alpha^2} - \frac{3c^2\mu}{\alpha}\tan^2\left[\frac{c\xi + C}{2}\right], \\ \bar{\xi} &= \lambda x + \omega y \pm \frac{\sqrt{\lambda\alpha^2\left(-2Q\alpha + \lambda\Delta^2\mu^2\right)}}{\alpha}t + C. \end{aligned} \tag{22}$$

Case 2. $a = a, b = b, c = c, \Delta = a^2 + b^2 - c^2, \mu = \lambda^2\beta + \omega^2\gamma,$
$\bar{\xi} = \xi + C,$

$$\begin{aligned} A_0 &= -\frac{\lambda\alpha\mu\left(a^2-2b^2+2c^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\Delta^2\mu^2\right)}}{\lambda\alpha^2}, A_1 = 0, A_2 = 0, \\ B_1 &= -\frac{6a\mu(b+c)}{\alpha}, B_2 = \frac{-3\mu(b+c)^2}{\alpha}, \sigma = \pm\frac{\sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\Delta^2\mu^2\right)}}{\alpha}, \\ u(\xi) &= A_0 + B_1\cot\left(\frac{\varphi(\xi)}{2}\right) + B_2\cot^2\left(\frac{\varphi(\xi)}{2}\right). \end{aligned} \tag{23}$$

With 19 kinds of results of reference [49], we obtain solutions of equation (3)

$$u_{16}(\xi) = -\frac{\lambda\alpha\mu\left(a^2 - 2b^2 + 2c^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\Delta^2\mu^2\right)}}{\lambda\alpha^2} - \frac{3\mu}{\alpha}\left\{2a\left[a - \sqrt{-\Delta}\,\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]^{-1} + \left[a - \sqrt{-\Delta}\,\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]^{-2}\right\}, \tag{24}$$

$$u_{17}(\xi) = -\frac{\lambda\alpha\mu\left(a^2 - 2b^2 + 2c^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\Delta^2\mu^2\right)}}{\lambda\alpha^2} - \frac{3\mu}{\alpha}\left\{2a\left[a + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right]^{-1} + \left[a + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right]^{-2}\right\}, \tag{25}$$

$$u_{18}(\xi) = -\frac{\lambda\alpha\mu\left(a^2 - 2b^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\mu^2\left(b^2 + a^2\right)^2\right)}}{\lambda\alpha^2} - \frac{3\mu}{\alpha}\left\{2a\left[a + \sqrt{b^2 + a^2}\tanh\left(\frac{\sqrt{b^2 + a^2}}{2}\bar{\xi}\right)\right]^{-1} + \left[a + \sqrt{b^2 + a^2}\tanh\left(\frac{\sqrt{b^2 + a^2}}{2}\bar{\xi}\right)\right]^{-2}\right\}, \tag{26}$$

$$u_{19}(\xi) = -\frac{\lambda\alpha\mu\left(a^2 + 2c^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\mu^2\left(c^2 - a^2\right)^2\right)}}{\lambda\alpha^2}, - \frac{3\mu}{\alpha}\left\{2a\left[-a + \sqrt{c^2 - a^2}\,\tan\left(\frac{\sqrt{c^2 - a^2}}{2}\bar{\xi}\right)\right]^{-1} + \left[-a + \sqrt{c^2 - a^2}\,\tan\left(\frac{\sqrt{c^2 - a^2}}{2}\bar{\xi}\right)\right]^{-2}\right\}, \tag{27}$$

$$u_{20}(\xi) = -\frac{2\lambda\alpha\mu\left(c^2 - b^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda\mu^2\left(b^2 - c^2\right)^2\right)}}{\lambda\alpha^2} - \frac{3\mu\left(b^2 - c^2\right)}{\alpha}\coth^2\left(\frac{\sqrt{b^2 - c^2}}{2}\bar{\xi}\right), \tag{28}$$

$$u_{21}(\xi) = -\frac{2\lambda\alpha b^2\mu \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda b^4\mu^2\right)}}{\lambda\alpha^2} - \frac{3b^2\mu}{\alpha}\cot^2\left\{\frac{1}{2}\arctan\left[\frac{e^{2b\bar{\xi}} - 1}{e^{2b\bar{\xi}} + 1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}} + 1}\right]\right\}, \tag{29}$$

$$u_{22}(\xi) = -\frac{3\lambda\alpha a^2\mu \pm \sqrt{-2\lambda\alpha^3 Q}}{\lambda\alpha^2} - \frac{3\mu\left(b^2 - c^2\right)}{\alpha}\left\{2a\left[\frac{2 + a\bar{\xi}}{\bar{\xi}}\right]^{-1} + \left(b^2 - c^2\right)\left[\frac{2 + a\bar{\xi}}{\bar{\xi}}\right]^{-2}\right\}, \tag{30}$$

$$u_{23}(\xi) = -\frac{\lambda\alpha a^2k^2\mu \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda a^4k^4\mu^2\right)}}{\lambda\alpha^2} - \frac{12a^2k^2\mu}{\alpha}\left\{\left[e^{ka\bar{\xi}} - 1\right]^{-1} - \left[e^{ka\bar{\xi}} - 1\right]^{-2}\right\}, \tag{31}$$

$$u_{24}(\xi) = -\frac{\lambda\alpha\left(3a^2 - 2b^2\right)\mu \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda b^4\mu^2\right)}}{\lambda\alpha^2} + \frac{3\mu}{\alpha}\left\{2a(a + b)\left[\frac{(a + b)e^{b\bar{\xi}} - 1}{(a - b)e^{b\bar{\xi}} - 1}\right]^{-1} - (a + b)^2\left[\frac{(a + b)e^{b\bar{\xi}} - 1}{(a - b)e^{b\bar{\xi}} - 1}\right]^{-2}\right\}, \tag{32}$$

