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Research Article

S-FP-Projective Modules and Dimensions

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Let R be a ring and let S be a multiplicative subset of R. An R-module M is said to be a u-S-absolutely pure module if $\mathrm{Ext}_R^1(N, M)$ is u-S-torsion for any finitely presented R-module N. This paper introduces and studies the notion of S-FP-projective modules, which extends the classical notion of FP-projective modules. An R-module M is called an S-FP-projective module if $\mathrm{Ext}_R^1(M, N) = 0$ for any u-S-absolutely pure R-module N. We also introduce the S-FP-projective dimension of a module and the global S-FP-projective dimension of a ring. Then, the relationship between the S-FP-projective dimension and other homological dimensions is discussed.

1. Introduction

Throughout the paper, all rings considered are commutative with identity, all modules are unitary, and S is always a multiplicative subset of R, that is, $1 \in S$ and $s_1 s_2 \in S$ for any $s_1, s_2 \in S$. A multiplicative subset S of R is said to be finite if the cardinal of S is finite. Let R be a ring and M be an R-module. As usual, we use $\mathrm{id}_R(M)$, $\mathrm{pd}_R(M)$, and $\mathrm{fd}_R(M)$ to denote the classical injective dimension, projective dimension, and flat dimension of M, respectively, and $\mathrm{gldim}(R)$ and $\mathrm{wdim}(R)$ to denote the global and weak homological dimensions of R, respectively. We also use "f.g." (resp., "f.p.") as shorthand for "finitely generated" (resp., "finitely presented").

From reference [1], we recall that an R-module M is said to be u-S-torsion if $sM = 0$ for some $s \in S$. An R-module M is said to be S-finite if M/F is u-S-torsion for some f.g. submodule F of M. Also, following Zhang [1, 2], a sequence $0 \longrightarrow A \longrightarrow^{\beta} B \longrightarrow^{\gamma} C \longrightarrow 0$ is said to be u-S-exact (at B) provided that there is an element $t \in S$ such that $t\mathrm{Ker}(\gamma) \subseteq \mathrm{Im}(\beta)$ and $t\mathrm{Im}(\beta) \subseteq \mathrm{Ker}(\gamma)$. A long R-sequence $\ldots \longrightarrow M_{i-1} \longrightarrow^{\beta} i\, M_i \longrightarrow^{\beta} i+1\, M_{i+1} \longrightarrow \cdots$ is called u-S-exact if for any i, there is an element $t \in S$ such that $t\mathrm{Ker}(\beta_{i+1}) \subseteq \mathrm{Im}(\beta_i)$ and $t\mathrm{Im}(\beta_i) \subseteq \mathrm{Ker}(\beta_{i+1})$. A u-S-exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ is called a short u-S-exact sequence. A homomorphism $\beta: A \longrightarrow B$ is a u-S-monomorphism (resp., u-S-epimorphism and u-S-isomorphism) provided that $0 \longrightarrow A \longrightarrow^{\beta} B$ (resp., $A \longrightarrow^{\beta} B \longrightarrow 0$ and $0 \longrightarrow A \longrightarrow^{\beta} B \longrightarrow 0$) is u-S-exact. It is easy to verify that a homomorphism $\beta: A \longrightarrow B$ is a u-S-monomorphism (resp., u-S-epimorphism and u-S-isomorphism) if $\mathrm{Ker}(\beta)$ is (resp., $\mathrm{CoKer}(\beta)$ is, both $\mathrm{Ker}(\beta)$ and $\mathrm{CoKer}(\beta)$ are) u-S-torsion.

Maddox [3] called a module absolutely pure if it is pure in every module containing it as a submodule. In reference [4], Megibben showed that an R-module A is absolutely pure if and only if $\mathrm{Ext}_R^1(F, A) = 0$ for every f.p. R-module F. Thus, an absolutely pure module is called an FP-injective module in [5]. Recently, the concept of u-S-absolutely pure modules (abbreviates uniformly S-absolutely pure) is introduced in reference [6] as a generalization of that of absolutely pure modules. As in reference [6], a u-S-exact sequence of R-modules $0 \longrightarrow M \longrightarrow N \longrightarrow X \longrightarrow 0$ is called u-S-pure provided that for every R-module F, the induced sequence $0 \longrightarrow F \otimes_R M \longrightarrow F \otimes_R N \longrightarrow F \otimes_R X \longrightarrow 0$ is also u-S-exact, and a submodule N of M is called a u-S-pure submodule if the exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ is u-S-pure exact. An R-module M is said to be u-S-absolutely pure provided that any short u-S-exact sequence $0 \longrightarrow M \longrightarrow B \longrightarrow C \longrightarrow 0$ beginning with M is u-S-pure. By reference [6], Theorem 3.2, an R-module E is u-S-absolutely pure if and only if there exists an element $s \in S$, satisfying that $\mathrm{Ext}_R^1(N, E)$ is u-S-torsion with respect to s for any f.p. R-module N.

Recently, Zhang in reference [1] defined the u-S-von Neumann regular ring as follows: A ring R is called a u-S-von Neumann regular ring if there exists an element $s \in S$ such that for any $a \in R$, there exists $r \in R$ with $sa = ra^2$. Thus, by reference [1], Theorem 3.13, R is a u-S-von Neumann regular ring if and only if every R-module is u-S-flat, and in reference [6] Theorem 3.5, it is proved that a ring R is u-S-von Neumann regular if and only if every R-module is u-S-absolutely pure.

In reference [7], the authors introduced and characterized the concept of the FP-projective dimension of modules and rings. The FP-projective dimension $\mathrm{fpd}_R(N)$ of an R-module N is the smallest integer $i \geq 0$ such that $\mathrm{Ext}_R^{i+1}(N, M) = 0$ for any absolutely pure R-module M. The FP-projective dimension $\mathrm{fpD}(R)$ of R is defined as the supremum of the FP-projective dimensions of the f.g. R-modules. These dimensions measure how far an f.g. module is from being f.p. and how far away a ring is from being Noetherian, respectively. For example, they proved that a ring R is Noetherian if and only if every R-module is FP-projective. Recall that R is called a hereditary ring (resp., an FP-hereditary ring) if every ideal of R is projective (resp., FP-projective) (see, ([8], Definition 3.1)). It is trivial that the projective module is FP-projective, and so, the hereditary ring is FP-hereditary. A natural question is whether a new class of modules (resp., rings) exists between the classes of these two modules (resp., rings). From this point of view, in reference [9], w-FP-projective modules and dimensions were introduced and studied using the torsion theory derived from the star operation w. The motivation of this paper is to unify these concepts in the module case and the ring case using a multiplicative subset of the ring.

