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Research Article

Applicability of Darbo's Fixed Point Theorem on the Existence of a Solution to Fractional Differential Equations of Sequential Type

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In this article, we study the existence of the solution for the fractional differential equations of sequential type with nonlocal integral boundary conditions. The main results are established with the aid of Darbo's fixed point theorem and Hausdorff's measure of noncompactness method. The stability of the proposed fractional differential equation is also investigated via the Ulam–Hyer technique. In addition, an applied example that supports the theoretical results reached through this study is included.

1. Introduction

In light of the numerous applications that it has in the fields of engineering, the social sciences, and the technical sciences, the study of fractional calculus has arisen as an important subject in which to do research. Due to their capacity to reveal the history of ongoing phenomena and processes, fractional order differential and integral operators-based mathematical models are considered more realistic and relevant than their integer-order equivalents. This area of mathematics analysis has advanced significantly in recent years and currently includes a wide range of intriguing finding, such as the studies of [1–7].

Nowadays, academic researchers deal with many physical phenomena in plasma physics, physical chemistry, geophysics, fluid mechanics, nonlinear optics, electromagnetic theory, and fluid motion, and their mathematical models are expressed by nonlinear fractional differential equations (NFDEs). These equations are commonly used in various scientific disciplines and have been investigated from different viewpoints. The exact solutions of these equations have gained more and more interest. For this reason, a lot of different techniques have been dealt with by researchers.

Several studies have been conducted over the years to investigate how stability concepts such as the Mittag–Leffler function and exponential and Lyapunov stability apply to various types of dynamical systems. Ulam and Hyers identified previously unknown types of stability known as Ulam–stability [8–10].

The study of boundary value problems for equations with nonlinear fractional differentials has a prominent and important role in the theory of fractional calculus and in the study of physical phenomena through the physical interpretation of boundary conditions. To pass quickly to the practical applications of fractional derivatives in various applied sciences, some valuable works in this field can be found in [11–19].

Through the in-depth and comprehensive study of fractional differential equations, the existence and uniqueness of solutions to fractional differential equations are proven using a set of fixed point theories, such as Banach's, the Leray–Schauder alternative, Darbo's theorem, and Mönch's fixed point theorem.

In [20], the authors used Darbo's fixed point theorem to study the existence and the stability of the solution of the following fractional differential equation (FDE) which involves the Hadamard fractional derivative (H-FD) of variable order:

$$\begin{cases} {}^{\mathscr{H}}\mathscr{D}_{1+}^{\varrho}\widehat{\mathscr{Q}_1}(\xi) = \mathscr{F}_1\left(\xi, \widehat{\mathscr{Q}_1}(\xi)\right), \xi \in [1, \mathscr{T}], \\ \widehat{\mathscr{Q}_1}(1) = \widehat{\mathscr{Q}_1}(\mathscr{T}) = 0, \end{cases} \quad (1)$$

where $1 < \varrho \leq 2, \mathscr{F}_1\colon [1, \mathscr{T}] \times \mathscr{R} \longrightarrow \mathscr{R}$ is a continuous function and ${}^{\mathscr{H}}\mathscr{D}_{1+}^{\varrho}, {}^{\mathscr{H}}\mathscr{I}_{1+}^{\varrho}$ are the Hadamard fractional derivative and integral of variable-order $\widehat{\mathscr{Q}_1}(\xi)$.

Recently, in 2022, the authors developed the existence theory for a new class of nonlinear coupled systems of sequential fractional differential equations supplemented with coupled, nonconjugate, Riemann–Stieltjes, and integro-multipoint boundary conditions [21]:

$$\begin{cases} \left({}^{c}\mathscr{D}^{\eta_1+1} + {}^{c}\mathscr{D}^{\eta_1}\right)\Phi_1(\xi) = \mathscr{G}_1(\xi, \Phi_1(\xi), \Psi_1(\xi)), \quad 2 < \eta_1 < 3, \quad \xi \in [0,1], \\ \left({}^{c}\mathscr{D}^{\varsigma_1+1} + {}^{c}\mathscr{D}^{\varsigma_1}\right)\Psi_1(\xi) = \mathscr{G}_2(\xi, \Phi_1(\xi), \Psi_1(\xi)), \quad 2 < \varsigma_1 < 3, \quad \xi \in [0,1], \end{cases} \quad (2)$$

subject to the coupled boundary conditions:

$$\begin{cases} \Phi_1(0) = 0, \quad \Phi_1'(0) = 0, \quad \Phi_1'(0) = 0, \quad \Phi_1(1) = k\int_0^{\rho}\Psi_1(\zeta)\mathrm{d}A\,s + \sum_{i=1}^{n-2}\varrho_i\Psi_1(\sigma_i) + k_1\int_{\nu}^{1}\Psi_1(\zeta)\mathrm{d}A(\zeta), \\ \Psi_1(0) = 0, \quad \Psi_1'(0) = 0, \quad \Psi_1'(0) = 0, \quad \Psi_1(1) = h\int_0^{\rho}\Phi_1(\zeta)\mathrm{d}A\,s + \sum_{i=1}^{n-2}\beta_i\Phi_1(\sigma_i) + h_1\int_{\nu}^{1}\Phi_1(\zeta)\mathrm{d}A(\zeta), \end{cases} \quad (3)$$

where ${}^{c}\mathscr{D}^{P}$ denotes the Caputo fractional derivative of order $P \in \eta_1, \varsigma_1, 0 < \rho < \sigma_i < \nu < 1, \mathscr{G}_1, \mathscr{G}_2\colon [0,1] \times \mathscr{R} \times \mathscr{R} \times \mathscr{R} \longrightarrow \mathscr{R}$ are given continuous functions, $k, k_1, h, h_1, \varrho_i, \beta_i \in \mathscr{R}, i = 1, 2, \cdots n-2$, and A is a function of bounded variation.

In [22], the authors studied the existence and uniqueness of a multipoint BVP with H-FD (sequential type):

$$\begin{cases} \left({}^{\mathscr{H}}\mathscr{D}^{\varrho} + \varpi\, {}^{\mathscr{H}}\mathscr{D}^{\varrho-1}\right)\widehat{\mathscr{Q}_1}(\xi) = \mathscr{F}_1\left(\xi, \widehat{\mathscr{Q}_1}(\xi)\right), \quad \xi \in [1, \mathscr{T}], 1 < \varrho \leq 2, \\ \widehat{\mathscr{Q}_1}(1) = 0, \quad \widehat{\mathscr{Q}_1}(\mathscr{T}) = \sum_{j=1}^{m}\delta_{1j}\mathscr{V}\left(\xi_j\right), \end{cases} \quad (4)$$

where ${}^{\mathscr{H}}\mathscr{D}^{\cdot}$ is the Hadamard fractional derivative of order $\varrho, \mathscr{F}_1\colon [1, \mathscr{T}] \times \mathscr{R} \longrightarrow \mathscr{R}$ is a continuous function, $\varpi \in \mathscr{R}^{+}, \xi_j, j = 1, 2, \cdots m$, are given points with $1 \leq \xi_1 \leq \cdots \leq \xi_m < \mathscr{T}$, and δ_{1j} are appropriate real numbers.

In [23], the authors considered the existence and uniqueness of solutions for the following coupled system of Hilfer-type fractional differential equations with Riemann–Stieltjes integral multistrip boundary conditions of the form:

$$\begin{cases} \left({}^{H}\mathscr{D}^{\alpha_1,\beta_1} + \sigma_1\mathscr{D}^{\alpha_1-1,\beta_1}\right)\mathfrak{u}(\xi) = \mathfrak{f}_1(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)), \quad \xi \in [c,d], \\ \left({}^{H}\mathscr{D}^{\alpha_2,\beta_2} + \sigma_2\mathscr{D}^{\alpha_2-1,\beta_2}\right)\mathfrak{u}(\xi) = \mathfrak{f}_2(\xi, \mathfrak{u}(\xi), \mathfrak{v}(\xi)), \quad \xi \in [c,d], \\ \mathfrak{u}(c) = 0, \quad \mathfrak{u}(d) = \lambda_1\int_c^d \mathfrak{v}(s)\mathrm{d}H_1(s) + \sum_{i=1}^{n}\mu_i\int_{\eta_i}^{\zeta_i}\mathfrak{v}(s)\mathrm{d}s, \\ \mathfrak{u}(c) = 0, \quad \mathfrak{u}(d) = \lambda_2\int_c^d \mathfrak{u}(s)\mathrm{d}H_2(s) + \sum_{r=1}^{n}\nu_i\int_{\chi_i}^{\theta_i}\mathfrak{u}(s)\mathrm{d}s, \end{cases} \quad (5)$$

in which ${}^{H}\mathscr{D}^{\alpha_1,\beta_1}$ and ${}^{H}\mathscr{D}^{\alpha_2,\beta_2}$ are the Hilfer fractional derivatives of orders $1 < \alpha_1, \alpha_2 < 2$ and parameters $\beta_1, \beta_2, 0 \leq \beta_1, \beta_2 \leq 1, \mathfrak{f}_1, \mathfrak{f}_2\colon [c,d] \times \mathscr{R} \times \mathscr{R} \longrightarrow \mathscr{R}$ are continuous functions, $\int_c^d(.)\mathrm{d}H_1(s), \int_c^d(.)\mathrm{d}H_2(s)$ are the Riemann–Stieltjes integrals with respect to the functions, and H_i: $[c,d] \longrightarrow \mathcal{R}, i=1,2, c\geq 0, \mu_i, \nu_i \in \mathcal{R}, \chi_i, \theta_i, \eta_i, \zeta_i, \in (c,d), i=1,2\cdots n, r=1,2,\ldots,p, \lambda_1,\lambda_2,\sigma_1,\sigma_2 \in \mathcal{R}$.

Due to the importance of the subject and the possibility of employing it in various scientific fields, many researchers in the field of fractional differential have studied the systems of fractional differentials equations with a variety of serious conditions accompanying them. For more information about, these scientific papers, the reader can see [24–31], and the stability of solutions was studied after the existence of them. To enrich the reader, it is possible to see [32–34].

We study the following nonlinear sequential fractional differential equation to nonseparated nonlocal integral fractional boundary conditions:

$$\begin{cases} \left({}^c\mathscr{D}^{\varrho} + \varpi\mathscr{D}^{\varrho}\right)\widehat{\mathscr{Q}_1}(\xi) = \mathscr{P}\left(\xi, \widehat{\mathscr{Q}_1}(\xi)\right), 1\leq\varrho\leq 2, 0\leq\xi\leq\mathscr{T}, \\ \omega_1\widehat{\mathscr{Q}_1}(\sigma) + \rho_1\widehat{\mathscr{Q}_1}(\mathscr{T}) = \upsilon_1\displaystyle\int_0^{\eta}\widehat{\mathscr{Q}_1}(\zeta)\mathrm{d}\zeta, \\ \omega_2{}^c\mathscr{D}^{\varrho-1}\widehat{\mathscr{Q}_1}(\sigma) + \rho_2{}^c\mathscr{D}^{\varrho-1}\widehat{\mathscr{Q}_1}(\mathscr{T}) = \upsilon_2\displaystyle\int_{\varsigma}^{\mathscr{T}}\widehat{\mathscr{Q}_1}(\zeta)\mathrm{d}\zeta, \end{cases} \tag{6}$$

where $0\leq\sigma\leq\mathscr{T}, 0<\eta<\varsigma<\mathscr{T}, \varpi\in\mathbb{R}_+, \omega_1,\omega_2,\rho_1,\rho_2,\upsilon_1,\upsilon_2\in\mathbb{R}$. The originality and distinction of this work are summarized in employing Darbo's fixed point theorem with the aid of the Hausdorff's measure of noncompactness technique, to verify the necessary condition for the existence of the solution of the fractional and nonlinear equation of sequential type. This work also examines the stability of the solution for the proposed fractional differential equation.