$$u_{25}(\bar{\xi}) = -\frac{\lambda\alpha\mu\left(3c^2 - 2b^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda b^4\mu^2\right)}}{\lambda\alpha^2} - \frac{3\mu}{\alpha}\left\{2(b+c)c\left[\frac{(b+c)e^{b\bar{\xi}} + 1}{(b-c)e^{b\bar{\xi}} - 1}\right]^{-1} - (b+c)^2\left[\frac{(b+c)e^{b\bar{\xi}} + 1}{(b-c)e^{b\bar{\xi}} - 1}\right]^{-2}\right\}, \tag{33}$$

$$u_{26}(\bar{\xi}) = -\frac{\lambda\alpha\mu\left(3a^2 - 2b^2\right) \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda b^4\mu^2\right)}}{\lambda\alpha^2} - \frac{3\mu}{\alpha}\left\{2a(b-a)\left[\frac{e^{b\bar{\xi}} + b - a}{e^{b\bar{\xi}} - b - a}\right]^{-1} + (b-a)^2\left[\frac{e^{b\bar{\xi}} + b - a}{e^{b\bar{\xi}} - b - a}\right]^{-2}\right\}, \tag{34}$$

$$u_{27}(\bar{\xi}) = -\frac{3\lambda\alpha c^2\mu \pm \sqrt{-2\lambda\alpha^3 Q}}{\lambda\alpha^2} - \frac{3c^2\mu}{\alpha}\left\{2\left[\frac{c\bar{\xi} + 2}{c\bar{\xi}}\right]^{-1} + \left[\frac{c\bar{\xi} + 2}{c\bar{\xi}}\right]^{-2}\right\}, \tag{35}$$

$$u_{28}(\bar{\xi}) = -\frac{2\lambda\alpha c^2\mu \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda c^4\mu^2\right)}}{\lambda\alpha^2} - \frac{3c^2\mu}{\alpha}\cot^2\left[\frac{c\xi + C}{2}\right], \tag{36}$$

$$u_{29}(\bar{\xi}) = -\frac{\lambda\alpha a^2\mu \pm \sqrt{\lambda\alpha^2\left(-2\alpha Q + \lambda a^4\mu^2\right)}}{\lambda\alpha^2} - \frac{12\mu}{\alpha}\left\{ac\left[\frac{e^{a\bar{\xi}} - c}{a}\right]^{-1} + c^2\left[\frac{e^{a\bar{\xi}} - c}{a}\right]^{-2}\right\},$$
$$\bar{\xi} = \lambda x + \omega y \pm \frac{\sqrt{\lambda\alpha^2\left(-2Q\alpha + \lambda\Delta^2\mu^2\right)}}{\alpha}t + C. \tag{37}$$

Case 3. $a = 0, b = b, c = c, \mu = \lambda^2\beta + \omega^2\gamma, \bar{\xi} = \xi + C,$

$$A_0 = \frac{2\lambda\alpha\mu\left(b^2 - c^2\right) \pm \sqrt{2\lambda\alpha^2\left(-\alpha Q + 8\lambda\mu^2\left(b^2 - c^2\right)^2\right)}}{\lambda\alpha^2}, A_1 = 0, A_2 = -\frac{3\mu(b-c)^2}{\alpha},$$
$$B_1 = 0, B_2 = -\frac{3\mu(b+c)^2}{\alpha}, \sigma = \mp\frac{\sqrt{2\lambda\alpha^2\left(-\alpha Q + 8\lambda\mu^2\left(b^2 - c^2\right)^2\right)}}{\alpha},$$
$$u(\xi) = A_0 + A_2\tan^2\left(\frac{\varphi(\xi)}{2}\right) + B_2\cot^2\left(\frac{\varphi(\xi)}{2}\right). \tag{38}$$

From the 19 results of reference [66], we can obtain solutions of equation (3) of the form

$$u_{30}(\xi) = \frac{2\lambda\alpha\mu\left(b^2 - c^2\right) \pm \sqrt{2\lambda\alpha^2\left(-\alpha Q + 8\lambda\mu^2\left(b^2 - c^2\right)^2\right)}}{\lambda\alpha^2} - \frac{3\mu\left(b^2 - c^2\right)}{\alpha}\left\{\tanh^2\left(\frac{\sqrt{b^2 - c^2}}{2}\bar{\xi}\right) + \coth^2\left(\frac{\sqrt{b^2 - c^2}}{2}\bar{\xi}\right)\right\}, \tag{39}$$

$$u_{31}(\xi) = \frac{2\lambda\alpha b^2\mu \pm \sqrt{2\lambda\alpha^2\left(-\alpha Q + 8\lambda b^4\mu^2\right)}}{\lambda\alpha^2} - \frac{3b^2\mu}{\alpha}\left\{\tan^2\left[\frac{1}{2}\arctan\left(\frac{e^{2b\bar{\xi}} - 1}{e^{2b\bar{\xi}} + 1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}} + 1}\right)\right] + \cot^2\left[\frac{1}{2}\arctan\left(\frac{e^{2b\bar{\xi}} - 1}{e^{2b\bar{\xi}} + 1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}} + 1}\right)\right]\right\}, \tag{40}$$