Section 2 introduces the concept of S-FP-projective modules and gives some characterizations of S-FP-projective modules. Using these results, we prove that a ring R is coherent if and only if every ideal of R is S-FP-projective; if and only if every f.g. submodule of a projective module is S-FP-projective. Also, we prove that R is an S-FP-hereditary ring if and only if every submodule of a projective R-module is S-FP-projective; if and only if every submodule of an S-FP-projective R-module is S-FP-projective.

Section 3 deals with the S-FP-projective dimension of a module M, denoted by $S\text{-}\mathrm{fpd}_R(M)$, and the global S-FP-projective dimension of a ring R, denoted by $S - \mathrm{fpD}(R)$. Among other results, we characterize when $S\text{-}\mathrm{fpd}_R(M) \leq n$ and when $S - \mathrm{fpD}(R) \leq n$, as is usually carried out in the study of the classical homology dimensions. In particular, it is shown that $S - \mathrm{fpD}(R) \leq 1$, if and only if every submodule of projective (resp., S-FP-projective) R-module is S-FP-projective; if and only if $\mathrm{id}_R(A) \leq 1$ for any u-S-absolutely pure R-module A; if and only if R is an S-FP-hereditary ring. Finally, a nontrivial example that FP-hereditary rings are not S-FP-hereditary, in general, is given.

2. S-FP-Projective Modules

In this section, we introduce a class called the S-FP-projective module, study their properties, and characterize them. We begin this section with the following definition:

Definition 1. An R-module M is called S-FP-projective if $\mathrm{Ext}_R^1(M, N) = 0$ for any u-S-absolutely pure R-module N.

Since every absolutely pure module is u-S-absolutely pure by reference [6], Proposition 3.3, we have the following implications:

$$\text{projective module} \Rightarrow S - \mathrm{FP} - \text{projectivemodule} \Rightarrow \mathrm{FP} - \text{projectivemodule}. \quad (1)$$

Remark 1

(1) If S consists of units, then u-S-absolutely pure modules and absolutely pure modules coincide. Thus, the S-FP-projective R-modules are just the FP-projective R-modules.
(2) If $0 \in S$, then every R-module is u-S-absolutely pure. So, S-FP-projective modules are exactly projective modules.
(3) Using reference [4], Theorem 5, it is easy to see that the three classes of modules previous coincide over a von Neumann regular ring.

In the following example, we show that there exists an FP-projective R-module but not S-FP-projective.

Example 1. Let $R = \mathbb{Z}$, the ring of integers, p a prime in $\mathbb{Z}$, and $S = \{1, p, p^2, \ldots\}$. Since R is Noetherian, all R-modules are FP-projective by ([7], Proposition 2.6). Since $M: = R/\langle p\rangle$ is u-S-torsion, it is also u-S-absolutely pure (see ([6], Example 3.8)). However, since $\mathrm{Ext}_R^1(M, M) \cong M \neq 0$ (see ([10], page 267)), M is not S-FP-projective.

Now, we characterize rings over which all S-FP-projective modules are projective.

Proposition 1. *Let R be a ring. Then, R is u-S-von Neumann regular if and only if all S-FP-projective modules are projective.*

Proof. Suppose that R is u-S-von Neumann regular. Then, all R-modules are u-S-absolutely pure by reference [6], Theorem 3.5. So, all S-FP-projective modules are projective. The sufficiency follows similarly. □

We know that every f.g. projective R-module is f.p.; we generalize this result to S-FP-projective modules in the following proposition:

Proposition 2. *Let R be a ring. Then, every f.g. S-FP-projective R-module is f.p.*

Proof. Let M be an f.g. S-FP-projective R-module. Then, $\mathrm{Ext}_R^1(M,N)=0$ for any u-S-absolutely pure module N. Hence, it follows by reference [11], Theorem 2.1.10 that M is f.p. □

Now, we give some characterizations of S-FP-projective modules.

Proposition 3. *Let R be a ring and M be an R-module. Then, the following are equivalent:*

(1) M is S-FP-projective

(2) M is projective with respect to every exact sequence $0\longrightarrow N\longrightarrow L\longrightarrow X\longrightarrow 0$, where N is u-S-absolutely pure

(3) For any exact sequence of R-modules of the form $0\longrightarrow L\longrightarrow E\longrightarrow M\longrightarrow 0$ and any u-S-absolutely pure module N, the sequence $0\longrightarrow Hom_R(M,N)\longrightarrow Hom_R(E,N)\longrightarrow Hom_R(L,N)\longrightarrow 0$ is exact

(4) Every exact sequence of the form $0\longrightarrow N\longrightarrow L\longrightarrow M\longrightarrow 0$, where N is u-S-absolutely pure, splits

(5) $M\otimes F$ is S-FP-projective for any projective R-module F

(6) $Hom_R(F,M)$ is S-FP-projective for any f.g. projective R-module F

Proof. (1)⇒(2) Let $0\longrightarrow N\longrightarrow L\longrightarrow X\longrightarrow 0$ be an exact sequence with N u-S-absolutely pure. Then, we have the exact sequence $0\longrightarrow \mathrm{Hom}_R(M,N)\longrightarrow \mathrm{Hom}_R(M,L)\longrightarrow \mathrm{Hom}_R(M,X)\longrightarrow \mathrm{Ext}_R^1(M,N)$. Since M is S-FP-projective and A is u-S-absolutely pure, $\mathrm{Ext}_R^1(M,N)=0$. Thus, $0\longrightarrow \mathrm{Hom}_R(M,N)\longrightarrow \mathrm{Hom}_R(M,L)\longrightarrow \mathrm{Hom}_R(M,X)\longrightarrow 0$ is exact.

