Hindawi
Journal of Mathematics
Volume 2023, Article ID 7178550, 5 pages
https://doi.org/10.1155/2023/7178550

Research Article

Some Properties of Unbounded M-Weakly and Unbounded L-Weakly Compact Operators

Zahra Niktab,[1] Kazem Haghnejad Azar ,[2] Razi Alavizadeh,[2] and Saba Sadeghi Gavgani[1]

[1]*Department of Mathematics, Sarab Branch, Islamic Azad University, Sarab, Iran*
[2]*Department of Mathematics and Applications, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran*

Correspondence should be addressed to Kazem Haghnejad Azar; haghnejad@uma.ac.ir

Received 19 July 2022; Revised 29 December 2022; Accepted 7 March 2023; Published 1 April 2023

Academic Editor: Ali Jaballah

We introduce the class of unbounded M-weakly compact operators and the class of unbounded L-weakly compact operators. We investigate some properties for this new classification of operators, and we study the relation between them and M-weakly compact and L-weakly compact operators. We also present an operator characterization of Banach lattices with an ordered continuous norm.

1. Introduction

Unbounded norm convergence was introduced by Troitsky in [1] and further considered in [2, 3]. Let E be a vector lattice and $x \in E$. A net $(x_\alpha)_{\alpha \in A} \subseteq E$ is said to be unbounded norm convergent (un-convergent, for short) to x if moreover E is a Banach lattice and $\| |x_\alpha - x| \wedge u \| \longrightarrow 0$ for all $u \in E^+$. We denote this convergence by $x_\alpha \xrightarrow{un} x$ and write that $(x_\alpha)_\alpha$ is un-convergent to x. A continuous operator $T\colon E \longrightarrow X$ from a Banach lattice E to a Banach space X is said to be M-weakly compact if $\|Tx_n\| \longrightarrow 0$ holds for every norm bounded disjoint sequence $\{x_n\}$ of E. A continuous operator $T\colon X \longrightarrow E$ from a Banach space X to a Banach lattice E is said to be L-weakly compact if $\|y_n\| \longrightarrow 0$ holds for every sequence $\{y_n\}$ of solid hull of $T(U)$, where U is the closed unit ball of the Banach space X. The classes of L-weakly and M-weakly compact operators were introduced by Meyer-Nieberg in [4]. He proved some interesting properties for these classifications of operators. For example, he proved that L-weakly and M-weakly compact operators are weakly compact (Theorem 5.61 in [3]). The properties of these classifications of operators have been investigated and extended to some general cases by some authors; see [5–8]. In this paper, we introduce an unbounded version for these classifications of operators as unbounded L-weakly and M-weakly (in short u-L- and u-M-weakly) compact operators. We show that these new versions are different from L-weakly and M-weakly compact operators and we prove some of their properties. We show that u-L- and u-M-weakly compact operators satisfy the domination problem, and we study their modulus properties, that is, T is u-M-weakly compact if and only if $|T|$ is u-M-weakly compact and if T or $|T|$ is u-L-weakly compact, then both of them are u-L- and u-M-weakly compact operators.

In this paper, by $E^\sim$ and $E^{\sim\sim}$ we will denote the order dual and order bidual of the vector lattice E, respectively. And by E' and E'' we will denote the topological dual and topological bidual of normed space E, respectively. The solid hull of a subset A of vector lattice E is the smallest solid set, including A. It is easy to see that

$$\mathrm{Sol}(A) = \{x \in E\colon \exists y \in A \text{ with } |x| \le |y|\}. \tag{1}$$

A net (x_α) in E is said to be unbounded order convergent (uo-convergent, for short) to x if for every $u \in E^+$ the net $(|x_\alpha - x| \wedge u)_\alpha$ converges to zero in order. An operator $T\colon E \longrightarrow F$ between two vector lattices is said to be a lattice (or Riesz) homomorphism whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in E$. An operator $T\colon E \longrightarrow F$ between two vector lattices is called disjointness-preserving if $Tx \perp Ty$ for all $x, y \in E$ satisfying $x \perp y$. By Meyer's theorem [see [11], Theorem 3.1.4], we know that, if an order bounded operator

$T: E \longrightarrow F$ between two Archimedean vector lattices preserves disjointness, then its modulus exists, and

$$|T|(|x|) = |T(|x|)| = |Tx|, \tag{2}$$

holds for all $x \in E$. Moreover, $|T|$ is a lattice homomorphism. We refer the reader to [9, 10] for any unexplained terms from Banach lattice theory and to read more about *uo*-convergence and *un*-convergence refer to [3, 11], respectively.