The rest of the article is as follows. Section 2 presents the basic definitions, lemmas, and theorems that underpin our main conclusions. In Section 3, we provide the solutions to the given fractional differential equations (6) using the Darbo's fixed point theorem. Section 4 looks at the Ulam–Hyers stability of the provided fractional differential equations (6). In Section 5, example is provided to further clarify of the study's finding. In Section 6, a conclusion and a future work are introduced.

2. Preliminaries

In this section, we state the most important definitions, lemmas, and theorems which are necessary in obtaining our main results. In addition, we introduce some useful notations that make our result less complicated; also, we finish this section by an auxiliary lemma which gives the solution of our proposed fractional differential equation.

Denote the Banach space of all continuous function by $\mathscr{C}([0,\mathscr{T}],\mathscr{E})$ with the norm:

$$\|\widehat{\mathscr{Q}_1}\|_{\infty} = \sup_{\xi\in\mathscr{J}}\|\mathscr{J}(\xi)\|. \tag{7}$$

Let $\mathscr{L}^1[0,\mathscr{T}]$ represent the space of integrable functions $\widehat{\mathscr{Q}_1}$: $[0,\mathscr{T}]\longrightarrow\mathscr{E}$, with the norm:

$$\|\widehat{\mathscr{Q}_1}\|_1 = \int_a^b\left\|\widehat{\mathscr{Q}_1(\xi)}\right\|\mathrm{d}\xi. \tag{8}$$

Definition 1 (see [35]). Let $\mathscr{E}$ be a Banach space and $\mathscr{V}_2$ a bounded subsets of $\mathscr{E}$. Then, Hausdorff measurable of noncompactness of $\mathscr{V}_2$ is defined by $\chi(\mathscr{V}_2) = \inf\{\tau>0\text{: } \mathscr{V}_2 \text{has a finite cover by balls of radius } \tau\}$.

To discuss the problem in this paper, we need the following lemmas.

Lemma 1 (see [35]). *Let $\mathscr{V}_1,\mathscr{V}_2\subset\mathscr{E}$ be bounded. Then, HMNC has the following properties:*

(1) $\mathscr{V}_1\subset\mathscr{V}_2\Rightarrow\chi(\mathscr{V}_1)\leq\chi(\mathscr{V}_2)$
(2) $\chi(\mathscr{V}_1)=0\Leftrightarrow\mathscr{V}_1$ is a relatively compact
(3) $\chi(\mathscr{V}_1\cup\mathscr{V}_2)=\max\{\chi(\mathscr{V}_1),\chi(\mathscr{V}_2)\}$
(4) $\chi(\mathscr{V}_1)=\chi(\overline{\mathscr{V}_1})=\chi(\mathrm{conv}(\mathscr{V}_1))$, where $\overline{\mathscr{V}_1}$ and $\mathscr{V}_1$ represent the closure and the convex hull of $\mathscr{V}_1$, respectively
(5) $\chi(\mathscr{A}+\mathscr{B})\leq\chi(\mathscr{V}_1)+\chi(\mathscr{V}_2)$, where $(\mathscr{V}_1)+(\mathscr{V}_2)=\{u+v\text{: } u\in\mathscr{V}_1, v\in\mathscr{V}_2\}$
(6) $\chi(\varpi\mathscr{V}_1)\leq|\varpi|\chi(\mathscr{V}_1)\forall\varpi\in\mathbb{R}$

Lemma 2 (see [35]). *If $\mathscr{G}_1\subseteq\mathscr{C}([0,\mathscr{T}],\mathscr{E})$ is bounded and equicontinuous, then $\chi(\mathscr{G}_1(\xi))$ is continuous on $[0,\mathscr{T}]$ and*

$$\chi(\mathscr{G}_1) = \sup_{\xi\in[0,\mathscr{T}]}\chi(\mathscr{G}_1(\xi)). \tag{9}$$

The set $\mathscr{V}_2\subset\mathscr{L}([0,\mathscr{T}],\mathscr{E})$ is called bounded (uniformly) if $\exists\sigma\in\mathscr{L}^1([0,\mathscr{T}],\mathbb{R}^+)$ such that

$$\|u(\zeta)\|\leq\sigma(\zeta),\forall\widehat{\mathscr{Q}_1}\in\mathscr{V}_2. \tag{10}$$

Lemma 3 (see [36]). *If $\left\{\widehat{\mathscr{Q}_{1n}}\right\}_{n=1}^{\infty}\subset\mathscr{L}^1([0,\mathscr{T}],\mathscr{E})$ is integrable (uniformly), then $\chi(\left\{\widehat{\mathscr{Q}_{1n}}\right\}_{n=1}^{\infty})$ is measurable, and*

$$\chi\left(\left\{\int_0^t\widehat{\mathscr{Q}_{1n}}(\zeta)d\zeta\right\}_{n=1}^{\infty}\right)\leq\int_0^t\chi\left(\left\{\widehat{\mathscr{Q}_{1n}}(\zeta)\right\}_{n=1}^{\infty}\right)d\zeta. \tag{11}$$

Lemma 4 (see [37]). *If a set $\mathscr{G}_1$ is bounded, then $\forall\epsilon$, $\exists\left\{\widehat{\mathscr{Q}_{1n}}\right\}_{n=1}^{\infty}\subset\mathscr{G}_1$, such that*

$$\chi\mathscr{G}_1\leq 2\chi\left(\left\{\widehat{\mathscr{Q}_{1n}}(\zeta)\right\}_{n=1}^{\infty}\right)+\delta. \tag{12}$$

Definition 2 (see [38]). A function $\mathscr{J}$: $[c,d]\times\mathscr{E}\longrightarrow\mathscr{E}$ satisfies the carathéodory conditions, if the following points are satisfied:

(i) $\mathscr{J}(\xi,\widehat{\mathscr{Q}_1})$ is continuous w. r. t. ξ for $\widehat{\mathscr{Q}_1}\in\mathscr{E}\,\forall\,\xi\in[c,d]$
(ii) $\mathscr{J}(\xi,\widehat{\mathscr{Q}_1})$ is measurable with respect to ξ for $\widehat{\mathscr{Q}_1}\in\mathscr{E}$

Definition 3 (see [39]). The function $\mathscr{J}$: $\Omega\subset\mathscr{E}\longrightarrow\mathscr{E}$ is a $\chi-$ contraction, if $\exists k$, $0<k<1$ such that,

$$\chi(\mathscr{J}(\mathscr{V}_1))\leq k\chi\mathscr{V}_1, \tag{13}$$

for all bounded $\mathscr{V}_1\subset\Omega$.

Next, we state the most important theory on which the results of this work are based. It is called the fixed point theory of "Darbo and Sadovskii" [35, 40].

Theorem 1. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space $\mathscr{E}$ and let $\mathscr{J}: \Omega \longrightarrow \Omega$ be a continuous operator. If $\mathscr{J}$ is a $\chi-$ contraction, then $\mathscr{J}$ has at least one fixed point.*

Definition 4 (see [41]). The RL fractional integral of order $\varrho > 0$ for a function $\mathscr{P}: [0, +\infty] \longrightarrow \mathscr{R}$ is defined as

$$\mathscr{I}^{\varrho}_{0+}\mathscr{P}(\xi) = \frac{1}{\Gamma(\varrho)}\int_0^{\xi} (\xi-\zeta)^{\varrho-1}\mathscr{P}(\zeta)\mathrm{d}\zeta. \tag{14}$$

Definition 5 (see [41]). The Caputo derivative of order $\varrho > 0$ for a function $\mathscr{P}: [0, +\infty] \longrightarrow \mathscr{R}$ is written as

$$\mathscr{D}^{\varrho}_{0+}\mathscr{P}(\xi) = \frac{1}{\Gamma(n-\varrho)}\int_0^{\xi} (\xi-\zeta)^{n-\varrho-1}\mathscr{P}^{(n)}(\zeta)\mathrm{d}\zeta, \tag{15}$$

where $n = [\varrho] + 1$, $[\varrho]$ is integral part of ϱ.

Lemma 5. *Let $\varrho > 0$. Then, the differential equation $\mathscr{D}^{\varrho}_{0+}\mathscr{P}(\xi) = 0$ has the solution:*

$$\mathscr{P}(\xi) = c_0 + c_1\xi + c_1\xi^2 + \cdots + c_{n-1}\xi^{n-1},$$
$$\mathscr{I}^{\varrho}_{0+}\mathscr{D}^{\varrho}_{0+}\mathscr{P}(\xi) = \mathscr{P}(\xi) + c_0 + c_1\xi + c_1\xi^2 + \cdots + c_{n-1}\xi^{n-1}, \tag{16}$$

where $c_i \in \mathbb{R}$ and $i = 1, 2\cdots, n = [\varrho] + 1$.