(2)⇒(1) Let M (2)⇒(1) be a u-S-absolutely pure module. Consider the following exact sequence $0\longrightarrow A\longrightarrow E\longrightarrow L\longrightarrow 0$ with E an injective module. So, we have the following exact sequence:

$$0\longrightarrow \mathrm{Hom}_R(M,A)\longrightarrow \mathrm{Hom}_R(M,E)\longrightarrow \mathrm{Hom}_R(M,L)\longrightarrow \mathrm{Ext}_R^1(M,A)\longrightarrow 0. \qquad (2)$$

Keeping in mind that $0\longrightarrow \mathrm{Hom}_R(M,A)\longrightarrow \mathrm{Hom}_R(M,E)\longrightarrow \mathrm{Hom}_R(M,L)\longrightarrow 0$ is exact, we deduce that $\mathrm{Ext}_R^1(M,A)=0$. Hence, M is S-FP-projective.

(1)⇒(3) Let $0\longrightarrow L\longrightarrow E\longrightarrow M\longrightarrow 0$ be an exact sequence. For any u-S-absolutely pure module N, it follows that $0\longrightarrow \mathrm{Hom}_R(M,N)\longrightarrow \mathrm{Hom}_R(E,N)\longrightarrow \mathrm{Hom}_R(L,N)\longrightarrow \mathrm{Ext}_R^1(M,N)$ is exact. Since M is S-FP-projective, $\mathrm{Ext}_R^1(M,N)=0$, and so (3) holds.

(3)⇒(1) Let $0\longrightarrow L\longrightarrow E\longrightarrow M\longrightarrow 0$ be an exact sequence with E projective. Hence, for any u-S-absolutely pure N, we have the following exact sequence:

$$0\longrightarrow \mathrm{Hom}_R(M,N)\longrightarrow \mathrm{Hom}_R(E,N)\longrightarrow \mathrm{Hom}_R(L,N)\longrightarrow \mathrm{Ext}_R^1(M,N)\longrightarrow 0. \qquad (3)$$

Since $0\longrightarrow \mathrm{Hom}_R(M,N)\longrightarrow \mathrm{Hom}_R(E,N)\longrightarrow \mathrm{Hom}_R(L,N)\longrightarrow 0$ is exact, we deduce that $\mathrm{Ext}_R^1(M,N)=0$. Hence, M is S-FP-projective.

(2)⇔(4) It is clear.

(1)⇒(5) Let N be a u-S-absolutely pure R-module and F be a projective R-module. By ([12], Theorem 3.3.10), we have the following isomorphism:

$$\mathrm{Ext}_R^1(F\otimes M,N)\cong \mathrm{Hom}\left(F,\mathrm{Ext}_R^1(M,N)\right). \qquad (4)$$

Thus, $\mathrm{Ext}_R^1(M,N)=0$ since M is S-FP-projective. Hence, $\mathrm{Ext}_R^1(F\otimes M,N)=0$, and so $F\otimes M$ is S-FP-projective.

(1)⇒(6) Let N be a u-S-absolutely pure R-module and F be an f.g. projective R-module. By reference [12], Theorem 3.3.12, we have the following isomorphism:

$$\mathrm{Ext}_R^1(\mathrm{Hom}(F,M),N)\cong F\otimes \mathrm{Ext}_R^1(M,N)=0. \qquad (5)$$

Thus, $\mathrm{Hom}(F,M)$ is an S-FP-projective R-module.

(5)⇒(1) and (6)⇒(1) These follow by setting $P:=R$. □

Recall that an R-module M is said to be S-torsion if for any $x\in M$, there exists $s\in S$ such that $sx=0$.

Lemma 1. *Let R be a ring and S be finite. Then, every S-torsion R-module is u-S-absolutely pure.*

Proof. Let F be an R-module and M be an f.p. R-module. Then, the natural homomorphism

$$\theta\colon \mathrm{Hom}_R(M,F)_S \longrightarrow \mathrm{Hom}_{R_S}(M_S,F_S), \quad (6)$$

induces a homomorphism

$$\theta_1\colon \mathrm{Ext}^1_R(M,F)_S \longrightarrow \mathrm{Ext}^1_{R_S}(M_S,F_S). \quad (7)$$

By reference [13], Proposition 1.10, θ_1 is a monomorphism. Let F be an S-torsion R-module. Then, $\mathrm{Ext}^1_R(M,F)_S = 0$ since $F_S = 0$ by reference [12], Example 1.6.13. Hence, $\mathrm{Ext}^1_R(M,F)$ is S-torsion by reference [12], Example 1.6.13 again. Hence, $\mathrm{Ext}^1_R(M,F)$ is u-S-torsion by ([1], Proposition 2.3). Consequently, F is a u-S-absolutely pure by reference [6], Theorem 3.2. □

The following proposition gives a condition that all S-FP-projective modules are projective:

Proposition 4. *Let R be a ring, S be finite, and M be an R-module. If M is S-FP-projective and $\mathrm{Ext}^1_R(M,K) = 0$ for any S-torsion-free R-module K, then M is projective.*

Proof. Let N be an R-module. The exact sequence,

$$0 \longrightarrow \mathrm{tor}_S(N) \longrightarrow N \longrightarrow \frac{N}{\mathrm{tor}_S(N)} \longrightarrow 0, \quad (8)$$

gives rise to the following exact sequence:

$$\mathrm{Ext}^1_R(M,\mathrm{tor}_S(N)) \longrightarrow \mathrm{Ext}^1_R(M,N) \longrightarrow \mathrm{Ext}^1_R\left(M,\frac{N}{\mathrm{tor}_S(N)}\right). \quad (9)$$

The left term is zero by Lemma 1 and the right term is zero since $N/\mathrm{tor}_S(N)$ is S-torsion-free (see ([12], Example 1.6.13)). Thus, $\mathrm{Ext}^1_R(M,N) = 0$, and so M is projective. □

Recall that a ring R is said to be coherent if every f.g. ideal of R is f.p.

Lemma 2. *Let R be a coherent ring, S be finite and E be an R-module. Then, the following conditions are equivalent:*

(1) E is u-S-absolutely pure

(2) There exists an element $s \in S$ satisfying that for any f.p. R-module N and any integer $n \geq 0$, $\mathrm{Ext}^{n+1}_R(N,E)$ is u-S-torsion with respect to s

Proof. (1)⇒(2) Suppose that E is a u-S-absolutely pure R-module and let F be an f.p. R-module. The case $n = 0$ is trivial by ([6], Theorem 3.2). Thus, we may assume that $n > 0$. Consider an exact sequence as follows:

$$0 \longrightarrow F' \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow F \longrightarrow 0, \quad (10)$$

where F' is an f.p., and free R-module and $F_0,\ldots,F_{n-1}$ are f.g.. Such sequence exists because R is coherent. Hence, we have the following isomorphism:

$$\left(\mathrm{Ext}^{n+1}_R(F,M)\right)_S \cong \left(\mathrm{Ext}^1_R\left(F',M\right)\right)_S = 0, \quad (11)$$

by reference [12], Example 1.6.13 since every u-S-torsion is S-torsion. Thus, $\mathrm{Ext}^{n+1}_R(N,E)_S = 0$, which implies that $\mathrm{Ext}^{n+1}_R(N,E)$ is an S-torsion R-module by reference [12], Example 1.6.13 and $\mathrm{Ext}^{n+1}_R(N,E)$ is u-S-torsion by reference [1], Proposition 2.3.