2. Main Results

In this section, we introduce two new concepts as unbounded M-weakly compact and unbounded L-weakly compact operators and we investigate some of their properties. We establish their relationships with M-weakly compact and L-weakly compact operators and we study lattice properties of these new concepts.

Definition 1. A continuous operator $T: E \longrightarrow F$ between two Banach lattices E and F is said to be unbounded M-weakly compact (or u-M-weakly compact for short) if $Tx_n \xrightarrow{un} 0$ holds for every norm bounded disjoint sequence $\{x_n\}$ of E.

Definition 2. A continuous operator $T: X \longrightarrow E$ from a Banach space X to a Banach lattice E is said to be unbounded L-weakly compact (or u-L-weakly compact for short) if $y_n \xrightarrow{un} 0$ holds for every disjoint sequence $\{y_n\}$ of solid hull of $T(U)$, where U is the closed unit ball of the Banach space X.

We know that every disjoint sequence in a Banach space with order continuous norm is *un*-null(Proposition 3.5 in [9]).

Example 1

(1) Since c_0 and ℓ_1 have order continuous norm, so every disjoint sequence in c_0 and ℓ_1 is un-null. Therefore, identity operators of ℓ_1 and c_0 are obvious examples of u-L-weakly and u-M-weakly compact operators.

(2) Let $T: \ell_1 \longrightarrow c_0$ be the inclusion operator. Let $\{a_n\}$ be a norm bounded disjoint sequence in ℓ_1. Clearly, for each $u \in c_0^+$, $\{a_n \wedge u\}$ is a disjoint sequence in c_0, so it follows from order continuity of c_0 that $a_n \wedge u \xrightarrow{\|\cdot\|} 0$. Therefore, T is u-M-weakly compact. On the other hand, every norm bounded disjoint sequence in solid hull of $T(U)$ is *un*-convergent to zero. Hence, T is u-L-weakly compact.

We do not use net in abovementioned definitions since we have the following propositions.

Proposition 1. *A continuous operator $T: E \longrightarrow F$ between two Banach lattices E and F is u-M-weakly compact iff $Tx_\alpha \xrightarrow{un} 0$ holds for every norm bounded disjoint net $\{x_\alpha\}$ of E.*

Proposition 2. *A continuous operator $T: X \longrightarrow E$ from a Banach space X to a Banach lattice E is u-L-weakly compact iff $y_\alpha \xrightarrow{un} 0$ holds for every disjoint net $\{y_\alpha\}$ of solid hull of $T(U)$, where U is the closed unit ball of the Banach space X.*

In the rest of this paper, we denote by

$L(X, Y)$: the class of all continuous operators between two normed vector spaces X and Y.

$MW(E, X)$: the class of all M-weakly compact operators from a Banach lattice E to a Banach space X.

$LW(X, E)$: the class of all L-weakly compact operators from a Banach space X to a Banach lattice E.

$MW_u(E, F)$: the class of all u-M-weakly compact operators between two Banach lattices E and F.

$LW_u(X, E)$: the class of all u-L-weakly compact operators from a Banach space X to a Banach lattice E.

For Banach lattices E and F and a Banach space X, we have the following inclusions:

$$\mathrm{LW}(X, E) \subset \mathrm{LW}_u(X, E), \mathrm{MW}(E, F) \subset \mathrm{MW}_u(E, F). \tag{3}$$

In the next remark, we give a condition that the reverse inclusions hold. But, in general the abovementioned inclusions are proper, as shown in the following example.

Example 2. Let $T: \ell_1 \longrightarrow c_0$ be the inclusion operator. In Example 1, we show that T is a u-M- and u-L-weakly compact operator. The sequence $\{e_n\}$ of the standard unit vectors is a norm bounded disjoint sequence of ℓ^1. We have $\|T(e_n)\| = 1$ for each n, therefore T is not M-weakly compact. On the other hand, we have $\{e_n\} \subset T(U)$, where U is the closed unit ball of ℓ_1, since $\|e_n\| = 1$ for each n, therefore T is not L-weakly compact.