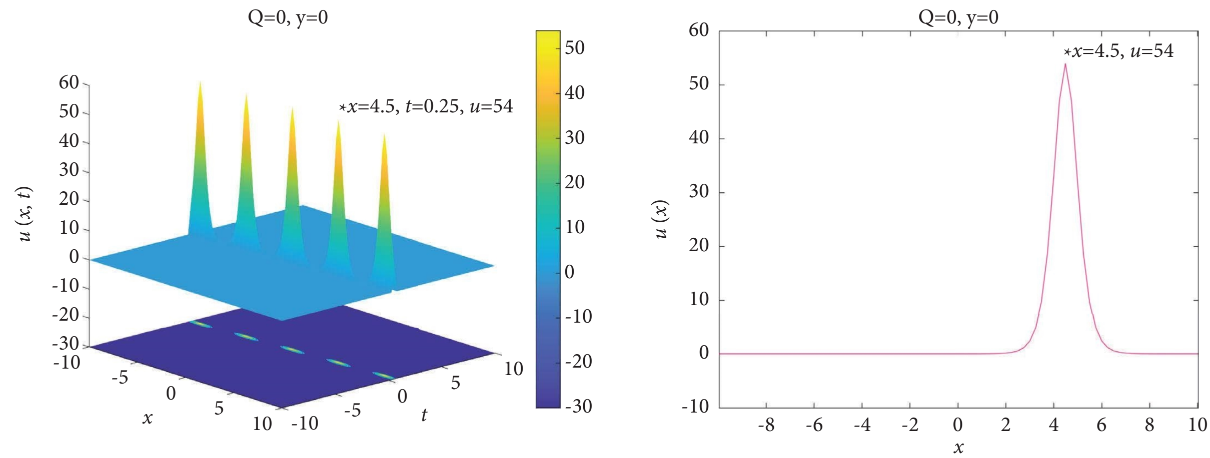

FIGURE 1: The three-dimensional figure and the plane figure of solution (8) with $a = 2, b = 3, c = 2, \lambda = \omega = \alpha = \beta = \gamma = 1, Q = 0, y = 0, C = 0, t = 0.25$.

$$u_{32}(\xi) = \frac{\pm\sqrt{-2\lambda\alpha^3 Q}}{\lambda\alpha^2} - \frac{3\mu}{\alpha}\left\{(b-c)^2\left[\frac{2}{(b-c)\overline{\xi}}\right]^2 + (b+c)^2\left[\frac{2}{(b-c)\overline{\xi}}\right]^{-2}\right\}, \tag{41}$$

$$u_{33}(\overline{\xi}) = \frac{\pm\sqrt{-2\lambda\alpha^3 Q}}{\lambda\alpha^2} - \frac{12c^2\mu}{\alpha}\cot^2[c\overline{\xi}], \tag{42}$$

$$u_{34}(\overline{\xi}) = \frac{\pm\sqrt{-2\lambda\alpha^3 Q}}{\lambda\alpha^2} - \frac{12c^2\mu}{\alpha}\tan^2\left[\frac{1}{c\overline{\xi}}\right]. \tag{43}$$

Here $\overline{\xi} = \lambda x + \omega y \mp \sqrt{2\lambda\alpha^2(-\alpha Q + 8\lambda\mu^2(b^2 - c^2)^2)}/\alpha t + C$.

3. Analysis and Discussion

In this section, we perform numerical simulations for the exact solution to the forced ZK equation. We use MATLAB drawing software to draw the three-dimensional and plane figures of the partial solutions. From the graph, the following conclusions can be drawn:

By depicting the exact periodic travelling wave solution $u_1(x, y, t)$ to equation (1), we can clearly see the effect of external sources Q and terms γu_{xyy} with coefficients on wave propagation. When the parameter is taken as $a = 2, b = 3, c = 2, \lambda = \omega = \alpha = \beta = \gamma = 1,\ y = 0, C = 0$, and $t = 0.25$, the external source is not considered, the corresponding wave amplitude height to Figure 1 is 54. When the external source is considered, that is, $Q = 0.1$ and $Q = 0.5$, it can be seen from the corresponding three-dimensional figure and plane figure in Figures 2 and 3, the amplitude heights are 53.9942 and 53.9663, respectively, so it can be concluded that with the increase in the external source Q, the wave amplitude height of the same time wave is decreasing. In Figure 4, we observe the influence of the longitude variable y on the wave propagation without considering the external source. Compared to the height of the moment $t = 0.25$ in Figure 1, the height changes from 54 to 52.803, which indicates that the wave amplitude increases with the increase in the longitude variable y and the height being reduced, which is consistent with the propagation effect of the external source on the wave amplitude. Through Figure 5, we also observed the case of considering the change of external source and longitude variable y at the same time. Compared to the height of the moment $t = 0.25$ in Figure 1, the height changed from 54 to 52.7645, which indicates that when the external source and longitude variables change at the same time, the drop in the height of the wave amplitude is obvious; compared with only changing the external source, the height change is more obvious. Combining the above analysis, we can conclude that both the external source and longitude variables play a very important role in the propagation of Rossby waves. Figures 1–5 and 6 correspond to $u_1(x, y, t)$ and $u_{16}(x, y, t)$ the three-dimensional and planar graphs of the solutions, which are the exact periodic solutions to the forced ZK equation (1).

We consider an external source $Q = 0.1$ and variable $y = 0.1$, Figures 7 and 8, respectively, represent, when $\beta = 0.5$ and $\beta = 1$ the corresponding three-dimensional graphs and with $t = 0.1, 0.25, 0.5$ the plane graphs of the exact solitary solution $u_5(x, y, t)$ of the forced ZK equation (1). It can be seen from the graphs that with the increase in the dispersion coefficient β, the wave amplitude height changed from 8.54017 to −9.2352, and the height obviously decreased, which was consistent with the previous results.