In what follows we use the following notations:

$$\begin{aligned}
a_{11} &= \omega_1 e^{-\varpi\sigma} + \rho_1 e^{-\varpi\mathscr{T}} - \frac{v_1}{\varpi}\left(1 - e^{-\varpi\eta}\right),\\
a_{12} &:= \omega_1 + \rho_1 - v_1\eta,\\
a_{21} &= \omega_2 \frac{\varpi}{\Gamma(2-\varrho)}\int_0^{\sigma} (\sigma-\zeta)^{1-\varrho} e^{-\varpi s}\mathrm{d}\zeta + \rho_2 \frac{\varpi}{\Gamma(2-\varrho)}\int_0^{\mathscr{T}} (\mathscr{T}-\zeta)^{1-\varrho} e^{-\varpi s}\mathrm{d}\zeta + v_2\int_{\varsigma}^{\mathscr{T}} e^{-\varpi\xi}\mathrm{d}\zeta,\\
a_{22} &= v_2(\mathscr{T} - \varsigma),\\
\Delta &= a_{11}a_{22} - a_{12}a_{21}, \Delta \neq 0,\\
\phi_1(\xi) &= \frac{a_{21} - a_{22}e^{-\varpi\xi}}{\Delta},\\
\phi_2(\xi) &= \frac{a_{11} - a_{12}e^{-\varpi\xi}}{\Delta},\\
\Theta_1(\xi,\zeta) &= \frac{1}{\Gamma(\varrho-1)}\int_s^{\xi} e^{-\varpi(\xi-r)}(r-\zeta)^{\varrho-2}\mathrm{d}r,\\
\Theta_2(\xi,r) &= \frac{1}{\Gamma(2-\varrho)}\int_r^{\xi} (\xi-\zeta)^{1-\varrho}\Theta_1(\zeta,r)\mathrm{d}r.
\end{aligned} \tag{17}$$

It is clear that

$$\left|\varphi_1(\xi)\right| \leq \max\left(\frac{\left|a_{21} - a_{22}\right|}{|\Delta|}, \frac{\left|a_{21} - a_{22}e^{-\varpi\mathscr{T}}\right|}{|\Delta|}\right) = \phi_1,$$
$$\left|\varphi_2(\xi)\right| \leq \max\left(\frac{\left|a_{11} - a_{12}\right|}{|\Delta|}, \frac{\left|a_{11} - a_{12}e^{-\varpi\mathscr{T}}\right|}{|\Delta|}\right) = \phi_2,$$

$$\int_0^{\xi} e^{-\varpi(\xi-r)}\mathcal{J}^{\varrho-1}h(r)\mathrm{d}r = \int_0^{\xi}\Theta_1(\xi,\zeta)h(\zeta)\mathrm{d}\zeta,$$
$$\frac{1}{\Gamma(2-\varrho)}\int_0^{\xi}(\xi-\zeta)^{1-\varrho}\int_0^{\xi} e^{-\varpi(\xi-r)}\mathcal{J}^{\varrho-1}h(r)\mathrm{d}r = \int_0^{\xi}\Theta_2(\xi,r)h(r)\mathrm{d}r. \quad (18)$$

Lemma 6. *Let* $h\in\mathscr{C}([0,\mathscr{T}],\mathbb{R})$*. Then, the following boundary value problem:*

$$\begin{cases} \left({}^{c}\mathscr{D}^{\varrho}+\varpi\mathscr{D}^{\varrho}\right)\widehat{\mathscr{Q}_1}(\xi)=h(\xi), 1\leq\varrho\leq 2, 0\leq\xi\leq\mathscr{T},\\ \omega_1\widehat{\mathscr{Q}_1}(\sigma)+\rho_1\widehat{\mathscr{Q}_1}(\mathscr{T})=\upsilon_1\int_0^{\eta}\widehat{\mathscr{Q}_1}(\zeta)\mathrm{d}\zeta,\\ \omega_2{}^{c}\mathscr{D}^{\varrho-1}\widehat{\mathscr{Q}_1}(\sigma)+\rho_2{}^{c}\mathscr{D}^{\varrho-1}\widehat{\mathscr{Q}_1}(\mathscr{T})=\upsilon_2\int_{\varsigma}^{\mathscr{T}}\widehat{\mathscr{Q}_1}(\zeta)\mathrm{d}\zeta, \end{cases} \quad (19)$$

is equivalent to the fractional integral equations:

$$\begin{aligned} \widehat{\mathscr{Q}_1}(\xi) = &\int_0^{\xi}\Theta_1(\xi,\zeta)h(\zeta)\mathrm{d}\zeta\\ &+\omega_1\phi_1(\xi)\int_0^{\sigma}\Theta_1(\sigma,\zeta)h(\zeta)\mathrm{d}\zeta\\ &+\rho_1\phi_1(\xi)\int_0^{\mathscr{T}}\Theta_1(\mathscr{T},\zeta)h(\zeta)\mathrm{d}\zeta\\ &-\upsilon_1\phi_1(\xi)\int_0^{\eta}\int_0^{r}\Theta_1(r,\zeta)h(\zeta)\mathrm{d}\zeta\,\mathrm{d}r\\ &-\upsilon_2\phi_2(\xi)\int_{\varsigma}^{\mathscr{T}}\int_0^{\xi}\Theta_1(\xi,\zeta)h(\zeta)\mathrm{d}\zeta\,\mathrm{d}\xi\\ &-\varpi\omega_2\phi_2(\xi)\int_0^{\sigma}\Theta_2(\sigma,\zeta)h(\zeta)\mathrm{d}\zeta\\ &-\varpi\rho_2\phi_2(\xi)\int_0^{\mathscr{T}}\Theta_2(\mathscr{T},\zeta)h(\zeta)\mathrm{d}\zeta\\ &+\omega_2\phi_2(\xi)\int_0^{\sigma}h(\zeta)\mathrm{d}\zeta\\ &+\rho_2\phi_2(\xi)\int_0^{\mathscr{T}}h(\zeta)d\zeta. \end{aligned} \quad (20)$$

Proof. Taking the operator $\mathscr{J}^{\varrho-1}$ to both sides of (19) we obtain

$$\begin{aligned} \mathscr{J}^{\varrho-1c}\mathscr{D}^{\varrho-1}(\mathscr{D}+\varpi)\widehat{\mathscr{Q}_1}(\xi) &= \mathscr{J}^{\varrho-1}h(\xi),\\ (\mathscr{D}+\varpi)\widehat{\mathscr{Q}_1}(\xi)-c_0 &= \mathscr{J}^{\varrho-1}h(\xi). \end{aligned} \quad (21)$$

We solve the above-given linear differential equations:

$$\begin{aligned} \widehat{\mathscr{Q}_1}(\xi) &= \left(\widehat{\mathscr{Q}_1}(0)-c_0\right)e^{-\varpi\xi}+c_0+\int_0^{\xi}e^{-\varpi(\xi-\zeta)}\mathscr{J}^{\varrho-1}h(\zeta)\mathrm{d}\zeta,\\ \widehat{\mathscr{Q}_1}(\xi) &= c_1e^{-\varpi\xi}+c_0+\int_0^{\xi}e^{-\varpi(\xi-\zeta)}\mathscr{J}^{\varrho-1}h(\zeta)\mathrm{d}\zeta. \end{aligned} \quad (22)$$

The condition $(\omega_1\widehat{\mathscr{Q}_1}(\sigma)+\rho_1\widehat{\mathscr{Q}_1}(\mathscr{T})=\upsilon_1\int_0^{\eta}\mathscr{Q}_1\widehat{(\zeta)}\mathrm{d}\zeta)$, leads to

$$\begin{aligned}
\omega_1\widehat{\mathcal{Q}_1}(\sigma)+\rho_1\widehat{\mathcal{Q}_1}(\mathcal{T}) &= \omega_1 c_1 e^{-\varpi\sigma}+\omega_1 c_0+\omega_1\int_0^\sigma e^{-\varpi(\sigma-\zeta)}\mathcal{I}^{\varrho-1}h(\zeta)\mathrm{d}\zeta\\
&\quad+\rho_1 c_1 e^{-\varpi\mathcal{T}}+\rho_1 c_0+\rho_1\int_0^{\mathcal{T}} e^{-\varpi(\mathcal{T}-\zeta)}\mathcal{I}^{\varrho-1}h(\zeta)\mathrm{d}\zeta\\
&= \upsilon_1\int_0^\eta\left(c_1 e^{-\varpi r}+c_0+\int_0^r e^{-\varpi(r-\zeta)}\mathcal{I}^{\varrho-1}h(\zeta)\mathrm{d}\zeta\right)\mathrm{d}r\\
&= \frac{\upsilon_1 c_1}{\varpi}\left(1-e^{-\varpi\eta}\right)+\upsilon_1 c_0\eta+\upsilon_1\int_0^\eta\int_0^r e^{-\varpi(r-\zeta)}\mathcal{I}^{\varrho-1}h(\zeta)\mathrm{d}\zeta\,\mathrm{d}r,\\
&\cdot\left(\omega_1 e^{-\varpi\sigma}+\rho_1 e^{-\varpi\mathcal{T}}-\frac{\upsilon_1}{\varpi}\left(1-e^{-\varpi\eta}\right)\right)c_1+\left(\omega_1+\rho_1-\upsilon_1\eta\right)c_0\\
&= \upsilon_1\int_0^\eta\int_0^r e^{-\varpi(r-\zeta)}\mathcal{I}^{\varrho-1}h(\zeta)\mathrm{d}\zeta\,\mathrm{d}r-\omega_1\int_0^\sigma e^{-\varpi(\sigma-\zeta)}\mathcal{I}^{\varrho-1}h(\zeta)\mathrm{d}\zeta\\
&\quad+\rho_1\int_0^{\mathcal{T}} e^{-\varpi(\mathcal{T}-\zeta)}\mathcal{I}^{\varrho-1}h(\zeta)\mathrm{d}\zeta.
\end{aligned}\tag{23}$$

While, condition $(\omega_2{}^c\mathcal{D}^{\varrho-1}\widehat{\mathcal{Q}_1}(\sigma)+\rho_2{}^c\mathcal{D}^{\varrho-1}\widehat{\mathcal{Q}_1}(\mathcal{T})=\upsilon_2\int_\varsigma^{\mathcal{T}}\widehat{\mathcal{Q}_1(\zeta)}\mathrm{d}\zeta)$, implies

$$\begin{aligned}
&\left(\omega_2\frac{\varpi}{\Gamma(2-\varrho)}\int_0^\sigma(\sigma-\zeta)^{1-\varrho}e^{-\varpi s}\mathrm{d}\zeta+\rho_2\frac{\varpi}{\Gamma(2-\varrho)}\int_0^{\mathcal{T}}(\mathcal{T}-\zeta)^{1-\varrho}e^{-\varpi s}\mathrm{d}\zeta+\upsilon_2\int_\varsigma^{\mathcal{T}}e^{-\varpi\xi}\mathrm{d}\xi\right)c_1+\upsilon_2(\mathcal{T}-\varsigma)c_0\\
&= \omega_2\frac{\varpi}{\Gamma(2-\varrho)}\int_0^\sigma(\sigma-\zeta)^{1-\varrho}\left(\mathcal{I}^{\varrho-1}h(\zeta)-\varpi\int_0^\zeta e^{-\varpi(\zeta-r)}\mathcal{I}^{\varrho-1}h(r)\mathrm{d}r\right)\mathrm{d}\zeta\\
&\quad+\rho_2\frac{\varpi}{\Gamma(2-\varrho)}\int_0^{\mathcal{T}}(\mathcal{T}-\zeta)^{1-\varrho}\left(\mathcal{I}^{\varrho-1}h(\zeta)-\varpi\int_0^\zeta e^{-\varpi(\zeta-r)}\mathcal{I}^{\varrho-1}h(r)\mathrm{d}r\right)\mathrm{d}\zeta\\
&\quad-\upsilon_2\int_\varsigma^{\mathcal{T}}\int_0^\xi e^{-\varpi(\xi-\zeta)}\mathcal{I}^{\varrho-1}h(\zeta)\mathrm{d}\zeta\,\mathrm{d}\xi.
\end{aligned}\tag{24}$$