Remark 1. If F is a Banach lattice with strong unit, then it follows from Theorem 2.3 of [3] that *un*-topology agrees with norm topology on F. That is, for a sequence $\{x_n\} \subset F$, we have $x_n \xrightarrow{un} 0$ iff $x_n a_n \wedge u \xrightarrow{\|\cdot\|} 0$. So

(1) For each Banach lattice E, we have $\mathrm{MW}_u(E, F) = \mathrm{MW}(E, F)$

(2) For each Banach space X, we have $\mathrm{LW}_u(X, F) = \mathrm{LW}(X, F)$

The following example shows that compact operator need not to be u-L- or u-M-weakly compact.

Example 3. Let $T: \ell^1 \longrightarrow \ell^\infty$ be an operator defined as follows:

$$T(a_n) = \left(\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} a_n, \cdots \right). \tag{4}$$

Clearly, T is of finite rank and so is a compact operator. Now, let $\{e_n\}$ be the standard basis of ℓ^1. We see that $\|Te_n \wedge (1, 1, 1, \ldots)\| = 1$ for all n, so T is not u-M-weakly compact.

On the other hand, we can easily see

$$\mathrm{Sol}(T(U)) = \{x \in \ell^{\infty}: |x| \le (1,1,1,\ldots)\}, \tag{5}$$

where U is the closed unit ball of ℓ^1. Therefore, $\{e_n\} \subset \mathrm{Sol}(T(U))$. For each $n \in \mathbb{N}$ we have $\|e_n \wedge (1,1,1,\ldots)\| = 1$, hence T is not u-L-weakly compact.

The notions of M- and L-weakly compact operators are in duality to each other (Theorem 5.64 in [3]). By the following example, we show that u-M- and u-L-weakly compact operators do not have the same duality properties.

Example 4. By $I_{\ell^1}, I_{\ell^\infty}$, and $I_{(\ell^\infty)'}$ we denote the identity operators of ℓ^1, ℓ^∞, and $(\ell^\infty)'$, respectively. We know that $I_{\ell^1}' = I_{\ell^\infty}$ and $I_{\ell^\infty}' = I_{(\ell^\infty)'}$. Since ℓ^1 and $(\ell^\infty)'$ are AL-spaces, so they have order continuous norm. Therefore, it follows from Theorem 6 that I_{ℓ^1} and $I_{(\ell^\infty)'}$ are both u-M- and u-L-weakly compact operators. On the other hand, I_{ℓ^∞} is neither u-M- nor u-L-weakly compact.

The u-L- and u-M-weakly compact operators have a lattice approximation properties like L- and M-weakly compact operators. To prove it, we need a slightly modified version of Theorem 4.36 of [9] that can be proved using the argument of the same theorem. Recall that a map $f: V \longrightarrow W$ from a vector space V to an ordered vector space W is called subadditive if for each $x, y \in V$ we have $f(x+y) \le f(x) + f(y)$ and $f(0) = 0$.

Lemma 1. *Let $T: E \longrightarrow X$ be a continuous operator from a Banach lattice E to a Banach space X, let A be a norm bounded solid subset of E, and let $f: X \longrightarrow \mathbb{R}$ be a norm continuous subadditive function. If $f(Tx_n) \longrightarrow 0$ holds for each disjoint sequence $\{x_n\}$ in A, then for each $\epsilon > 0$ there exists some $u \in E^+$ lying in the ideal generated by A such that*

$$f\left(T(|x| - u)^+\right) < \epsilon, \tag{6}$$

holds for all $x \in A$.

Theorem 1. *Let $T: E \longrightarrow F$ be a continuous operator between two Banach lattices E and F, let A be a norm bounded solid subset of E. If $Tx_n \xrightarrow{un} 0$ holds for each disjoint sequence $\{x_n\}$ in A, then for each $w \in F^+$ and each $\epsilon > 0$ there exists some $u \in E^+$ lying in the ideal generated by A such that*

$$\left\| \left| T(|x| - u)^+ \right| \wedge w \right\| < \epsilon, \tag{7}$$

holds for all $x \in A$.