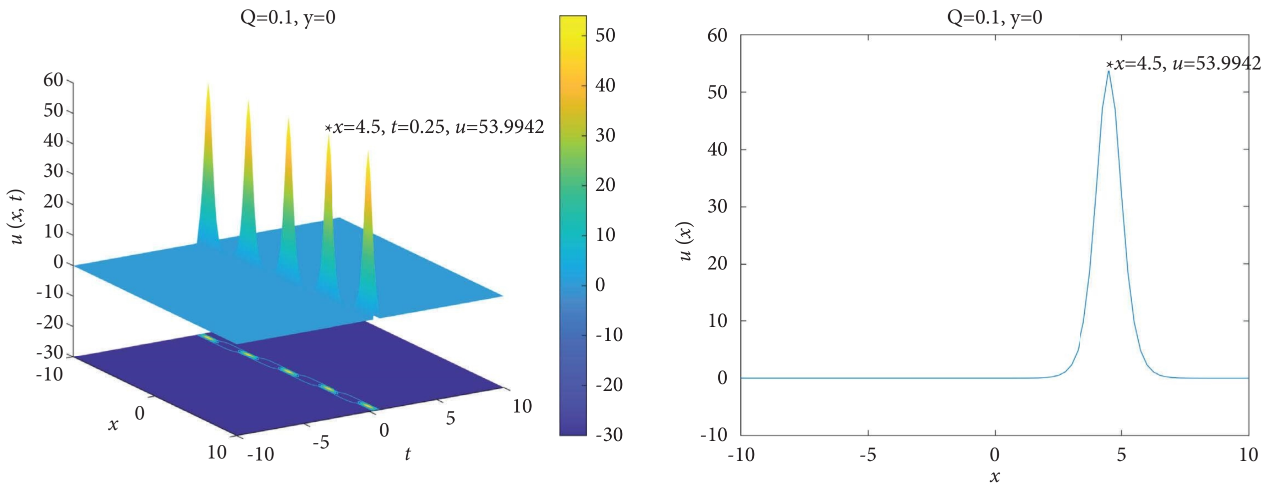

FIGURE 2: Three-dimensional figure and plane figure of solution (8) with $a = 2, b = 3, c = 2, \lambda = \omega = \alpha = \beta = \gamma = 1, Q = 0.1, y = 0, C = 0, t = 0.25$.

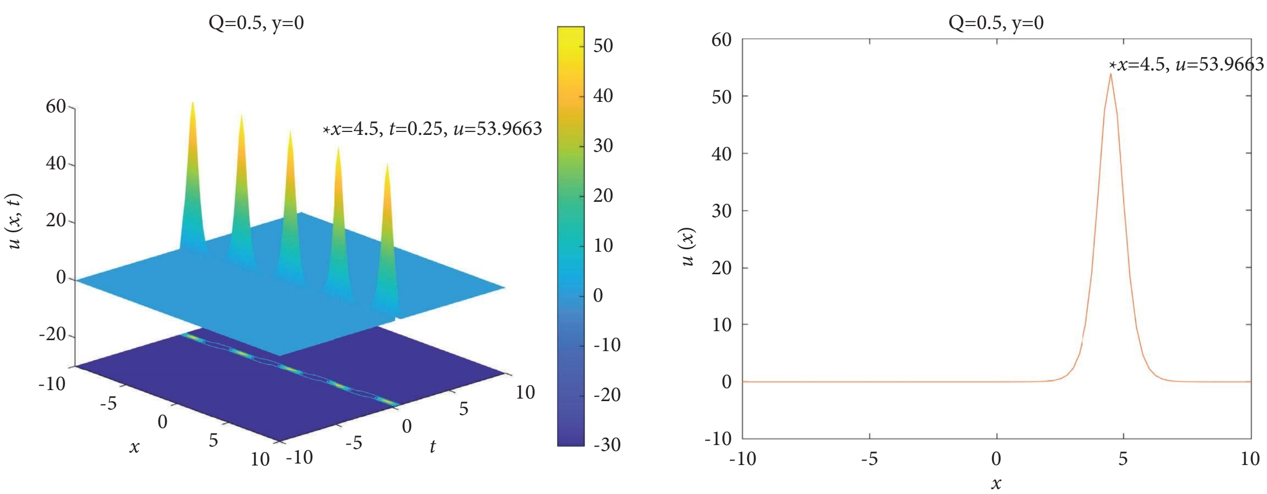

FIGURE 3: Three-dimensional figure and plane figure of solution (8) with $a = 2, b = 3, c = 2, \lambda = \omega = \alpha = \beta = \gamma = 1, Q = 0.5, y = 0, C = 0, t = 0.25$.

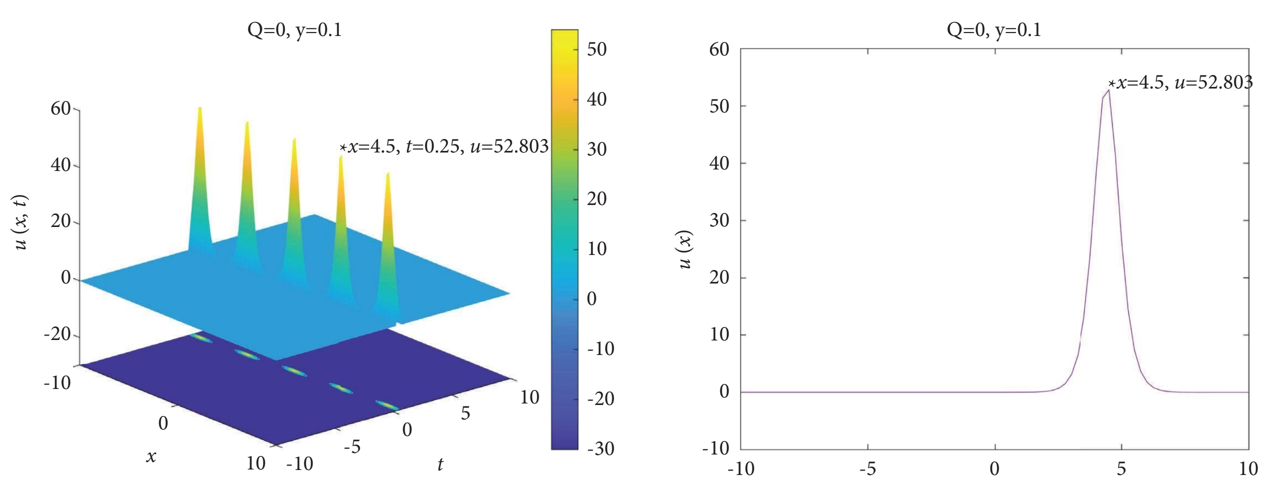

FIGURE 4: The three-dimensional figure and the plane figure of solution (8) with $a = 2, b = 3, c = 2, \lambda = \omega = \alpha = \beta = \gamma = 1, Q = 0, y = 0.1, C = 0, t = 0.25$.