Thus,

$$\begin{aligned}
a_{11}c_1+a_{12}c_0 &= \upsilon_1\int_0^\eta\int_0^r\Theta_1(r,\zeta)h(\zeta)\mathrm{d}\zeta\,\mathrm{d}r-\omega_1\int_0^\sigma\Theta_1(\sigma,\zeta)h(\zeta)\mathrm{d}\zeta-\rho_1\int_0^{\mathcal{T}}\Theta_1(\mathcal{T},\zeta)h(\zeta)\mathrm{d}\zeta,\\
a_{21}c_1+a_{22}c_0 &= \omega_2\int_0^\sigma h(\zeta)\mathrm{d}\zeta+\rho_2\int_0^{\mathcal{T}}h(\zeta)\mathrm{d}\zeta-\varpi\omega_2\int_0^\sigma\Theta_2(\sigma,\zeta)h(\zeta)\mathrm{d}\zeta-\varpi\rho_2\int_0^{\mathcal{T}}\Theta_2(\mathcal{T},\zeta)h(\zeta)\mathrm{d}\zeta\\
&\quad-\upsilon_2\int_\varsigma^{\mathcal{T}}\int_0^\xi\Theta_1(\xi,\zeta)h(\zeta)\mathrm{d}\zeta\,\mathrm{d}\xi.
\end{aligned}\tag{25}$$

A simultaneous solution for the above-given system gives

$$
\begin{aligned}
c_0 = {} & \frac{a_{11}}{\Delta}\omega_2 \int_0^{\sigma} h(\zeta)\mathrm{d}\zeta + \frac{a_{11}}{\Delta}\rho_2 \int_0^{\mathcal{T}} h(\zeta)\mathrm{d}\zeta \\
& - \frac{a_{11}}{\Delta}\varpi\omega_2 \int_0^{\sigma} \Theta_2(\sigma,\zeta)h(\zeta)\mathrm{d}\zeta - \frac{a_{11}}{\Delta}\varpi\rho_2 \int_0^{\mathcal{T}} \Theta_2(\mathcal{T},\zeta)h(\zeta)\mathrm{d}\zeta \\
& - \frac{a_{11}}{\Delta}\upsilon_2 \int_{\varsigma}^{\mathcal{T}} \int_0^{\xi} \Theta_1(\xi,\zeta)h(\zeta)\mathrm{d}\zeta\,\mathrm{d}\xi \\
& - \frac{a_{21}}{\Delta}\upsilon_1 \int_0^{\eta} \int_0^{r} \Theta_1(r,\zeta)h(\zeta)\mathrm{d}\zeta\,\mathrm{d}r \\
& - \frac{a_{21}}{\Delta}\omega_1 \int_0^{\sigma} \Theta_1(\sigma,\zeta)h(\zeta)\mathrm{d}\zeta \\
& + \frac{a_{21}}{\Delta}\rho_1 \int_0^{\mathcal{T}} \Theta_1(\mathcal{T},\zeta)h(\zeta)\mathrm{d}\zeta, \\
c_1 = {} & \frac{a_{22}}{\Delta}\upsilon_1 \int_0^{\eta} \int_0^{r} \Theta_1(r,\zeta)h(\zeta)\mathrm{d}\zeta\,\mathrm{d}r \\
& - \frac{a_{22}}{\Delta}\omega_1 \int_0^{\sigma} \Theta_1(\sigma,\zeta)h(\zeta)\mathrm{d}\zeta \\
& - \frac{a_{22}}{\Delta}\rho_1 \int_0^{\mathcal{T}} \Theta_1(\mathcal{T},\zeta)h(\zeta)\mathrm{d}\zeta \\
& - \frac{a_{12}}{\Delta}\omega_2 \int_0^{\sigma} h(\zeta)\mathrm{d}\zeta \\
& - \frac{a_{12}}{\Delta}\rho_2 \int_0^{\mathcal{T}} h(\zeta)\mathrm{d}\zeta \\
& + \frac{a_{12}}{\Delta}\varpi\omega_2 \int_0^{\sigma} \Theta_2(\sigma,\zeta)h(\zeta)\mathrm{d}\zeta + \frac{a_{12}}{\Delta}\varpi\rho_2 \int_0^{\mathcal{T}} \Theta_2(\mathcal{T},\zeta)h(\zeta)\mathrm{d}\zeta \\
& + \frac{a_{12}}{\Delta}\upsilon_2 \int_{\varsigma}^{\mathcal{T}} \int_0^{\xi} \Theta_1(\xi,\zeta)h(\zeta)\mathrm{d}\zeta\,\mathrm{d}\xi.
\end{aligned} \tag{26}
$$

Substituting the obtained values of $c_i, i = 1,2$ in (22) leads to (20). □

Lemma 7. *Given* $\mathfrak{f}_1, \mathfrak{f}_2 \in \mathcal{C}([0,\mathcal{T}],\mathbb{R})$ *we have*

$$
\begin{aligned}
& \left| \int_0^{\xi} \Theta_1(\xi,\zeta)\mathfrak{f}_1(\zeta)d\zeta - \int_0^{\xi} \Theta_1(\xi,\zeta)\mathfrak{f}_2(\zeta)d\zeta \right| \leq \frac{\xi^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1 - e^{-\varpi\xi}\right)\|\mathfrak{f}_1 - \mathfrak{f}_2\|_{\mathcal{C}}, \\
& \left| \int_0^{\xi} \Theta_2(\xi,\zeta)\mathfrak{f}_1(\zeta)d\zeta - \int_0^{\xi} \Theta_2(\xi,\zeta)\mathfrak{f}_2(\zeta)d\zeta \right| \leq \frac{\xi}{\varpi}\left(1 - e^{-\varpi\xi}\right)\|\mathfrak{f}_1 - \mathfrak{f}_2\|_{\mathcal{C}}.
\end{aligned} \tag{27}
$$

Proof. Indeed,

$$
\begin{aligned}
&\left|\int_0^{\xi}\Theta_1(\xi,\zeta)\mathfrak{f}_1(\zeta)d\zeta-\int_0^{\xi}\Theta_1(\xi,\zeta)\mathfrak{f}_2(\zeta)d\zeta\right|\\
&\le\int_0^{\xi}\Theta_1(\xi,\zeta)\left|\mathfrak{f}_1(\zeta)-\mathfrak{f}_2(\zeta)\right|d\zeta\\
&\le\int_0^{\xi}\Theta_1(\xi,\zeta)d\zeta\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\\
&\le\frac{1}{\Gamma(\varrho-1)}\int_0^{\xi}\left(\int_s^{\xi}e^{-\varpi(\xi-r)}(r-\zeta)^{\varrho-2}dr\right)d\zeta\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\\
&\le\frac{1}{\Gamma(\varrho-1)}\int_0^{\xi}\int_0^{r}e^{-\varpi(\xi-r)}(r-\zeta)^{\varrho-2}d\zeta\, dr\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\\
\frac{\xi^{\varrho-1}}{\varpi(\varrho-1)\Gamma(\varrho-1)}\left(1-e^{-\varpi\xi}\right)\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\quad&\le\frac{\mathscr{T}^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\mathscr{T}}\right)\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}.
\end{aligned}
\tag{28}
$$

On the other hand,

$$
\begin{aligned}
&\left|\int_0^{\xi}\Theta_2(\xi,\zeta)\mathfrak{f}_1(\zeta)d\zeta-\int_0^{\xi}\Theta_2(\xi,\zeta)\mathfrak{f}_2(\zeta)d\zeta\right|\\
&=\left|\frac{1}{\Gamma(2-\varrho)}\int_0^{\xi}(\xi-\zeta)^{1-\varrho}\int_0^{\zeta}e^{-\varpi(\zeta-r)}\mathscr{I}^{\varrho-1}(\mathfrak{f}_1(r)-\mathfrak{f}_2(r))dr\, d\zeta\right|\\
&=\frac{1}{\Gamma(2-\varrho)\Gamma(\varrho-1)}\left|\int_0^{\xi}(\xi-\zeta)^{1-\varrho}\int_0^{\zeta}e^{-\varpi(\zeta-r)}\int_0^{r}(r-l)^{\varrho-2}(\mathfrak{f}_1(l)-\mathfrak{f}_2(l))dl\, dr\, d\zeta\right|\\
&\le\frac{1}{(\varrho-1)\Gamma(2-\varrho)\Gamma(\varrho-1)}\int_0^{\xi}(\xi-\zeta)^{1-\varrho}\int_0^{\zeta}e^{-\varpi(\zeta-r)}r^{\varrho-2}dr\, d\zeta\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\\
&\le\frac{1}{(\varrho-1)\Gamma(2-\varrho)\Gamma(\varrho-1)}\int_0^{\xi}(\xi-\zeta)^{1-\varrho}\zeta^{\varrho-1}\int_0^{\zeta}e^{-\varpi(\zeta-r)}dr\, d\zeta\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\\
&\le\frac{\left(1-e^{-\varpi\xi}\right)}{\varpi(\varrho-1)\Gamma(2-\varrho)\Gamma(\varrho-1)}\int_0^{\xi}(\xi-\zeta)^{1-\varrho}\zeta^{\varrho-1}d\zeta\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\\
&\le\frac{\left(1-e^{-\varpi\xi}\right)\xi}{\varpi(\varrho-1)\Gamma(2-\varrho)\Gamma(\varrho-1)}\int_0^{1}(1-\zeta)^{1-\varrho}\zeta^{\varrho-1}d\zeta\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\\
&\le\frac{\left(1-e^{-\varpi\xi}\right)\xi}{\varpi(\varrho-1)\Gamma(2-\varrho)\Gamma(\varrho-1)}\mathscr{B}(\varrho,2-\varrho)\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\\
&\le\frac{\left(1-e^{-\varpi\xi}\right)\xi}{\varpi(\varrho-1)\Gamma(2-\varrho)\Gamma(\varrho-1)}\frac{\Gamma(\varrho)\Gamma(2-\varrho)}{\Gamma(2)}\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}\\
&=\frac{\xi}{\varpi}\left(1-e^{-\varpi\xi}\right)\left\|\mathfrak{f}_1-\mathfrak{f}_2\right\|_{\mathscr{C}}.
\end{aligned}
\tag{29}
$$

□

3. Existence Results via DFPT

To set our main results we introduce the following assumptions:

(1) $(\mathscr{A}_1)$ The function $\mathscr{P}$: $[0,\mathscr{T}]\times\mathscr{E}\longrightarrow\mathscr{E}$ satisfies carth e odory conditions.