(2)⇒(1) This is obvious. □

Lemma 3. *Let R be a coherent ring, S be finite, and $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules, where M' is u-S-absolutely pure. Then, M is u-S-absolutely pure if and only if M'' is u-S-absolutely pure.*

Proof. Let F be an f.p. R-module. We have the following exact sequence:

$$\mathrm{Ext}^1_R\left(F,M'\right) \longrightarrow \mathrm{Ext}^1_R(F,M) \longrightarrow \mathrm{Ext}^1_R\left(F,M''\right) \longrightarrow \mathrm{Ext}^2_R\left(F,M'\right). \quad (12)$$

By Lemma 2, ([1], Proposition 2.3), and ([12], Example 1.6.13), we have the following exact sequence:

$$0 = \mathrm{Ext}^1_R\left(F,M'\right)_S \longrightarrow \mathrm{Ext}^1_R(F,M)_S \longrightarrow \mathrm{Ext}^1_R\left(F,M''\right)_S \longrightarrow \mathrm{Ext}^2_R\left(F,M'\right)_S = 0. \quad (13)$$

Hence, $\mathrm{Ext}^1_R(F,M)_S \cong \mathrm{Ext}^1_R(F,M'')_S$. So, $\mathrm{Ext}^1_R(F,M)$ is an S-torsion R-module if and only if $\mathrm{Ext}^1_R(F,M'')$ is an S-torsion R-module by ([12], Example 1.6.13). By ([1], Proposition 2.3), $\mathrm{Ext}^1_R(F,M)$ is a u-S-torsion R-module if and only if $\mathrm{Ext}^1_R(F,M'')$ is a u-S-torsion R-module. Thus, M is u-S-absolutely pure if and only if M'' is u-S-absolutely pure. □

Proposition 5. *Let R be a coherent ring, S be finite, and M be an R-module. Then, the following conditions are equivalent:*

(1) M is S-FP-projective

(2) $\mathrm{Ext}^{n+1}_R(M,N) = 0$ for any u-S-absolutely pure R-module N and any integer $n \geq 0$

Proof. (1)⇒(2) Let N be a u-S-absolutely pure R-module. The case $n = 0$ is trivial. So, we may assume $n > 0$. We consider the following exact sequence:

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow N' \longrightarrow 0, \quad (14)$$

where $E^0, \ldots, E^{n-1}$ are injective R-modules. By Lemma 3, N' is u-S-absolutely pure. Hence, $\mathrm{Ext}_R^{n+1}(M, N) \cong \mathrm{Ext}_R^1(M, N') = 0$.

(2)⇒(1) This is trivial. □

Proposition 6. *Let R be a coherent ring, S be finite, and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules, where C is S-FP-projective. Then, A is S-FP-projective if and only if B is S-FP-projective.*

Proof. Let L be a u-S-absolutely pure R-module. Then, we have the following exact sequence:

$$\mathrm{Ext}_R^1(C, L) \longrightarrow \mathrm{Ext}_R^1(B, L) \longrightarrow \mathrm{Ext}_R^1(A, L) \longrightarrow \mathrm{Ext}_R^2(C, L). \quad (15)$$

Since C is S-FP-projective, $\mathrm{Ext}_R^1(C, L) = 0$, and by Proposition 5 we have $\mathrm{Ext}_R^2(C, L) = 0$. Thus, $\mathrm{Ext}_R^1(B, L) \cong \mathrm{Ext}_R^1(A, L)$. Hence, A is S-FP-projective if and only if B is S-FP-projective. □

Proposition 7. *Let R be a ring, S be finite, and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules. If A and C are S-FP-projective, then B is S-FP-projective.*

Proof. For any u-S-absolutely pure R-module L, we have the following exact sequence $\mathrm{Ext}_R^1(C, L) \longrightarrow \mathrm{Ext}_R^1(B, L) \longrightarrow \mathrm{Ext}_R^1(A, L)$. Since A and C are S-FP-projective, we have $\mathrm{Ext}_R^1(A, L) = 0 = \mathrm{Ext}_R^1(C, L)$, and so $\mathrm{Ext}_R^1(B, L) = 0$. Therefore, B is S-FP-projective. □

Proposition 8. *Let S be finite. Then, the class of all $S - FP$-projective modules is closed under arbitrary direct sums and under direct summands.*

Proof. It follows by ([12], Theorem 3.3.9(2)). □

Proposition 9. *Let R be a ring. If every u-S-absolutely pure R-module has injective dimension ≤ 1, then R is a coherent ring.*

Proof. Let J be an f.g. ideal of R and A be a u-S-absolutely pure R-module. Then, by hypothesis, $\mathrm{Ext}_R^2((R/J), A) = 0$. Consider the following exact sequence:

$$\mathrm{Ext}_R^1(R, A) \longrightarrow \mathrm{Ext}_R^1(J, A) \longrightarrow \mathrm{Ext}_R^2((R/J), A). \quad (16)$$

Hence, $\mathrm{Ext}_R^1(J, A) = 0$ since $\mathrm{Ext}_R^1(R, A) = 0 = \mathrm{Ext}_R^2((R/J), A)$. Thus, J is S-FP-projective. Then, by Proposition 2, J is f.p., which implies that R is a coherent ring. □

We recall from reference [5] that the FP-injective dimension of M, denoted by FP-$\mathrm{id}_R M$, is defined to be the least nonnegative integer n such that $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for any f.p. R-module N.

Proposition 10. *Let R be a ring. We consider the following conditions:*

(1) R is a coherent ring

(2) Every f.g. submodule of a projective R-module is S-FP-projective

(3) Every f.g. ideal of R is S-FP-projective

Then, (2)⇒(3)⇒(1), and if S is composed of units, we have (1)⇒(2).