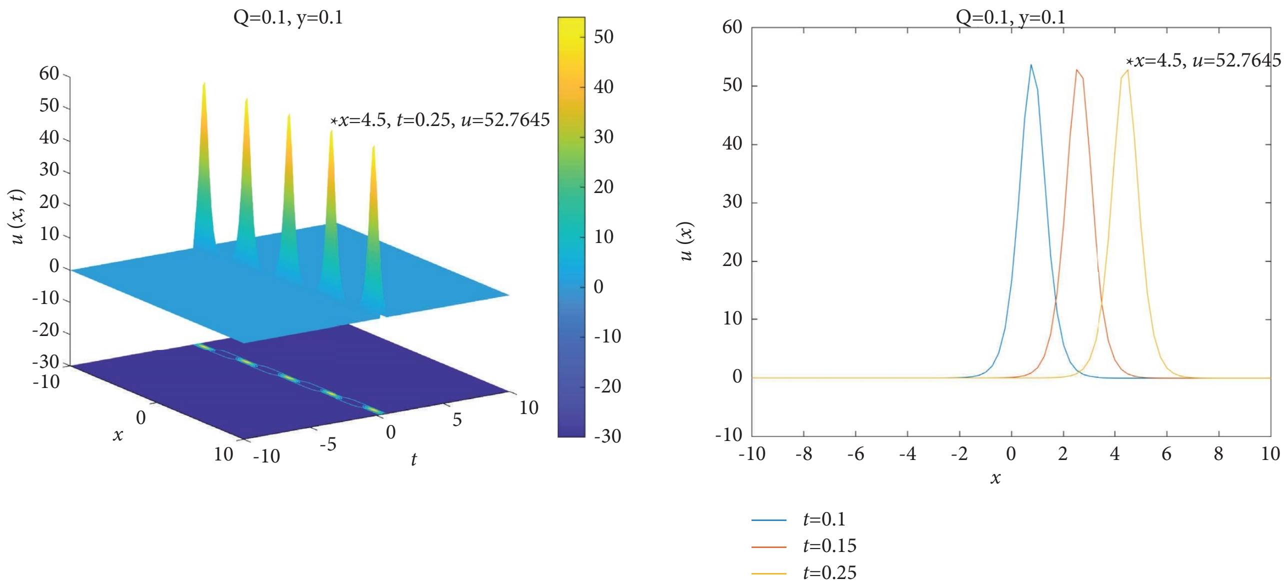

FIGURE 5: The three-dimensional figure and the plane figure of the solution (8) with $a = 2, b = 3, c = 2, \lambda = \omega = \alpha = \beta = \gamma = 1, Q = y = 0.1, C = 0, t = 0.05, 0.15, 0.25$.

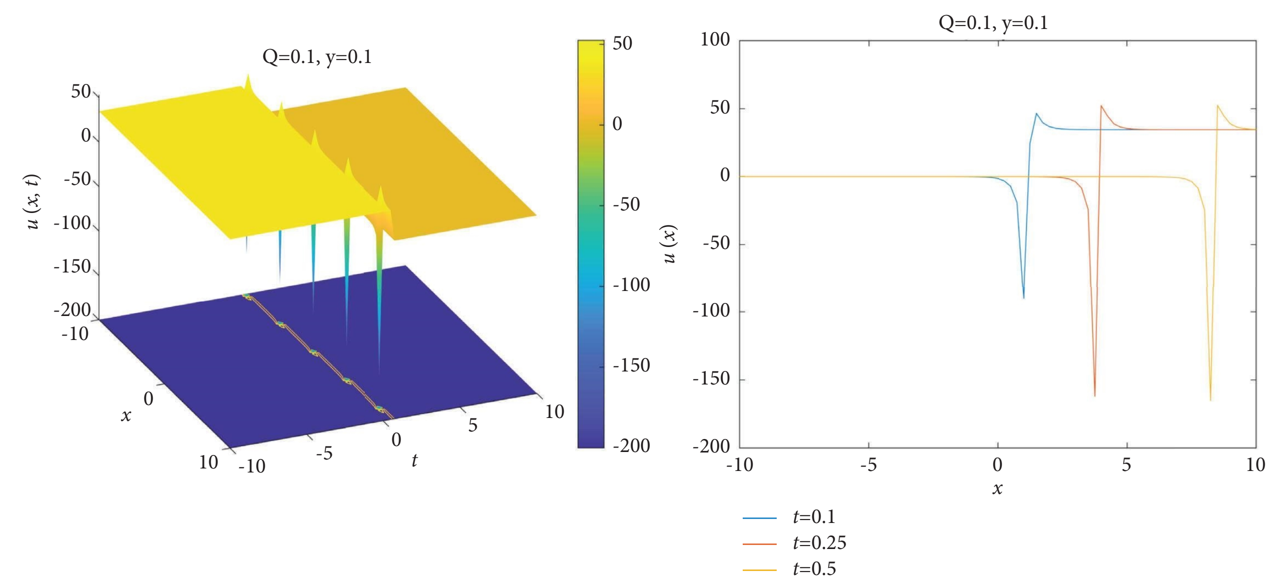

FIGURE 6: The three-dimensional figure and the plane figure of solution (24) with $a = 2, b = 3, c = 2, \lambda = \omega = \alpha = \beta = \gamma = 1, Q = y = 0.1, C = 0, t = 0.1, 0.25, 0.5$.