(2) $(\mathscr{A}_2)$ There exists a function $\Psi\in\mathscr{L}^{\infty}([0,\mathscr{T}],\mathbb{R}_+)$ show that

$$\left\|\mathscr{P}\left(\xi,\widehat{\mathscr{Q}_1}(\xi)\right)\right\| \leq \Psi(\xi)\left(1+\left\|\widehat{\mathscr{Q}_1}\right\|\right), \forall\widehat{\mathscr{Q}_1}\in\mathscr{C}([0,\mathscr{T}],\mathscr{E}). \tag{30}$$

(3) $(\mathscr{A}_3)$ Assume $\mathscr{W}\subset\mathscr{E}$ is any bounded set $\forall\xi\in[0,\mathscr{T}]$, then

$$\chi(\mathscr{P}(\xi,\mathscr{W}))\leq\Psi(\xi)\chi(\mathscr{W}). \tag{31}$$

For easy computations we let

$$\begin{aligned}\mathscr{R} &:= \frac{\mathscr{T}^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\mathscr{T}}\right)+|\omega_1|\Phi_1\frac{\sigma^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\sigma}\right)\\ &\quad+|\rho_1|\Phi_1\frac{\mathscr{T}^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\mathscr{T}}\right)+|v_1|\Phi_1\int_0^{\eta}\frac{r^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi r}\right)\mathrm{d}r\\ &\quad+|v_2|\Phi_2\int_{\varsigma}^{\mathscr{T}}\frac{\xi^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\xi}\right)d\xi+\varpi|\omega_2|\Phi_2\frac{\sigma}{\varpi}\left(1-e^{-\varpi\sigma}\right)\\ &\quad+\varpi|\rho_2|\Phi_2\frac{\mathscr{T}}{\varpi}\left(1-e^{-\varpi\mathscr{T}}\right)+|\omega_2|\Phi_2\sigma+|\rho_2|\Phi_2\mathscr{T},\end{aligned} \tag{32}$$

Theorem 2. *Assume that the assumptions $(\mathscr{A}_1)-(\mathscr{A}_3)$ hold true and let $\mathscr{R}_\Psi=\|\Psi\|\mathscr{R}$. If*

$$4\mathscr{R}_\Psi<1, \tag{33}$$

then the BVP (6) has at least one solution defined on $[0,\mathscr{T}]$.

Proof. Consider the operator $\mathscr{H}:\mathscr{C}([0,\mathscr{T}],\mathscr{E})\longrightarrow\mathscr{C}([0,\mathscr{T}],\mathscr{E})$

$$\begin{aligned}\left(\mathscr{H}\widehat{\mathscr{Q}_1}\right)(\xi) &= \int_0^{\xi}\Theta_1(\xi,\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta+\omega_1\phi_1(\xi)\int_0^{\sigma}\Theta_1(\sigma,\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta\\ &\quad+\rho_1\phi_1(\xi)\int_0^{\mathscr{T}}\Theta_1(\mathscr{T},\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta\\ &\quad-v_1\phi_1(\xi)\int_0^{\eta}\int_0^{r}\Theta_1(r,\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta\,\mathrm{d}r\\ &\quad-v_2\phi_2(\xi)\int_{\varsigma}^{\mathscr{T}}\int_0^{\xi}\Theta_1(\xi,\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta\,\mathrm{d}\xi\\ &\quad-\varpi\omega_2\phi_2(\xi)\int_0^{\sigma}\Theta_2(\sigma,\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta\\ &\quad-\varpi\rho_2\phi_2(\xi)\int_0^{\mathscr{T}}\Theta_2(\mathscr{T},\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta\\ &\quad+\omega_2\phi_2(\xi)\int_0^{\sigma}\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta+\rho_2\phi_2(\xi)\int_0^{\mathscr{T}}\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta.\end{aligned} \tag{34}$$

The operator $\mathscr{H}$ is well defined as a result of $\mathscr{A}_1$ and $\mathscr{A}_2$. Therefore, () is equivalent to the following operator equation:

$$\widehat{\mathscr{Q}_1}=\mathscr{H}\widehat{\mathscr{Q}_1}. \tag{35}$$

Subsequently, showing the existence of fixed point for (35) is equivalent to existence of a solution for (20).

Let

$$\mathscr{B}_\epsilon=\left\{\widehat{\mathscr{Q}_1}\in\mathscr{C}([0,\mathscr{T}],\mathscr{E}): \left\|\widehat{\mathscr{Q}_1}\right\|_\infty\leq\epsilon\right\}, \tag{36}$$

be a closed convex set with $\epsilon>0$, such that

$$\epsilon\geq\frac{\mathscr{R}_\Psi}{1-\mathscr{R}_\Psi}. \tag{37}$$

The applicability of the DFPT will be shown in four steps. □

Step 1. We show that HB $\mathscr{H}\mathscr{B}_\epsilon \subset \mathscr{B}_\epsilon$, by $(\mathscr{A}_2)$, we have

$$
\begin{aligned}
\left|\mathscr{H}\widehat{\mathscr{Q}_1}(\xi)\right| \leq & \int_0^\xi \Theta_1(\xi,\zeta)\Psi(\zeta)\left(1+\left\|\widehat{\mathscr{Q}_1}(\zeta)\right\|\right)\mathrm{d}\zeta \\
& +\left|\omega_1\right|\left|\phi_1(\xi)\right| \int_0^\sigma \Theta_1(\sigma,\zeta)\Psi(\zeta)\left(1+\left\|\widehat{\mathscr{Q}_1}(\zeta)\right\|\right)\mathrm{d}\zeta \\
& +\left|\rho_1\right|\left|\phi_1(\xi)\right| \int_0^{\mathscr{T}} \Theta_1(\mathscr{T},\zeta)\Psi(\zeta)\left(1+\left\|\widehat{\mathscr{Q}_1}(\zeta)\right\|\right)\mathrm{d}\zeta \\
& -\left|\upsilon_1\right|\left|\phi_1(\xi)\right| \int_0^\eta \int_0^r \Theta_1(r,\zeta)\Psi(\zeta)\left(1+\left\|\widehat{\mathscr{Q}_1}(\zeta)\right\|\right)\mathrm{d}\zeta\,\mathrm{d}r \\
& -\left|\upsilon_2\right|\left|\phi_2(\xi)\right| \int_\varsigma^{\mathscr{T}} \int_0^\xi \Theta_1(\xi,\zeta)\Psi(\zeta)\left(1+\left\|\widehat{\mathscr{Q}_1}(\zeta)\right\|\right)\mathrm{d}\zeta\,\mathrm{d}\xi \\
& -\varpi\left|\omega_2\right|\left|\phi_2(\xi)\right| \int_0^\sigma \Theta_2(\sigma,\zeta)\Psi(\zeta)\left(1+\left\|\widehat{\mathscr{Q}_1}(\zeta)\right\|\right)\mathrm{d}\zeta \\
& -\varpi\left|\rho_2\right|\left|\phi_2(\xi)\right| \int_0^{\mathscr{T}} \Theta_2(\mathscr{T},\zeta)\Psi(\zeta)\left(1+\left\|\widehat{\mathscr{Q}_1}(\zeta)\right\|\right)\mathrm{d}\zeta \\
& +\left|\omega_2\right|\left|\phi_2(\xi)\right| \int_0^\sigma \Psi(\zeta)\left(1+\left\|\widehat{\mathscr{Q}_1}(\zeta)\right\|\right)\mathrm{d}\zeta \\
& +\left|\rho_2\right|\left|\phi_2(\xi)\right| \int_0^{\mathscr{T}} \Psi(\zeta)\left(1+\left\|\widehat{\mathscr{Q}_1}(\zeta)\right\|\right)\mathrm{d}\zeta, \\
\leq \|\Psi\|\left(1+\left\|\widehat{\mathscr{Q}_1}\right\|\right) & \begin{bmatrix}
\frac{\mathscr{T}^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\mathscr{T}}\right)+\left|\omega_1\right|\phi_1\frac{\sigma^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\sigma}\right)+\left|\rho_1\right|\phi_1\frac{\mathscr{T}^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\mathscr{T}}\right)+ \\
\left|\upsilon_1\right|\phi_1\int_0^\eta \frac{r^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi r}\right)\mathrm{d}r+\left|\upsilon_2\right|\phi_2\int_\varsigma^{\mathscr{T}} \frac{\xi^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\xi}\right)\mathrm{d}\xi \\
+\varpi\left|\omega_2\right|\phi_2\frac{\sigma}{\varpi}\left(1-e^{-\varpi\sigma}\right)+\varpi\left|\rho_2\right|\phi_2\frac{\mathscr{T}}{\varpi}\left(1-e^{-\varpi\mathscr{T}}\right)+\left|\omega_2\right|\phi_2\sigma+\left|\rho_2\right|\phi_2\mathscr{T}
\end{bmatrix}, \\
\left\|\mathscr{H}\widehat{\mathscr{Q}_1}\right\| \leq & \|\Psi\|\left(1+\left\|\widehat{\mathscr{Q}_1}\right\|\right)\mathscr{R} \leq (1+\epsilon)\mathscr{R}_\Psi \leq \epsilon.
\end{aligned}
\tag{38}
$$

Thus, $\|\mathscr{H}\widehat{\mathscr{Q}_1}\| \leq \epsilon$. That is $\mathscr{H}\mathscr{B}_\epsilon \subset \mathscr{B}_\epsilon$.

Step 2. The operator $\mathscr{H}$ is continuous. Let $\{\widehat{\mathscr{Q}_{1n}}\}$ be a sequence in $\mathscr{B}_\epsilon$ such that $\widehat{\mathscr{Q}_{1n}} \longrightarrow \widehat{\mathscr{Q}_1}$ as $n \longrightarrow \infty$. Then, $\mathscr{P}(\zeta, \widehat{\mathscr{Q}_{1n}}(\zeta)) \longrightarrow \mathscr{P}(\zeta, \widehat{\mathscr{Q}_1}(\zeta))$ as $n \longrightarrow \infty$, as a sequel of the Carathéodory continuity of $\mathscr{P}$. $(\mathscr{A}_2)$ implies

$$\begin{aligned}
\left\|\mathscr{H}\widehat{\mathscr{Q}_{1n}}(\xi)-\mathscr{H}\widehat{\mathscr{Q}_1}(\xi)\right\| \leq & \int_0^\xi \Theta_1(\xi,\zeta)\Psi(\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_{1n}}(\zeta)\right)-\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& +|\omega_1||\phi_1(\xi)|\int_0^\sigma \Theta_1(\sigma,\zeta)\Psi(\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_{1n}}(\zeta)\right)-\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& +|\rho_1||\phi_1(\xi)|\int_0^{\mathscr{T}} \Theta_1(\mathscr{T},\zeta)\Psi(\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_{1n}}(\zeta)\right)-\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& -|\upsilon_1||\phi_1(\xi)|\int_0^\eta\int_0^r \Theta_1(r,\zeta)\Psi(\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_{1n}}(\zeta)\right)-\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta\,\mathrm{d}r \\
& -|\upsilon_2||\phi_2(\xi)|\int_\varsigma^{\mathscr{T}}\int_0^\xi \Theta_1(\xi,\zeta)\Psi(\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_{1n}}(\zeta)\right)-\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta\,\mathrm{d}\xi \\
& -\varpi|\omega_2||\phi_2(\xi)|\int_0^\sigma \Theta_2(\sigma,\zeta)\Psi(\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_{1n}}(\zeta)\right)-\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& -\varpi|\rho_2||\phi_2(\xi)|\int_0^{\mathscr{T}} \Theta_2(\mathscr{T},\zeta)\Psi(\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_{1n}}(\zeta)\right)-\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& +|\omega_2||\phi_2(\xi)|\int_0^\sigma \Psi(\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_{1n}}(\zeta)\right)-\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& +|\rho_2||\phi_2(\xi)|\int_0^{\mathscr{T}} \Psi(\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_{1n}}(\zeta)\right)-\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta, \\
\leq & \mathscr{R}\left\|\mathscr{P}\left(\cdot,\widehat{\mathscr{Q}_{1n}}(\cdot)\right)-\mathscr{P}\left(\cdot,\widehat{\mathscr{Q}_1}(\cdot)\right)\right\|.
\end{aligned} \tag{39}$$