Proof. (2)⇒(3) It is obvious.

(3)⇒(1) Let J be an f.g. ideal of R. Then, J is S-FP-projective by (3). Hence, J is f.p. by Proposition 2. Thus, R is coherent.

(1)⇒(2) Let N be an f.g. submodule of a projective R-module M. Hence, by ([9], Theorem 3.7), we have N is absolutely pure (FP-injective), and so S-FP-projective since S is composed of units. □

In the following definition, we define the S-FP-hereditary ring, which is an extension of the FP-hereditary ring.

Definition 2. A ring R is said to be *S-FP-hereditary* if every ideal of R is S-FP-projective.

Note that every S-FP-hereditary ring is FP-hereditary since every S-FP-projective module is FP-projective. Therefore, we have the following implications:

$$\text{hereditary rings} \Rightarrow S-\text{FP}-\text{hereditary rings} \Rightarrow \text{FP}-\text{hereditary rings}. \quad (17)$$

Remark 2

(1) If S is composed of units, then the class of S-FP-hereditary rings and the class of FP-hereditary rings coincide.

(2) If $0 \in S$, then every S-FP-projective module is projective. Hence, all S-FP-hereditary rings are exactly hereditary rings.

Later, we will provide an example of an FP-hereditary ring but not S-FP-hereditary (Example 2).

Lemma 4. *Let R be a ring. If every submodule of an S-FP-projective R-module is absolutely pure, then every FP-projective module is S-FP-projective.*

Proof. Let M be an FP-projective R-module. Then, there exists an exact sequence η: $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$, where P is an S-FP-projective R-module. Hence, by hypothesis, N is absolutely pure, and so by ([9], Proposition 3.3), the exact sequence η splits. Hence, by Proposition 3, M is S-FP-projective since every absolutely pure is u-S-absolutely pure. □

Corollary 1. *If R is an FP-hereditary ring and any submodule of an S-FP-projective R-module is absolutely pure, then R is S-FP-hereditary.*

Proof. Let J be an ideal of R. Then, J is FP-projective since R is FP-hereditary. Hence, J is S-FP-projective by Lemma 4, which implies that R is S-FP-hereditary. □

In the following result, we characterize S-FP-hereditary rings.

Proposition 11. *The following conditions are equivalent for a ring R:*

(1) R is S-FP-hereditary

(2) Every submodule of a projective R-module is S-FP-projective

(3) Every submodule of an S-FP-projective R-module is S-FP-projective

(4) Every u-S-absolutely pure R-module has injective dimension ≤ 1

(5) For any u-S-pure submodule A of an injective module B, the factor module B/A is injective

Proof. (3)⇒(2)⇒(1) These are obvious.

(1)⇒(4) Let N be a u-S-absolutely pure R-module and J be an ideal of R. The exact sequence $0 \longrightarrow J \longrightarrow R \longrightarrow (R/J) \longrightarrow 0$ gives the following exact sequence:

$$0 = \mathrm{Ext}_R^1(J, N) \longrightarrow \mathrm{Ext}_R^2(R/J, N) \longrightarrow \mathrm{Ext}_R^2(R, N) = 0. \quad (18)$$

Thus, $\mathrm{Ext}_R^2((R/J), N) = 0$, which implies that $\mathrm{id}_R(N) \leq 1$.

(4)⇒(3) Let N be a submodule of an S-FP-projective R-module M. By (4), for any u-S-absolutely pure R-module E, we have the following exact sequence:

$$\mathrm{Ext}_R^1(M, E) \longrightarrow \mathrm{Ext}_R^1(N, E) \longrightarrow \mathrm{Ext}_R^2((M/N), E), \quad (19)$$

where the left term is zero since M is an S-FP-projective R-module and the right term is zero since $\mathrm{id}_R(E) \leq 1$. Thus, $\mathrm{Ext}_R^1(N, E) = 0$, which implies that N is an S-FP-projective R-module.

(4)⇒(5) Let A be a u-S-pure submodule of an injective module B. Then, A is u-S-absolutely pure by ([6], Theorem 3.2), and so $\mathrm{id}_R(A) \leq 1$ by (4). Thus, the exactness of $0 \longrightarrow A \longrightarrow B \longrightarrow (B/A) \longrightarrow 0$ implies the injectivity of B/A.

(5)⇒(4) Let A be a u-S-absolutely pure R-module. Then, A is a u-S-pure submodule of its injective envelope $E(A)$ by ([6], Theorem 3.2). Hence, $E(A)/A$ is injective by (5). Therefore, $\mathrm{id}_R(A) \leq 1$. □

Corollary 2. *Every S-FP-hereditary ring is a coherent ring.*

Proof. This is a consequence of Proposition 11 and Proposition 9. □

The converse of Corollary 2 is not true in general (see, ([9], Example 3.9)).

Proposition 12. *The following conditions are equivalent for a ring R:*

(1) Every R-module is S-FP-projective

(2) R/I is S-FP-projective for any ideal I of R

(3) Every u-S-absolutely pure R-module is injective. If S is composed of units, then the previous conditions are also equivalent to

(4) R is a Noetherian ring

Proof. (1)⇒(2) This is trivial.

(2)⇒(3) Let N be a u-S-absolutely pure R-module. Then, for any ideal I of R, we have $\mathrm{Ext}_R^1((R/I), N) = 0$ since R/I is an S-FP-projective R-module. Hence, N is an injective R-module by ([12], Theorem 3.3.8).

(3)⇒(1) Let M be an R-module. Then, for any u-S-absolutely pure R-module N, we have $\mathrm{Ext}_R^1(M, N) = 0$ since N is an injective R-module. Thus, M is S-FP-projective.

(1)⇒(4) This follows by ([7], Proposition 2.6) since every S-FP-projective R-module is FP-projective.

(4)⇒(1) Let M be an R-module. Then, M is FP-projective since R is Noetherian by ([7], Proposition 2.6) again. Hence, M is S-FP-projective since S is composed of units. □

Proposition 13. *The following conditions are equivalent for a ring R:*

(1) Every f.p. R-module is S-FP-projective

(2) Every u-S-absolutely pure R-module is absolutely pure

(3) Every FP-projective R-module is S-FP-projective

Proof. (1)⇒(2) Let N be a u-S-absolutely pure R-module. Then, for any f.p. R-module F, we have $\mathrm{Ext}_R^1(F, N) = 0$ since F is an S-FP-projective R-module. Hence, N is an absolutely pure R-module.