Figures 7, 8, 9 and 10 show the corresponding three-dimensional graphs and plane graphs of the exact solitary-like solutions $u_5(x, y, t), u_{20}(x, y, t)$, and $u_{30}(x, y, t)$, respectively, for forcing the ZK equation (1).

We also use Figures 11 and 12 to plot the three-dimensional graphs and the corresponding planar graphs at different times for $t = 0.1, 0.25, 0.5$ and $t = 0.05, 0.15, 0.25$ of the exact singular kink-type travelling wave solutions $u_8(x, y, t)$ and $u_{23}(x, y, t)$ of the forced ZK. From Figure 11, it can be seen from the graph that with the change of time, the propagation height of the wave is changing, especially from $t = 0.05$ to $t = 0.15$ the propagation height of the arriving wave has dropped significantly, indicating that the propagation of the wave is significantly affected by the time variable during this period, while the propagation of the arriving wave is significantly affected by the time variable. But from $t = 0.15$ to $t = 0.25$, the height drop is not very obvious, indicating that the wave propagation is not significantly affected by time variables during this period and basically tends to be stable.

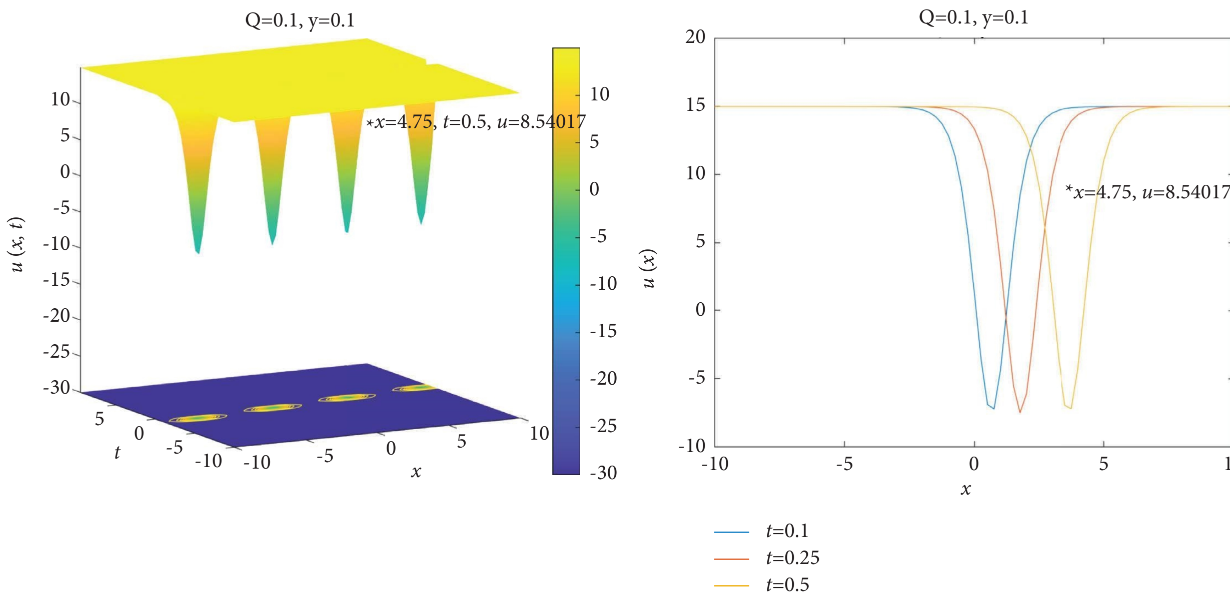

FIGURE 7: The three-dimensional figure and the plane figure of solution (13) with $b = 3, c = 2, \lambda = \omega = \alpha = \gamma = 1, \beta = 0.5, Q = y = 0.1, C = 0, t = 0.1, 0.25, 0.5$.

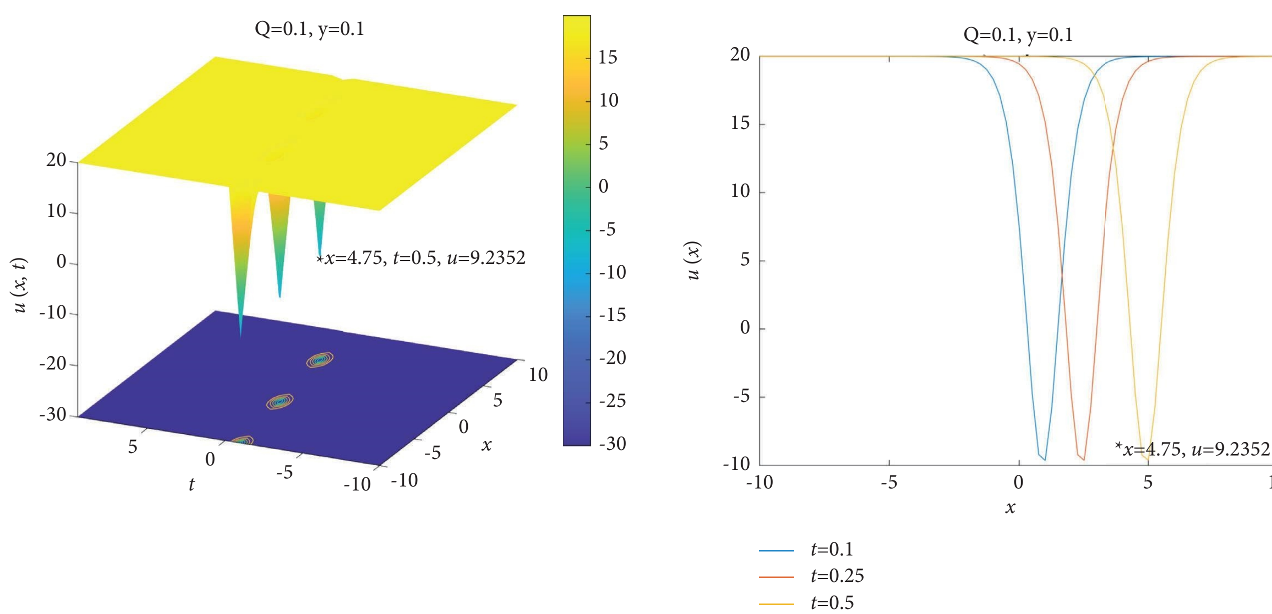

FIGURE 8: The three-dimensional figure and the plane figure of solution (13) with $b = 3, c = 2, \lambda = \omega = \alpha = \gamma = 1, \beta = 1, Q = y = 0.1, C = 0, t = 0.1, 0.25, 0.5$.