Now, from Lebesgue dominated convergence theorem, it is obvious that $\|\mathscr{H}\mathscr{P}_n(\xi)-\mathscr{H}\widehat{\mathscr{Q}_1}(\xi)\| \longrightarrow 0$ as $n\longrightarrow\infty$, $\forall\xi\in[0,\mathscr{T}]$, consequently, we have

$$\left\|\mathscr{H}\widehat{\mathscr{Q}_{1n}}-\mathscr{H}\widehat{\mathscr{Q}_1}\right\| \longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{40}$$

Step 3. The operator $\mathscr{H}$ is equicontinuous. For any $0<\xi_1<\xi_2<\mathscr{T}$ and $\widehat{\mathscr{Q}_1}\in\mathscr{B}_\epsilon$, we obtain

$$\begin{aligned}
\left\|\mathscr{H}\left(\widehat{\mathscr{Q}_1}\right)(\xi_2)-\mathscr{H}\left(\widehat{\mathscr{Q}_1}\right)(\xi_1)\right\| \leq & \left|\int_0^{\xi_1} \Theta_1(\xi_1,\zeta)-\Theta_2(\xi_2,\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right| + \left|\int_{\xi_2}^{\xi_1} \Theta_1(\xi_1,\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta\right| \\
& +|\omega_1||\phi_1(\xi_1)-\phi_1(\xi_2)|\int_0^\sigma \Theta_1(\sigma,\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& +|\rho_1||\phi_1(\xi_1)-\phi_1(\xi_2)|\int_0^{\mathscr{T}} \Theta_1(\mathscr{T},\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& +|\upsilon_1||\phi_1(\xi_1)-\phi_1(\xi_2)|\int_0^\eta\int_0^r \Theta_1(r,\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta\,\mathrm{d}r \\
& +|\upsilon_2||\phi_2(\xi_1)-\phi_2(\xi_2)|\int_\varsigma^{\mathscr{T}}\int_0^\xi \Theta_1(\xi,\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta\,\mathrm{d}\xi \\
& +\varpi|\omega_2||\phi_2(\xi_1)-\phi_2(\xi_2)|\int_0^\sigma \Theta_2(\sigma,\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& +\varpi|\rho_2||\phi_2(\xi_1)-\phi_2(\xi_2)|\int_0^{\mathscr{T}} \Theta_2(\mathscr{T},\zeta)\left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& +|\omega_2||\phi_2(\xi_1)-\phi_2(\xi_2)|\int_0^\sigma \left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta \\
& +|\rho_2||\phi_2(\xi_1)-\phi_2(\xi_2)|\int_0^{\mathscr{T}} \left\|\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right\|\mathrm{d}\zeta,
\end{aligned} \tag{41}$$

as $\xi_1 \longrightarrow \xi_2$ the RHS of the above approaches to zero and it is free of $\widehat{\mathcal{Q}_1} \in \mathcal{B}_\epsilon$. Hence, operator $\mathcal{H}$ is bounded and equicontinuous.

Step 4. We show that $\mathcal{H}$ is $\chi-$ contraction on $\mathcal{B}_\epsilon$. For all bounded subset $\mathcal{W} \subset \mathcal{B}_\epsilon$ and $\delta > 0$. By the aid of Lemma 4 and the properties of χ, $\exists \{\widehat{\mathcal{Q}_{1k}}\}_{k=1}^{\infty} \subset \mathcal{W}$ such that

$$
\begin{aligned}
\chi(\mathcal{H}\mathcal{W}(\xi)) \leq 2\chi\Bigg\{ & \int_0^\xi \Theta_1(\xi,\zeta)\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\mathrm{d}\zeta \\
& + \omega_1\phi_1(\xi)\int_0^\sigma \Theta_1(\sigma,\zeta)\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\mathrm{d}\zeta \\
& + \rho_1\phi_1(\xi)\int_0^{\mathcal{T}} \Theta_1(\mathcal{T},\zeta)\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\mathrm{d}\zeta \\
& - \upsilon_1\phi_1(\xi)\int_0^\eta\int_0^r \Theta_1(r,\zeta)\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\mathrm{d}\zeta\,\mathrm{d}r \\
& - \upsilon_2\phi_2(\xi)\int_\varsigma^{\mathcal{T}}\int_0^\xi \Theta_1(\xi,\zeta)\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\mathrm{d}\zeta\,\mathrm{d}\xi \\
& - \varpi\omega_2\phi_2(\xi)\int_0^\sigma \Theta_2(\sigma,\zeta)\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\mathrm{d}\zeta \\
& - \varpi\rho_2\phi_2(\xi)\int_0^{\mathcal{T}} \Theta_2(\mathcal{T},\zeta)\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\mathrm{d}\zeta \\
& + \omega_2\phi_2(\xi)\int_0^\sigma \mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\mathrm{d}\zeta \\
& + \rho_2\phi_2(\xi)\int_0^{\mathcal{T}} \mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)d\zeta, \Bigg\} + \delta.
\end{aligned} \tag{42}
$$

The properties of χ, $(\mathcal{A}_3)$, and Lemma 3 we obtain

$$
\begin{aligned}
\chi(\mathcal{H}\mathcal{W}(\xi)) \leq 4\Bigg\{ & \int_0^\xi \Theta_1(\xi,\zeta)\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\mathrm{d}\zeta \\
& + |\omega_1||\phi_1(\xi)|\int_0^\sigma \Theta_1(\sigma,\zeta)\chi\left(\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\right)\mathrm{d}\zeta \\
& + |\rho_1||\phi_1(\xi)|\int_0^{\mathcal{T}} \Theta_1(\mathcal{T},\zeta)\chi\left(\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\right)\mathrm{d}\zeta \\
& - |\upsilon_1||\phi_1(\xi)|\int_0^\eta\int_0^r \Theta_1(r,\zeta)\chi\left(\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\right)\mathrm{d}\zeta\,\mathrm{d}r \\
& - |\upsilon_2||\phi_2(\xi)|\int_\varsigma^{\mathcal{T}}\int_0^\xi \Theta_1(\xi,\zeta)\chi\left(\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\right)\mathrm{d}\zeta\,\mathrm{d}\xi \\
& - \varpi|\omega_2||\phi_2(\xi)|\int_0^\sigma \Theta_2(\sigma,\zeta)\chi\left(\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\right)\mathrm{d}\zeta \\
& - \varpi|\rho_2||\phi_2(\xi)|\int_0^{\mathcal{T}} \Theta_2(\mathcal{T},\zeta)\chi\left(\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\right)\mathrm{d}\zeta \\
& + |\omega_2||\phi_2(\xi)|\int_0^\sigma \chi\left(\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\right)\mathrm{d}\zeta \\
& + |\rho_2||\phi_2(\xi)|\int_0^{\mathcal{T}} \chi\left(\mathcal{P}\left(\zeta,\left\{\widehat{\mathcal{Q}_{1k}}(\zeta)\right\}_{k=1}^{\infty}\right)\right)\mathrm{d}\zeta\Bigg\} + \delta,
\end{aligned}
$$

$$\begin{aligned}
&\leq 4\Bigg\{\int_0^{\xi}\Theta_1(\xi,\zeta)\Psi(\zeta)\mathscr{P}\Big(\zeta,\big\{\widehat{\mathscr{Q}_{1k}}(\zeta)\big\}_{k=1}^{\infty}\Big)d\zeta\\
&\quad+|\omega_1\|\phi_1(\xi)|\int_0^{\sigma}\Theta_1(\sigma,\zeta)\Psi(\zeta)\chi\Big(\mathscr{P}\Big(\zeta,\big\{\widehat{\mathscr{Q}_{1k}}(\zeta)\big\}_{k=1}^{\infty}\Big)\Big)d\zeta\\
&\quad+|\rho_1\|\phi_1(\xi)|\int_0^{\mathscr{T}}\Theta_1(\mathscr{T},\zeta)\Psi(\zeta)\chi\Big(\mathscr{P}\Big(\zeta,\big\{\widehat{\mathscr{Q}_{1k}}(\zeta)\big\}_{k=1}^{\infty}\Big)\Big)d\zeta\\
&\quad-|\upsilon_1\|\phi_1(\xi)|\int_0^{\eta}\int_0^{r}\Theta_1(r,\zeta)\Psi(\zeta)\chi\Big(\mathscr{P}\Big(\zeta,\big\{\widehat{\mathscr{Q}_{1k}}(\zeta)\big\}_{k=1}^{\infty}\Big)\Big)d\zeta\, dr\\
&\quad-|\upsilon_2\|\phi_2(\xi)|\int_{\varsigma}^{\mathscr{T}}\int_0^{\xi}\Theta_1(\xi,\zeta)\Psi(\zeta)\chi\Big(\mathscr{P}\Big(\zeta,\big\{\widehat{\mathscr{Q}_{1k}}(\zeta)\big\}_{k=1}^{\infty}\Big)\Big)d\zeta\, d\xi\\
&\quad-\varpi|\omega_2\|\phi_2(\xi)|\int_0^{\sigma}\Theta_2(\sigma,\zeta)\Psi(\zeta)\chi\Big(\mathscr{P}\Big(\zeta,\big\{\widehat{\mathscr{Q}_{1k}}(\zeta)\big\}_{k=1}^{\infty}\Big)\Big)d\zeta\\
&\quad-\varpi|\rho_2\|\phi_2(\xi)|\int_0^{\mathscr{T}}\Theta_2(\mathscr{T},\zeta)\Psi(\zeta)\chi\Big(\mathscr{P}\Big(\zeta,\big\{\widehat{\mathscr{Q}_{1k}}(\zeta)\big\}_{k=1}^{\infty}\Big)\Big)d\zeta\\
&\quad+|\omega_2\|\phi_2(\xi)|\int_0^{\sigma}\Psi(\zeta)\chi\Big(\mathscr{P}\Big(\zeta,\big\{\widehat{\mathscr{Q}_{1k}}(\zeta)\big\}_{k=1}^{\infty}\Big)\Big)d\zeta\\
&\quad+|\rho_2\|\phi_2(\xi)|\int_0^{\mathscr{T}}Psi(\zeta)\chi\Big(\mathscr{P}\Big(\zeta,\big\{\widehat{\mathscr{Q}_{1k}}(\zeta)\big\}_{k=1}^{\infty}\Big)\Big)d\zeta\Bigg\}+\delta.\\
&\leq 4\mathscr{R}_{\Psi}\chi(\mathscr{R}_{\epsilon})+\delta,\forall\delta>0.
\end{aligned}\tag{43}$$

Then,

$$\chi(\mathscr{H}(\mathscr{W}))=\sup_{\xi\in[0,\mathscr{T}]}\chi(\mathscr{H}\mathscr{W}(\xi))\leq 4\mathscr{R}_{\Psi}\chi(\mathscr{R}_{\epsilon}).\tag{44}$$

By Theorem 2, we conclude the existence of a fixed point for the operator equation given by (35). This completes the proof.