Proof. Let $w \in F^+$ and $\epsilon > 0$ be arbitrary but fixed. Define map $f_w: F \longrightarrow \mathbb{R}$ as

$$f_w(x) = \| |x| \wedge w \|, \tag{8}$$

for each $x \in F$. It is obvious that f_w is subadditive and norm continuous.

Let $\{x_n\}$ be a disjoint sequence in A. It follows from assumption and $f_w(Tx_n) = \| |Tx_n| \wedge w \|$ that $f_w(Tx_n) \longrightarrow 0$. Therefore, by Lemma 1, there exists some $u \in E^+$ lying in the ideal generated by A such that

$$f_w\left(T(|x| - u)^+\right) < \epsilon, \tag{9}$$

for all $x \in A$. That is,

$$\left\| \left| T(|x| - u)^+ \right| \wedge w \right\| < \epsilon, \tag{10}$$

for all $x \in A$. Thus, the proof is complete. □

Corollary 1. *For two Banach lattices E and F, and a Banach space X the following statements hold:*

(1) If $T: E \longrightarrow F$ is a u-M-weakly compact operator, then for each $w \in F^+$ and for each $\epsilon > 0$ there exists some $u \in E^+$ such that

$$\left\| \left| T(|x| - u)^+ \right| \wedge w \right\| < \epsilon. \tag{11}$$

holds for all $x \in E$ with $\|x\| \le 1$.

(2) If $T: X \longrightarrow E$ is a u-L-weakly compact operator, then for each $w \in E^+$ and for each $\epsilon > 0$ there exists some $u \in E^+$ lying in the ideal generated by $T(X)$ satisfying

$$\left\| \left[(|Tx| - u)^+ \right] \wedge w \right\| < \epsilon, \tag{12}$$

for all $x \in X$ with $\|x\| \le 1$.

In the following theorem, we show that the class of u-M- and u-L-weakly compact operators are closed subspaces of the vector space of continuous operators.

Theorem 2. *For Banach lattice E and F, and a Banach space X the following hold. $\mathrm{LW}_u(X,E)$ and $\mathrm{MW}_u(E,F)$ are closed vector subspaces of $L(X,E)$ and $L(E,F)$, respectively.*

Proof. To see that the sum of two u-L-weakly compact operators is u-L-weakly compact, let $S, T: X \longrightarrow E$ be two u-L-weakly compact operators. Let $\{y_n\} \subset \mathrm{Sol}((T+S)(U))$ be a disjoint sequence, where U is the closed unit ball of X. For each n choose some $u_n \in U$ with $|y_n| \le |Tu_n + Su_n| \le |Tu_n| + |Su_n|$. It follows from (Theorem 1.13 in [3]) that for each n there exist $a_n, b_n \in E^+$ with $a_n \le |Tu_n|$ and $b_n \le |Su_n|$ such that $|y_n| = a_n + b_n$. Clearly, $\{a_n\}$ and $\{b_n\}$ are disjoint sequences in $\mathrm{Sol}(T(U))$ and $\mathrm{Sol}(S(U))$, respectively. So by assumption we have $a_n \xrightarrow{un} 0$ and $b_n \xrightarrow{un} 0$. Thus, $|y_n| = a_n + b_n \xrightarrow{un} 0$ (Lemma 2.1 in [5]). Therefore, $T + S \in \mathrm{LW}_u(X,E)$.

To see that $LW_u(X,E)$ is a closed vector subspace of $L(X,E)$, let $T \in L(X,E)$ be in the closure of the set of all u-L-weakly compact operators of $L(X,E)$. Assume that $\{y_n\} \subset \mathrm{Sol}(T(U))$ be a disjoint sequence. To this end, let $\epsilon > 0$ and $w \in E^+$ be fixed. Pick an u-L-weakly compact operator $S: X \longrightarrow E$ with $\|T - S\| < \epsilon$. For each n choose some $u_n \in U$ such that $|y_n| \le |Tu_n| \le |(T-S)u_n| + |Su_n|$. It follows from [see [3], Theorem 1.13] that for each n there exist $a_n, b_n \in E^+$ with $a_n \le |(T-S)u_n|$ and $b_n \le |Su_n|$ such that $|y_n| = a_n + b_n$. Clearly, $\{b_n\}$ is a disjoint sequence in $\mathrm{Sol}(S(U))$. So, by assumption, we have $\|b_n \wedge w\| \longrightarrow 0$. Now, it follows from the inequalities