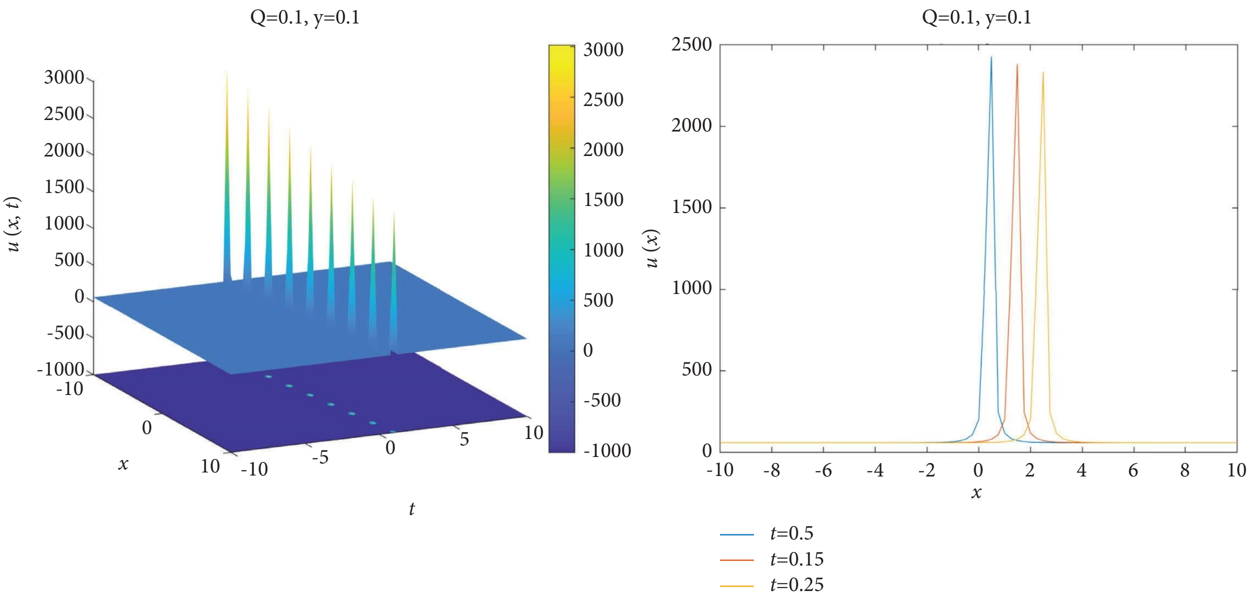

Figure 9: The three-dimensional figure and plane figure of solution (28) with $a = 0, b = 3, c = 2, \lambda = \omega = \alpha = \beta = \gamma = 1, Q = y = 0.1, C = 0, t = 0.05, 0.15, 0.25$.

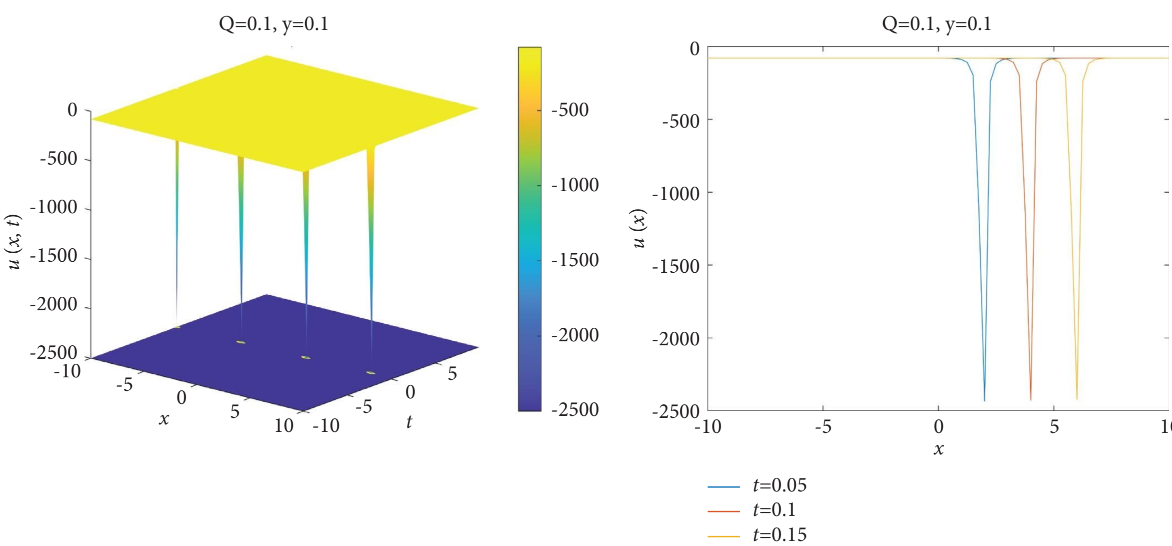

Figure 10: The three-dimensional figure and plane figure of solution (39) with $b = 3, c = 2, \lambda = \omega = \alpha = \beta = \gamma = 1, Q = y = 0.1, C = 0, t = 0.05, 0.1, 0.15$.