4. Stability Results

Let $\vartheta>0$ and Θ: $[0,\mathscr{T}]\longrightarrow[0,\infty]$ be a continuous function. We consider the following inequalities:

$$\Big|({}^{c}\mathscr{D}^{\varrho}+\varpi\mathscr{D}^{\varrho})\widehat{\mathscr{Q}_1}(\xi)-\mathscr{P}\big(\xi,\widehat{\mathscr{Q}_1}(\xi)\big)\Big|\leq\vartheta,\xi\in[0,\mathscr{T}],\tag{45}$$

$$\Big|({}^{c}\mathscr{D}^{\varrho}+\varpi\mathscr{D}^{\varrho})\widehat{\mathscr{Q}_1}(\xi)-\mathscr{P}\big(\xi,\widehat{\mathscr{Q}_1}(\xi)\big)\Big|\leq\vartheta\Theta(\xi),\xi\in[0,\mathscr{T}].\tag{46}$$

Definition 6 (see [42]). Problem (6) is U-H stable if $\exists\mathscr{M}>0$ such that, $\forall\vartheta>0$, $\forall\widehat{\mathscr{Q}_1}\in\mathscr{C}$ of the inequality (45), $\exists$ solution $\widehat{\mathscr{Q}_1}^{*}\in\mathscr{C}$ of problem (6) with

$$\Big|\widehat{\mathscr{Q}_1}(\xi)-\widehat{\mathscr{Q}_1}^{*}(\xi)\Big|\leq\mathscr{M}\vartheta,\xi\in[0,\mathscr{T}].\tag{47}$$

Definition 7 (see [42]). Problem (6) is generalized U-H stable if $\exists\Theta_{P\mathscr{P}}\in\mathscr{C}(\mathbb{R}^{+},\mathbb{R}^{+})$ and $\Theta_{\mathscr{P}}(0)=0$ such that, $\forall\widehat{\mathscr{Q}_1}\in\mathscr{C}$ of the inequality (46), $\exists$ a solution $\widehat{\mathscr{Q}_1}^{*}\in\mathscr{C}$ of problem (6) with

$$\Big|\widehat{\mathscr{Q}_1}(\xi)-\widehat{\mathscr{Q}_1}^{*}(\xi)\Big|\leq\Theta_{P}(\vartheta),\xi\in[0,\mathscr{T}].\tag{48}$$

Remark 1 (see [42]). A function $\widehat{\mathscr{Q}_1}\in\mathscr{C}$ is a solution of the equality (47) $\Longleftrightarrow\exists$ a function $\mathscr{Z}\in\mathscr{C}$, such that

(1) $|\mathscr{Z}(\xi)|\leq\vartheta,\xi\in[0,\mathscr{T}]$,
(2) $(\mathscr{D}^{c\varrho}+\varpi\mathscr{D}^{\varrho})\widehat{\mathscr{Q}_1}(\xi)=\mathscr{P}(\xi,\widehat{\mathscr{Q}_1}(\xi))|+\mathscr{Z}(\xi),\xi\in[0,\mathscr{T}]$.

Lemma 8. *Let $1\leq\varrho\leq 2$, if a function $\widehat{\mathscr{Q}_1}\in\mathscr{C}$ is a solution of the inequality, then $\widehat{\mathscr{Q}_1}$ is a solution of the following integral inequality:*

$$\Big|\widehat{\mathscr{Q}_1}(\xi)-\mathfrak{E}_{\widehat{\mathscr{Q}_1}}\Big|\leq\mathscr{R}\vartheta,\tag{49}$$

where

$$\begin{aligned}
\mathfrak{E}_{\widehat{\mathcal{Q}_1}} = & \int_0^{\xi} \Theta_1(\xi,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta + \omega_1\phi_1(\xi)\int_0^{\sigma} \Theta_1(\sigma,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta \\
& + \rho_1\phi_1(\xi)\int_0^{\mathcal{T}} \Theta_1(\mathcal{T},\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta \\
& - v_1\phi_1(\xi)\int_0^{\eta}\int_0^{r} \Theta_1(r,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta\, dr \\
& - v_2\phi_2(\xi)\int_{\varsigma}^{\mathcal{T}}\int_0^{\xi} \Theta_1(\xi,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta\, d\xi \\
& - \varpi\omega_2\phi_2(\xi)\int_0^{\sigma} \Theta_2(\sigma,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta \\
& - \varpi\rho_2\phi_2(\xi)\int_0^{\mathcal{T}} \Theta_2(\mathcal{T},\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta \\
& + \omega_2\phi_2(\xi)\int_0^{\sigma} \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta + \rho_2\phi_2(\xi)\int_0^{\mathcal{T}} \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta.
\end{aligned} \tag{50}$$

Proof. Using Remark 1 that

$$\begin{aligned}
\widehat{\mathcal{Q}_1}(\xi) = & \int_0^{\xi} \Theta_1(\xi,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta + \omega_1\phi_1(\xi)\int_0^{\sigma} \Theta_1(\sigma,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta \\
& + \rho_1\phi_1(\xi)\int_0^{\mathcal{T}} \Theta_1(\mathcal{T},\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta \\
& - v_1\phi_1(\xi)\int_0^{\eta}\int_0^{r} \Theta_1(r,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta\, dr \\
& - v_2\phi_2(\xi)\int_{\varsigma}^{\mathcal{T}}\int_0^{\xi} \Theta_1(\xi,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta\, d\xi \\
& - \varpi\omega_2\phi_2(\xi)\int_0^{\sigma} \Theta_2(\sigma,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta \\
& - \varpi\rho_2\phi_2(\xi)\int_0^{\mathcal{T}} \Theta_2(\mathcal{T},\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta \\
& + \omega_2\phi_2(\xi)\int_0^{\sigma} \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta + \rho_2\phi_2(\xi)\int_0^{\mathcal{T}} \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right)d\zeta \\
& + \int_0^{\xi} \Theta_1(\xi,\zeta)\mathcal{Z}(\zeta)d\zeta + \omega_1\phi_1(\xi)\int_0^{\sigma} \Theta_1(\sigma,\zeta)\mathcal{Z}(\zeta)d\zeta \\
& + \rho_1\phi_1(\xi)\int_0^{\mathcal{T}} \Theta_1(\mathcal{T},\zeta)\mathcal{Z}(\zeta)d\zeta \\
& - v_1\phi_1(\xi)\int_0^{\eta}\int_0^{r} \Theta_1(r,\zeta)\mathcal{Z}(\zeta)d\zeta\, dr \\
& - v_2\phi_2(\xi)\int_{\varsigma}^{\mathcal{T}}\int_0^{\xi} \Theta_1(\xi,\zeta)\mathcal{Z}(\zeta)d\zeta\, d\xi \\
& - \varpi\omega_2\phi_2(\xi)\int_0^{\sigma} \Theta_2(\sigma,\zeta)\mathcal{Z}(\zeta)d\zeta \\
& - \varpi\rho_2\phi_2(\xi)\int_0^{\mathcal{T}} \Theta_2(\mathcal{T},\zeta)\mathcal{Z}(\zeta)d\zeta \\
& + \omega_2\phi_2(\xi)\int_0^{\sigma} \mathcal{Z}(\zeta)d\zeta + \rho_2\phi_2(\xi)\int_0^{\mathcal{T}} \mathcal{Z}(\zeta)d\zeta,
\end{aligned} \tag{51}$$

implies

$$\begin{aligned}\left|\widehat{\mathscr{Q}_1}(\xi)-\mathfrak{C}_{\widehat{\mathscr{Q}_1}}\right| &= \Bigg|\int_0^{\xi}\Theta_1(\xi,\zeta)\mathscr{Z}(\zeta)\mathrm{d}\zeta+\omega_1\phi_1(\xi)\int_0^{\sigma}\Theta_1(\sigma,\zeta)\mathscr{Z}(\zeta)\mathrm{d}\zeta \\ &\quad+\rho_1\phi_1(\xi)\int_0^{\mathscr{T}}\Theta_1(\mathscr{T},\zeta)\mathscr{Z}(\zeta)\mathrm{d}\zeta \\ &\quad-\upsilon_1\phi_1(\xi)\int_0^{\eta}\int_0^{r}\Theta_1(r,\zeta)\mathscr{Z}(\zeta)\mathrm{d}\zeta\,\mathrm{d}r \\ &\quad-\upsilon_2\phi_2(\xi)\int_{\varsigma}^{\mathscr{T}}\int_0^{\xi}\Theta_1(\xi,\zeta)\mathscr{Z}(\zeta)\mathrm{d}\zeta\,\mathrm{d}\xi \\ &\quad-\varpi\omega_2\phi_2(\xi)\int_0^{\sigma}\Theta_2(\sigma,\zeta)\mathscr{Z}(\zeta)\mathrm{d}\zeta \\ &\quad-\varpi\rho_2\phi_2(\xi)\int_0^{\mathscr{T}}\Theta_2(\mathscr{T},\zeta)\mathscr{Z}(\zeta)\mathrm{d}\zeta \\ &\quad+\omega_2\phi_2(\xi)\int_0^{\sigma}\mathscr{Z}(\zeta)\mathrm{d}\zeta+\rho_2\phi_2(\xi)\int_0^{\mathscr{T}}\mathscr{Z}(\zeta)\mathrm{d}\zeta\Bigg| \\ &\leq\vartheta\left[\begin{array}{c}\dfrac{\mathscr{T}^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\mathscr{T}}\right)+|\omega_1|\phi_1\dfrac{\sigma^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\sigma}\right)+|\rho_1|\phi_1\dfrac{\mathscr{T}^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\mathscr{T}}\right) \\ +|\upsilon_1|\phi_1\displaystyle\int_0^{\eta}\frac{r^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi r}\right)\mathrm{d}r+|\upsilon_2|\phi_2\int_{\varsigma}^{\mathscr{T}}\frac{\xi^{\varrho-1}}{\varpi\Gamma(\varrho)}\left(1-e^{-\varpi\xi}\right)\mathrm{d}\xi \\ +\varpi|\omega_2|\phi_2\dfrac{\sigma}{\varpi}\left(1-e^{-\varpi\sigma}\right)+\varpi|\rho_2|\phi_2\dfrac{\mathscr{T}}{\varpi}\left(1-e^{-\varpi\mathscr{T}}\right)+|\omega_2|\phi_2\sigma+|\rho_2|\phi_2\mathscr{T}\end{array}\right] \\ &\leq\mathscr{R}\vartheta.\end{aligned} \tag{52}$$

We now state the main theorem as follows. □

Theorem 3. *Assume that* $(\mathscr{A}_1)$ *and* $(\mathscr{A}_2)$ *are satisfied with* $\mathscr{R}_{\Psi}<1$. *Then problem (6) is U-H and according is generalized U-H stable.*