$$\begin{aligned}\||y_n|\wedge w\| &\le \|a_n\wedge w\| + \|b_n\wedge w\| \\ &\le \|T-S\| + \|b_n\wedge w\| \\ &\le \epsilon + \|b_n\wedge w\|,\end{aligned} \quad (13)$$

that $\limsup \||y_n|\wedge w\| \le \epsilon$. Since $\epsilon > 0$ is arbitrary, we see that $\||y_n|\wedge w\| \longrightarrow 0$. Therefore, $T \in LW_u(X,E)$.

Now, we prove that $MW_u(E,F)fa$ is a norm closed vector subspace of $L(E,F)$. It is obvious that the set of all u-M-weakly compact operators between E and F is a vector subspace of $L(E,F)$. Let T be in the closure of the set of all u-M-weakly compact operators, and let $\{x_n\}$ be a disjoint sequence of E satisfying $\|x_n\| \le 1$ for all n. We have to show that $Tx_n \xrightarrow{un} 0$. Let $w \in F^+$ and $\epsilon > 0$ be arbitrary but fixed from now on. There exists some u-M-weakly compact operator $S: E \longrightarrow F$ such that $\|T-S\| < \epsilon$. We have $|Tx_n| \le |(T-S)x_n| + |Sx_n|$. Therefore,

$$\begin{aligned}|Tx_n|\wedge w &\le \left(|(T-S)x_n| + |Sx_n|\right)\wedge w \\ &\le |(T-S)x_n| + |Sx_n|\wedge w.\end{aligned} \quad (14)$$

So, it follows that $\||Tx_n|\wedge w\| \le \|(T-S)x_n\| + \||Sx_n|\wedge w\| < \epsilon + \||Sx_n|\wedge w\|$. Since $\epsilon > 0$ is arbitrary and S is u-M-weakly compact we see that $\||Tx_n|\wedge w\| \longrightarrow 0$ holds. Now, the proof follows from the fact that $w \in F^+$ is arbitrary. □

Theorem 3. *Let $T: E \longrightarrow F$ be a positive operator between two Banach lattices. Assume that E has order continuous norm and let G be a superorder dense sublattice of E. If $Tx_n \xrightarrow{un} 0$ holds for every norm bounded disjoint sequence $\{x_n\}$ of G, then $T \in MW_u(E,F)$.*

Proof. Let $\{x_n\}$ be a norm bounded disjoint sequence in E. It follows that $\{x_n^-\}$ and $\{x_n^+\}$ are norm bounded disjoint sequences in E. Therefore, we may assume without loss of generality that $x_n > 0$ for all $n \in \mathbb{N}$. Since G is superorder dense in E, for each $n \in \mathbb{N}$, there is a sequence $\{x_{n,k}\}$ such that $0 < x_{n,k} \uparrow x_n$. It follows that $x_n - x_{n,k} \downarrow 0$, and so by assumption we have $\|x_n - x_{n,k}\| \longrightarrow 0$. By continuity of T, we have $\|T(x_n - x_{n,k})\| \longrightarrow 0$, and so $\|T(x_n - x_{n,k})\wedge w\| \longrightarrow 0$ whenever $w \in F^+$. Let $w \in F^+$, $\epsilon > 0$ are fixed. For each $n \in \mathbb{N}$, there is some k_n such that $\|T(x_n - x_{n,k_n})\wedge w\| < \epsilon/2$. On the other hand, the sequence $\{x_{n,k_n}\}_{n=1}^{+\infty}$ is a norm bounded disjoint sequence in G, by assumption we have $\|T(x_{n,k_n})\wedge w\| \longrightarrow 0$, and so there is some $N \in \mathbb{N}$ such that $\|T(x_{n,k_n})\wedge w\| < \epsilon/2$ for all $n \ge N$. Thus, we have

$$\|Tx_n \wedge w\| \le \left\|T\left(x_n - x_{n,k_n}\right)\wedge w\right\| + \left\|T\left(x_{n,k_n}\right)\wedge w\right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad (15)$$

for all $n \ge N$, and so the proof follows.