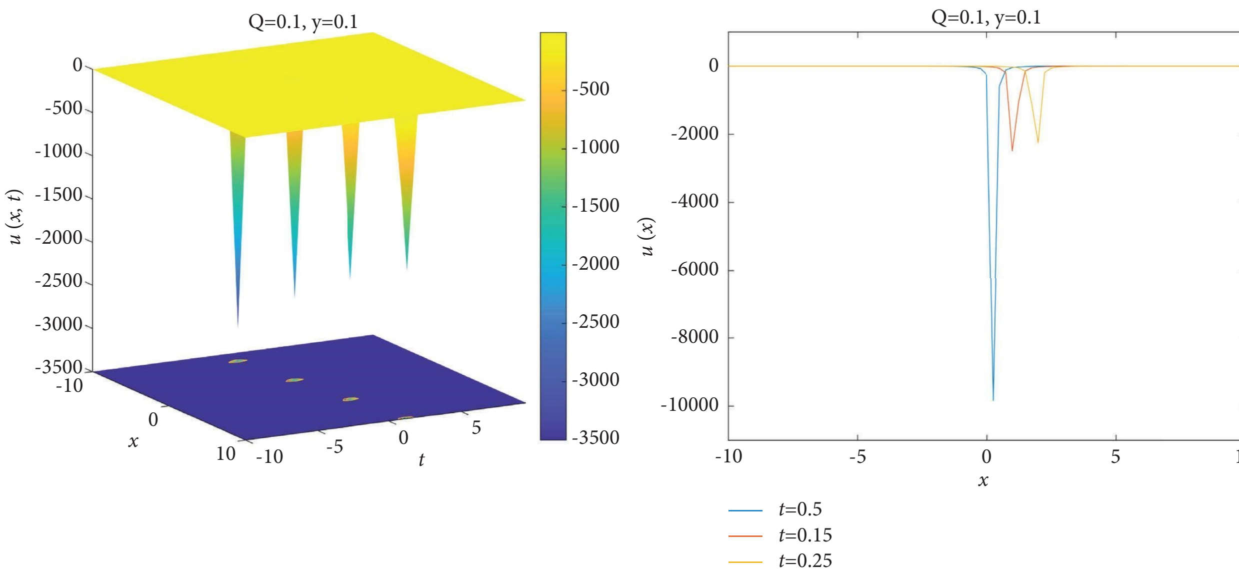

Figure 11: The three-dimensional figure and the plane figure of solution (15) with $a = 2, \lambda = k = \omega = \alpha = \beta = \gamma = 1, Q = y = 0.1, C = 0, t = 0.05, 0.15, 0.25$.

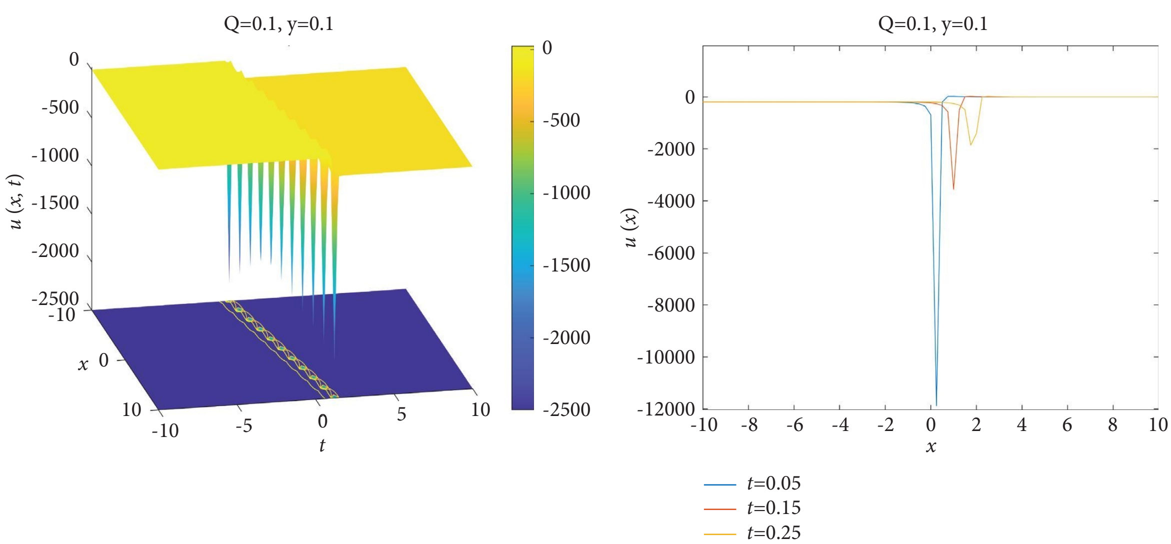

Figure 12: The three-dimensional figure and plane figure of solution (31) with $a = 2, \lambda = k = \omega = \alpha = \beta = \gamma = 1, Q = y = 0.1, C = 0, t = 0.05, 0.15, 0.25$.

4. Conclusions

In this paper, the method of the auxiliary equation is used to solve nonlinear equations, and the improved tan $(\varphi/2)$ expansion method is used to obtain various exact solutions of the forced ZK equation. More new forms of exact solutions, including exact periodic solutions, exact solitary waves solutions, rational function solutions, and singular kink-type travelling wave solutions have been obtained. This shows that the improved tan $(\varphi/2)$ expansion method is an effective mathematical method and tool for solving nonlinear partial differential equations. Then, with the help of MATLAB image processing software, we depict the three-dimensional graphics and plane graphics corresponding to these exact solutions, analyse the influence of external sources, longitude variables, and dispersion coefficient on Rossby wave propagation, and illustrate the influence of y and dispersion coefficient on Rossby wave amplitude. Compared to reference [11], we only use one solution method to take into account the influence of external sources on wave propagation and describe the influence of external sources on wave propagation more vividly through graphics, and we obtained many forms of solutions. These results have a certain practical significance for researchers exploring Rossby propagation of oceanography and atmospheric motions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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