Proof. Suppose that $\widehat{\mathscr{Q}_1}\in\mathscr{C}$ is a solution of inequality (47) and $\widehat{\mathscr{Q}_1}^{*}\in\mathscr{C}$ is a unique solution of problem (6). Then, it follows from Lemma 8 that

$$\begin{aligned}\left|\widehat{\mathscr{Q}_1}(\xi)-\widehat{\mathscr{Q}_1}^{*}(\xi)\right| &= \Bigg|\int_0^{\xi}\Theta_1(\xi,\zeta)\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\mathrm{d}\zeta \\ &\quad+|\omega_1||\phi_1(\xi)|\int_0^{\sigma}\Theta_1(\sigma,\zeta)\left(\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right)\mathrm{d}\zeta \\ &\quad+|\rho_1||\phi_1(\xi)|\int_0^{\mathscr{T}}\Theta_1(\mathscr{T},\zeta)\left(\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right)\mathrm{d}\zeta \\ &\quad-|\upsilon_1||\phi_1(\xi)|\int_0^{\eta}\int_0^{r}\Theta_1(r,\zeta)\left(\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right)\mathrm{d}\zeta\,\mathrm{d}r \\ &\quad-|\upsilon_2||\phi_2(\xi)|\int_{\varsigma}^{\mathscr{T}}\int_0^{\xi}\Theta_1(\xi,\zeta)\left(\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right)\mathrm{d}\zeta\,\mathrm{d}\xi \\ &\quad-\varpi|\omega_2||\phi_2(\xi)|\int_0^{\sigma}\Theta_2(\sigma,\zeta)\left(\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right)\mathrm{d}\zeta \\ &\quad-\varpi|\rho_2||\phi_2(\xi)|\int_0^{\mathscr{T}}\Theta_2(\mathscr{T},\zeta)\left(\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right)\mathrm{d}\zeta \\ &\quad+|\omega_2||\phi_2(\xi)|\int_0^{\sigma}\left(\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right)\mathrm{d}\zeta \\ &\quad+|\rho_2||\phi_2(\xi)|\int_0^{\mathscr{T}}\left(\mathscr{P}\left(\zeta,\widehat{\mathscr{Q}_1}(\zeta)\right)\right)\mathrm{d}\zeta\Bigg|\end{aligned}$$

$$\begin{aligned}
&\leq \left|\widehat{\mathcal{Q}_1}(\xi) - \mathfrak{E}_{\widehat{\mathcal{Q}_1}}\right| + \left|\int_0^{\xi} \Theta_1(\xi,\zeta)\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right) - \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}^*(\zeta)\right)\mathrm{d}\zeta\right.\\
&\quad + |\omega_1\|\phi_1(\xi)| \int_0^{\sigma} \Theta_1(\sigma,\zeta)\left(\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right) - \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}^*(\zeta)\right)\right)\mathrm{d}\zeta\\
&\quad + |\rho_1\|\phi_1(\xi)| \int_0^{\mathcal{T}} \Theta_1(\mathcal{T},\zeta)\left(\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right) - \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}^*(\zeta)\right)\right)\mathrm{d}\zeta\\
&\quad - |\upsilon_1\|\phi_1(\xi)| \int_0^{\eta}\int_0^{r} \Theta_1(r,\zeta)\left(\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right) - \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}^*(\zeta)\right)\right)\mathrm{d}\zeta\,\mathrm{d}r\\
&\quad - |\upsilon_2\|\phi_2(\xi)| \int_{\varsigma}^{\mathcal{T}}\int_0^{\xi} \Theta_1(\xi,\zeta)\left(\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right) - \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}^*(\zeta)\right)\right)\mathrm{d}\zeta\,\mathrm{d}\xi\\
&\quad - \varpi|\omega_2\|\phi_2(\xi)| \int_0^{\sigma} \Theta_2(\sigma,\zeta)\left(\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right) - \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}^*(\zeta)\right)\right)\mathrm{d}\zeta\\
&\quad - \varpi|\rho_2\|\phi_2(\xi)| \int_0^{\mathcal{T}} \Theta_2(\mathcal{T},\zeta)\left(\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right) - \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}^*(\zeta)\right)\right)\mathrm{d}\zeta\\
&\quad + |\omega_2\|\phi_2(\xi)| \int_0^{\sigma}\left(\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right) - \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}^*(\zeta)\right)\right)\mathrm{d}\zeta\\
&\quad \left.+ |\rho_2\|\phi_2(\xi)| \int_0^{\mathcal{T}}\left(\mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}(\zeta)\right) - \mathcal{P}\left(\zeta,\widehat{\mathcal{Q}_1}^*(\zeta)\right)\right)\mathrm{d}\zeta\right|\\
&\leq \mathcal{R}\vartheta + \Psi\mathcal{R}\left|\widehat{\mathcal{Q}_1}(\xi) - \widehat{\mathcal{Q}_1}^*(\xi)\right|,
\end{aligned} \tag{53}$$

which implies

$$\left\|\widehat{\mathcal{Q}_1}(\xi) - \widehat{\mathcal{Q}_1}^*\right\| \leq \mathcal{M}\vartheta, \tag{54}$$

where

$$\mathcal{M} = \frac{\mathcal{R}}{1 - \Psi\mathcal{R}} > 0. \tag{55}$$

Hence, we conclude that problem (6) is U-H stable. In addition, denoting $\Theta_{\mathcal{P}}(\vartheta) = \mathcal{M}\vartheta$, such that $\Theta_{\mathcal{P}}(0) = 0$, then problem (6) is generalized U-H stable. □

5. Example

Example 1. Denote the Banach space of real sequences by $\mathcal{C}_0 = \left\{\widehat{\mathcal{Q}_1} = (\widehat{\mathcal{Q}_{11}}, \widehat{\mathcal{Q}_{12}}, \cdots)\colon \widehat{\mathcal{Q}_{1n}} \longrightarrow 0\,(n \longrightarrow \infty)\right\}$, with the norm

$$\left\|\widehat{\mathcal{Q}_1}\right\|_{\infty} = \sup_{n\geq 1} \left|\widehat{\mathcal{Q}_{1n}}\right|. \tag{56}$$

Consider the following BVP

$$\begin{cases}
\left({}^{c}\mathcal{D}^{(3/2)} + {}^{c}\mathcal{D}^{(1/2)}\right)\widehat{\mathcal{Q}_1}(\xi) = \dfrac{1}{\sqrt{\xi^2+49}}\left(\dfrac{1}{49} + \cos\left|\widehat{\mathcal{Q}_{1n}}\right|\right), \xi \in [0,4],\\
2\widehat{\mathcal{Q}_1}(1) + 3\widehat{\mathcal{Q}_1}(4) = -\displaystyle\int_0^2 \widehat{\mathcal{Q}_1}(\zeta)\mathrm{d}\zeta,\\
{}^{c}\mathcal{D}^{(1/2)}\widehat{\mathcal{Q}_1}(1) + 5\,{}^{c}\mathcal{D}^{(1/2)}\widehat{\mathcal{Q}_1}(1) = -\displaystyle\int_0^2 \widehat{\mathcal{Q}_1}(\zeta)\mathrm{d}\zeta,
\end{cases} \tag{57}$$

where $\mathcal{P}(\xi,\widehat{\mathcal{Q}_1}) = (1/\sqrt{\xi^2+49})((\xi\sin\widehat{\mathcal{Q}_1}/49) + e^{\xi}\cos\xi)$, $T = 4, \sigma = 1, \varpi = 2, \varrho = (3/2), \rho_1 = 3, \rho_2 = 5, \eta = 2, \varsigma = 3, \upsilon_1 = -1, \upsilon_2 = 5, \omega_1 = 2, \omega_2 = 1a_{11} \approx 0.76, a_{21} \approx 27.967, a_{22} \approx 5, a_{12} \approx 6, \Delta = -145(3/2)$,

$$\varphi_1 = \max\left(\frac{27.967-5}{145},\frac{27.967-5e^{-8}}{145}\right)\cong 0.17,$$
$$\varphi_2 = \max\left(\frac{0.76-6}{145},\frac{0.76-6e^{-8}}{145}\right)\cong 0.036, \tag{58}$$
$$\mathscr{R} < 1.083.$$

where $\mathscr{P}\colon [0,4]\times\mathscr{C}_0 \longrightarrow \mathscr{C}_0$ given by

$$\mathscr{P}\left(\xi,\widehat{\mathscr{Q}_1}\right) = \left\{\frac{1}{\sqrt{\xi^2+49}}\left(\frac{\xi\sin\widehat{\mathscr{Q}_1}}{49}+e^{\xi}\cos\xi\right)\right\}_{n\geq 1}, \text{for } \xi\in[0,4], \widehat{\mathscr{Q}_1}=\left\{\widehat{\mathscr{Q}_{1n}}\right\}_{n\geq 1}\in\mathscr{C}_0. \tag{59}$$

It is clear that condition $(\mathscr{A}_1)$ holds, and as

$$\begin{aligned}\left\|\mathscr{P}\left(\xi,\widehat{\mathscr{Q}_1}\right)\right\| &= \left\|\frac{1}{\sqrt{\xi^2+49}}\left(\frac{\xi\sin\widehat{\mathscr{Q}_1}}{49}+e^{\xi}\cos\xi\right)\right\| \\ &\leq \frac{1}{\sqrt{\xi^2+49}}\left(1+\left\|\widehat{\mathscr{Q}_1}\right\|\right) \\ &= \Psi(\xi)\left(1+\left\|\widehat{\mathscr{Q}_1}\right\|\right).\end{aligned} \tag{60}$$

Therefore, $(\mathscr{A}_2)$ satisfied, with

$$\Psi(\xi) = \frac{1}{\sqrt{\xi^2+49}}, \xi\in[0,4]. \tag{61}$$

And the bounded set $\mathscr{G}_1\subset\mathscr{C}_0$, we have

$$\chi(\mathscr{P}(\xi,\mathscr{G}_1)) \leq \frac{1}{\sqrt{\xi^2+49}}\chi(\mathscr{G}_1), \forall\xi\in[0,4]. \tag{62}$$

So, $(\mathscr{A}_3)$ holds true. Indeed, $4\mathscr{R}_\Psi = 0.618857$ and $(1+\epsilon)\mathscr{R}_\Psi = \epsilon$. Thus,

$$\epsilon \geq \frac{\mathscr{R}_\Psi}{1-\mathscr{R}_\Psi} = \frac{0.15471}{1-0.15471} = \frac{0.15471}{0.84529}. \tag{63}$$

Then, ϵ can be chosen as $\epsilon = 0.2 > 0.183$. Consequently, all conditions of Theorem 2 hold true, yields the existence of a solution $\widehat{\mathscr{Q}_1}\in\mathscr{C}([0,4],\mathscr{C}_0)$ for the problem (57).

6. Conclusions

We discussed the existence results for a fractional differential equation of sequential type with nonlocal integral boundary conditions. The main results are established with the aid of Darbo's fixed point theorem and Hausdorff's measure of noncompactness method. Using standard functional analysis, we showed Ulam–Hyers stability. Our results in this configuration are novel and add to the body of knowledge on the theory of fractional differential equations. For future work, we suggest using other types of fractional derivative operators such as the generalized Hilfer fractional derivative, the one who is interested in the subject can also investigate the existence and uniqueness of the solutions for the coupled or tripled systems via several fixed points theorems such as Banach contraction, mapping principle, Leray–Schuader's alternative, and Mönch's fixed point theorem.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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