(2)⇒(3) Let M be an FP-projective R-module. Then, for any u-S-absolutely pure R-module N, we have $\mathrm{Ext}_R^1(M, N) = 0$ since N is an absolutely pure R-module. Thus, M is S-FP-projective.

(3)⇒(1) This follows from the fact that f.p. R-modules are always FP-projective. □

In the following proposition, we will prove that $M_1 \times M_2$ is S-FP-projective if and only if M_1 and M_2 are S-FP-projective. However, we need the following lemmas. For brevity's sake, when R_1, R_2 are rings and M_1 (resp., M_2) is an

R_1-module (resp., R_2-module), until the end of this section, we will sometimes set $R: = R_1 \times R_2$ and $M: = M_1 \times M_2$.

Lemma 5. *Let M_1 be an R_1-module and M_2 be an R_2-module and set $S: = S_1 \times S_2$. Then, $M_1 \times M_2$ is a u-S-absolutely pure $(R_1 \times R_2)$-module if and only if each M_i is a u-S-absolutely pure R_i-module, $i = 1, 2$.*

Proof. Suppose that $M_1 \times M_2$ is a u-S-absolutely pure $(R_1 \times R_2)$-module and let F be a finitely presented R_1-module. It is clear that F is also an $(R_1 \times R_2)$-module (via the canonical projection $R_1 \times R_2 \hookrightarrow R_1$). With this modulation, and by using ([11], Theorem 2.1.8), F is an f.p. $(R_1 \times R_2)$-module. Thus, there exists $s: = (s_1, s_2) \in S$ such that $s\mathrm{Ext}^1_{R_1\times R_2}(F, M_1 \times M_2) = 0$. From ([14], Theorem 10.75), $0 = s\mathrm{Ext}^1_R(F, M) = s_1\mathrm{Ext}^1_{R_1}(F, \mathrm{Hom}_R(R_1, M)) = s_1\mathrm{Ext}^1_{R_1}(F, M_1)$. Hence, $\mathrm{Ext}^1_{R_1}(F, R_1)$ is u-S-torsion with respect to s_1. Consequently, M_1 is a u-S-absolutely pure R_1-module. Similarly, M_2 is a u-S-absolutely pure R_2-module.

Conversely, assume that each M_i is a u-S-absolutely pure R_i-module, $i = 1, 2$, and let F be an f.p. $(R_1 \times R_2)$-module. Then, there exists $s_1 \in S$ and $s_2 \in S$ and set $s: = (s_1, s_2)$ such that

$$\begin{aligned} s\mathrm{Ext}^1_{R_1\times R_2}(F, M_1 \times M_2) &= s\mathrm{Ext}^1_{R_1\times R_2}((F, M_1 \times 0)\oplus(0 \times M_2)) \\ &= s_1\mathrm{Ext}^1_{R_1\times R_2}(F, M_1 \times 0)\oplus s_2\mathrm{Ext}^1_{R_1\times R_2}(F, 0 \times M_2) \\ &= s_1\mathrm{Ext}^1_{R_1\times R_2}(F, M_1)\oplus s_2\mathrm{Ext}^1_{R_1\times R_2}(F, M_2) \\ &= s_1\mathrm{Ext}^1_{R_1}(F \otimes_R R_1, M_1)\oplus s_2\mathrm{Ext}^1_{R_2}(F \otimes_R R_2, M_2), \end{aligned} \tag{20}$$

(by ([14], Theorem 10.74)).

On the other hand, by ([11], Theorem 2.1.8), $F \otimes_{R_1\times R_2} R_1 \cong (F/(0 \times R_2)F)$ (resp. $F \otimes_{R_1\times R_2} R_2 \cong F/(R_1 \times 0)F$) is an f.p. R_1-module (resp., R_2-module). Thus, $s_1\mathrm{Ext}^1_{R_1}(F \otimes_R R_1, M_1) = 0$ and $s_2\mathrm{Ext}^1_{R_2}(F \otimes_R R_2, M_2) = 0$, which imply that $\mathrm{Ext}^1_{R_1}(F \otimes_{R_1\times R_2} R_1, M_1)$ is u-S-torsion with respect to s_1 and $\mathrm{Ext}^1_{R_2}(F \otimes_{R_1\times R_2} R_2, M_2)$ is u-S-torsion with respect to s_2. Consequently, $\mathrm{Ext}^1_{R_1\times R_2}(F, M_1 \times M_2)$ is u-S-torsion with respect to s, and so $M_1 \times M_2$ is a u-S-absolutely pure $(R_1 \times R_2)$-module. □

Lemma 6. *Let $\phi: R_1 \longrightarrow R_2$ be a surjective ring homomorphism, where R_2 is projective as an R_1-module. If M is a u-S-absolutely pure R_2-module, then M is a u-$\phi^{-1}(S)$-absolutely pure R_1-module.*

Proof. Let N be an f.p. R_1-module. Then, there exists an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ of R_1-modules with K f.g. and P f.g. and projective. Since R_2 is a projective R_1-module, we have the following exact sequence $0 \longrightarrow K \otimes_{R_1} R_2 \longrightarrow P \otimes_{R_1} R_2 \longrightarrow N \otimes_{R_1} R_2 \longrightarrow 0$ of R_2-modules. Note that $K \otimes_{R_1} R_2$ is an f.g. R_2-module, and $P \otimes_{R_1} R_2$ is an f.g. and projective R_2-module. Thus, $N \otimes_{R_1} R_2$ is an f.p. R_2-module. Since M is a u-S-absolutely pure R_2-module, there exists $s_2 \in S$ such that $s_2\mathrm{Ext}^1_{R_2}(N \otimes_{R_1} R_2, M) = 0$, and so $\mathrm{Ext}^1_{R_2}(N \otimes_{R_1} R_2, M)$ is u-S-torsion with respect to s_2. Therefore, by ([14], Theorem 10.74), $\mathrm{Ext}^1_{R_1}(N, M)$ is u-S-torsion with respect to some $s_1 \in \phi^{-1}(s_2)$. Thus, M is a u-$\phi^{-1}(S)$-absolutely pure R_1-module. □

Proposition 14. *Let M_i be an R_i-module and let S_i be a multiplicative subset of R_i for $i = 1, 2$, and set $S: = S_1 \times S_2$. Then, $M_1 \times M_2$ is an S-FP-projective $(R_1 \times R_2)$-module if and only if each M_i is an S_i-FP-projective R_i-module for $i = 1, 2$.*

Proof. Suppose that $M_1 \times M_2$ is an S-FP-projective $(R_1 \times R_2)$-module, and let N be a u-S-absolutely pure R_1-module. It is clear that N is also an $(R_1 \times R_2)$-module (via the canonical projection $R_1 \times R_2 \twoheadrightarrow R_1$). With this modulation, N is a u-S-absolutely pure $R_1 \times R_2$-module by Lemma 6. Then, by reference [14], Theorem 10.74, we obtain the following isomorphisms:

$$\mathrm{Ext}^1_{R_1}(M_1, N) \cong \mathrm{Ext}^1_{R_1}((M_1 \times M_2)\otimes R_1, N) \cong \mathrm{Ext}^1_{R_1\times R_2}(M_1 \times M_2, N) = 0. \tag{21}$$

Consequently, M_1 is an S_1-FP-projective R_1-module. Similarly, M_2 is an S_2-FP-projective R_2-module.

Conversely, we assume that each M_i is an S_i-FP-projective R_i-module for $i = 1, 2$. Let N be a u-S-absolutely pure $(R_1 \times R_2)$-module, and set $N_i: = N \otimes R_i$ for $i = 1, 2$. It is clear that $N \cong N_1 \times N_2$. By reference [14], Theorem 10.74, we have the following isomorphisms:

$$\begin{aligned} &\mathrm{Ext}^1_{R_1}(M_1, N_1) \times \mathrm{Ext}^1_{R_2}(M_2, N_2) \\ &\cong \mathrm{Ext}^1_R(M, N_1) \times \mathrm{Ext}^1_R(M, N_2) \\ &\cong \mathrm{Ext}^1_R(M, N_1 \times 0) \times \mathrm{Ext}^1_R(M, 0 \times N_2) \\ &\cong \mathrm{Ext}^1_R(M, N_1 \times N_2) \\ &\cong \mathrm{Ext}^1_R(M, N). \end{aligned} \tag{22}$$

On the other hand, by Lemma 5, N_1 (resp., N_2) is a u-S-absolutely R_1-module (resp., R_2-module). Thus, $\mathrm{Ext}^1_{R_1}(M_1, N_1) = 0$ and $\mathrm{Ext}^1_{R_2}(M_2, N_2) = 0$. Consequently, $\mathrm{Ext}^1_{R_1\times R_2}(M_1 \times M_2, N) = 0$, and so $M_1 \times M_2$ is S-FP-projective. □

3. S-FP-Projective Dimension of a Module and Global S-FP-Projective Dimension of a Ring

This section introduces and investigates the S-FP-projective dimension of a module and the global S-FP-projective dimension of a ring.

Definition 3. Let R be a ring. For any R-module M, the S-FP-projective dimension of M, denoted by $S\text{-fpd}_R(M)$, is the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(M, A) = 0$ for any u-S-absolutely pure R-module A. If no such integer exists, we set $S\text{-fpd}_R(M) = \infty$.

The global S-FP-projective dimension of R is defined as follows:

$$S - \mathrm{fpD}(R) = \sup\{S\text{–}\mathrm{fpd}_R(M) \mid M \text{ is an f.g. } R - \text{module}\}. \tag{23}$$

Clearly, an R-module M is S-FP-projective if and only if $S\text{–fpd}_R(M) = 0$ and $\mathrm{fpd}_R(M) \leq S\text{–fpd}_R(M)$, with equality when S consists of units. However, this inequality may be strict (Example 1). Also, $\mathrm{fpD}(R) \leq S - \mathrm{fpD}(R)$ with equality when S consists of units, and this inequality may be strict. For example, consider a ring $R = \mathbb{Z}$, the ring of integers. Since R is Noetherian, we get $\mathrm{fpD}(R) = 0$ (by ([7], Proposition 2.6)). Moreover, by Example 1, there exists an (FP-projective) R-module M which is not S-FP-projective. Thus, $S - \mathrm{fpD}(R) > 0$.

First, we describe the S-FP-projective dimension of a module over a coherent ring.

Proposition 15. *Let R be a coherent ring and S be finite. The following conditions are equivalent for any R-module N:*

(1) $S - fpd(N) \leq n$
(2) $Ext^{n+1}_R(N, M) = 0$ *for any u-S-absolutely pure R-module M*
(3) $Ext^{n+i}_R(N, M) = 0$ *for any u-S-absolutely pure R-module M and any* $i \geq 1$
(4) *If a sequence* $0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0$ *is exact, where* $F_0, \ldots, F_{n-1}$ *are S – FP-projective R-modules, then* F_n *is S – FP-projective*
(5) *If a sequence* $0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0$ *is exact, where* $F_0, \ldots, F_{n-1}$ *are projective R-modules, then* F_n *is S – FP-projective*
(6) *There exists an exact sequence* $0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0$ *where each* F_i *is S – FP-projective*

Proof. (3)⇒(2)⇒(1) These are clear.

(1)⇒(4) Let $0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules, where $F_0, \ldots, F_{n-1}$ are S – FP-projective, and set $K_0 := \mathrm{Ker}(F_0 \longrightarrow N)$ and $K_j := \mathrm{Ker}(F_j \longrightarrow F_{j-1})$, where $j = 1, \ldots, n-1$. Using Proposition 5, we get the following isomorphisms:

$$0 = \mathrm{Ext}^{n+1}_R(N, M) \cong \mathrm{Ext}^n_R(K_0, M) \cong \cdots \cong \mathrm{Ext}^1_R(F_n, M), \tag{24}$$

for any u-S-absolutely pure R-module M. Thus, F_n is S – FP-projective.

(4)⇒(5)⇒(6) These are obvious.

(6)⇒(3) We proceed by induction on $n \geq 0$. For the $n = 0$ case, M is an S – FP-projective module, and so (3) holds by Proposition 5. If $n \geq 1$, then there is an exact sequence $0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0$, where each F_j is S – FP-projective. Set $K_0 := \mathrm{Ker}(F_0 \longrightarrow N)$. Then, we have the following exact sequences:

$$\begin{gathered} 0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow K_0 \longrightarrow 0, \\ 0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow N \longrightarrow 0. \end{gathered} \tag{25}$$

Hence, by induction, $\mathrm{Ext}^{n-1+i}_R(K_0, M) = 0$ for any R-module M and all $i \geq 1$. Thus, $\mathrm{Ext}^{n+j}_R(N, M) = 0$. Hence, the desired result follows. □

Proposition 16. *Let R be a coherent ring, S be finite, and* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be an exact sequence of R-modules. If two of* $S\text{–}fpd_R(A)$, $S\text{–}fpd_R(B)$, *and* $S\text{–}fpd_R(C)$ *are finite, so is the third. Moreover, we have the following conditions:*

(1) $S\text{–}fpd_R(A) \leq \sup\{S\text{–}fpd_R(B), S\text{–}fpd_R(C) - 1\}$.
(2) $S\text{–}fpd_R(B) \leq \sup\{S\text{–}fpd_R(A), S\text{–}fpd_R(C)\}$.
(3) $S\text{–}fpd_R(C) \leq \sup\{S\text{–}fpd_R(B), S\text{–}fpd_R(A) + 1\}$.

Proof. This follows from the standard of homological algebra. □

Corollary 3. *Let R be a coherent ring, S be finite, and* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be an exact sequence of R-modules. If B is S-FP-projective and* $S\text{-}fpd_R(C) > 0$, *then* $S\text{–}fpd_R(C) = S\text{–}fpd_R(A) + 1$.

The proof of the following result is straightforward.

Proposition 17. *Let R be a coherent ring, S be a finite multiplicative subset of R, and* $\{M_i\}$ *be a family of R-modules. Then,* $S\text{–}fpd_R(\oplus_i M_i) = \sup_i\{S\text{–}fpd_R(M_i)\}$.

Proposition 18. *Let R be a ring and* $n \geq 0$ *be an integer. Then, the following statements are equivalent:*

(1) $S - fpD(R) \leq n$
(2) $S - fpd(N) \leq n$ *for any R-module N*
(3) $S - fpd(R/J) \leq n$ *for any ideal J of R*
(4) $id_R(E) \leq n$ *for any u-S-absolutely pure R-module E*

Consequently, we have the following equalities:

$$S-\mathrm{fpD}(R)=\sup\{S\text{–}\mathrm{fpd}_R(N)\mid N\ \text{isan}\ R-\text{module}\}$$
$$=\sup\{S\text{–}\mathrm{fpd}_R(R/J)\mid J\ \text{isanideal of}\ R\},$$
$$=\sup\{\mathrm{id}_R(E)\mid E\text{isa}u-S-\text{abosolutelypure}\ R-\text{module}\}. \quad (26)$$

Proof. (4)⇒(2) Let N be an R-module. For every u-S-absolutely pure R-module E, we have $\mathrm{Ext}_R^{n+1}(N,E)=0$. Hence, $S-\mathrm{fpd}(N)\leq n$.

(2)⇒(1)⇒(3) These are clear.

(3)⇒(4) Let E be a u-S-absolutely pure R-module. For every ideal J of R, we have $\mathrm{Ext}_R^{n+1}((R/J),E)=0$. Thus, $\mathrm{id}_R(E)\leq n$. □

Next, we show that rings R with $S-\mathrm{fpD}(R)=0$ are exactly Noetherian rings if S is composed of units.

Proposition 19. *Let R be a ring and S be composed of units. Then, the following conditions are equivalent:*

(1) $S-fpD(R)=0$

(2) Every R-module is S-FP-projective

(3) R/I is S-FP-projective for any ideal I of R

(4) Every u-S-absolutely pure R-module is injective

(5) R is a Noetherian ring

Proof. The equivalence of (1), (2), (3), and (4) follow from Proposition 18.

(2)⇔(5) This follows from Proposition 12. □

Finally, we show that rings R with $S-\mathrm{fpD}(R)\leq 1$ are exactly S-FP-hereditary rings.

Proposition 20. *The following conditions are equivalent for a ring R:*

(1) $S-fpD(R)\leq 1$

(2) Every submodule of S-FP-projective R-module is S-FP-projective

(3) Every submodule of projective R-module is S-FP-projective

(4) $id_R(E)\leq 1$ for any u-S-absolutely pure R-module E

(5) R is an S-FP-hereditary ring

Proof. (2)⇒(3)⇒(5) These are obvious.

(1)⇔(4) This follows by Proposition 18.

(5)⇒(4) Let E be a u-S-absolutely pure R-module and I be an ideal of R. The exact sequence $0\longrightarrow I\longrightarrow R\longrightarrow (R/I)\longrightarrow 0$ gives rise to the following exact sequence:

$$0=\mathrm{Ext}_R^1(I,E)\longrightarrow \mathrm{Ext}_R^2(R/I,E)\longrightarrow \mathrm{Ext}_R^2(R,E)=0. \quad (27)$$

Thus, $\mathrm{Ext}_R^2((R/I),E)=0$, and so $\mathrm{id}_R(E)\leq 1$.

(4)⇒(5) Let I be an ideal of R. For any u-S-absolutely pure R-module E, we have the following exact sequence:

$$0=\mathrm{Ext}_R^1(R,E)\longrightarrow \mathrm{Ext}_R^1(I,E)\longrightarrow \mathrm{Ext}_R^2((R/I),E)=0. \quad (28)$$

Thus, $\mathrm{Ext}_R^1(I,E)=0$, which implies that I is S-FP-projective. Therefore, R is an S-FP-hereditary ring.

(5)⇒(2) This follows from Proposition 11. □

Proposition 21. *Let $R=R_1\times R_2$ be the product of rings R_1 and R_2 and S_i a multiplicative subset of R_i for each $i=1,2$, and set $S:=S_1\times S_2$. Then, R is an S-FP-hereditary ring if and only if R_i is an S_i-FP-hereditary ring for each $i=1,2$.*

Proof. This follows from Proposition 14 and Proposition 20. □

The following nontrivial example shows that FP-hereditary rings are not S-FP-hereditary in general.

Example 2. Let $R=R_1\times R_2$, where R_1 is an FP-hereditary ring and R_2 is an FP-hereditary ring that is not hereditary (see ([9], Example 3.2(3)) for a concrete example for R_2). Then, R is certainly FP-hereditary. We set $S:=\{1\}\times\{0,1\}$. Then, R is not S-FP-hereditary by Proposition 1.9 since $\{0,1\}$-FP-hereditary rings are exactly hereditary rings.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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