The following theorem shows that u-L- and u-M-weakly compact operators satisfy the domination problem. □

Theorem 4. *u - L- and u-M-weakly compact operators satisfy the domination problem. That is, for Banach lattice E and F, and a Banach space X, if $T \in MW_u(E,F)$ (resp. $T \in LW_u(X,E)$) and $S \in L(E,F)$ (resp. $S \in L(X,E)$) such that $0 \le S \le T$, then $S \in MW_u(E,F)$ (resp. $S \in LW_u(X,E)$).*

Proof. At first, we prove that u-M-weakly compact operators satisfy the domination problem. Let $\{x_n\}$ be a norm bounded disjoint sequence in E. Since T is u-M-weakly compact we have $\{T(x_n)\}$ is *un*-convergent to zero. For each $u \in F^+$, we have

$$|Sx_n|\wedge u \le S|x_n|\wedge u \le T|x_n|\wedge u. \quad (16)$$

Since $\{|x_n|\}$ is a norm bounded disjoint sequence, we have $T|x_n|\wedge u \xrightarrow{\|\cdot\|} 0$. Therefore, $Sx_n \xrightarrow{un} 0$ and so S is u-M-weakly compact.

Now, we show that u-L-weakly compact operators satisfy the domination problem. We claim that $\mathrm{Sol}(S(U)) \subset \mathrm{Sol}(T(U))$ where U is the closed unit ball of X. Let $y \in \mathrm{Sol}(S(U))$. Thus, for some $u \in U$ we have $|y| \le |Su|$. On the other hand, $|Su| \le S|u| \le T|u|$. Since $\||u|\| = \|u\| \le 1$, we have $|u| \in U$. Hence, it follows from $|y| \le T|u|$ that $y \in \mathrm{Sol}(T(U))$. Now, let $\{y_n\}$ be a disjoint sequence in $\mathrm{Sol}(S(U))$ by the above argument we conclude that $\{y_n\} \subset \mathrm{Sol}(T(U))$. As T is u-L-weakly compact, therefore $y_n \xrightarrow{un} 0$ and the proof is complete. □

Proposition 3. *The following assertions hold:*

(1) Let T be a lattice homomorphism from Banach lattice E with order continuous norm into Banach lattice F, and let Y denotes the range of T. Each of the following conditions implies that T is u-M-weakly compact.

(a) Y is majorizing in F
(b) Y is norm dense in F
(c) Y is a projection band in F

(2) If $T: X \longrightarrow E$ from a Banach space X to a Banach lattice E is continuous and E has order continuous norm then T is u-L-weakly compact.

Proof

(1) Let $\{x_n\}$ be a norm bounded disjoint sequence in E. Since E has order continuous norm, $x_n \xrightarrow{un} 0$. As T is lattice homomorphism, then it is easy to check that $Tx_n \xrightarrow{un} 0$ in Y, and hence $Tx_n \xrightarrow{un} 0$ in F by Theorem 4.3 in [9]

(2) The proof is clear by Proposition 3.5 of [3]

Recall that if $T: E \longrightarrow F$ is an order bounded disjointness-preserving operator between two Banach lattices then $|T|$ exists and is a lattice homomorphism. Therefore, by using Theorem 2.14 in [3] and Theorem 3.1.4 in [11], we have $||T|x| = |Tx|$ for all $x \in E$. In particular, for each $u \in F^+$ we have $\|T|x|\wedge u = |Tx|\wedge u$. □

Theorem 5. *Let $T: E \longrightarrow F$ be an order bounded disjointness-preserving operator between two Banach lattices. Then, the following assertions are hold:*

(1) T is u-M-weakly compact if and only if $|T|$ is u-M-weakly compact

(2) If T or $|T|$ is u-L-weakly compact then both of them are u-L- and u-M-weakly compact

Proof

(1) Let $\{x_n\}$ be a norm bounded disjoint sequence in E. For each $u \in F^+$ and for each $n \in \mathbb{N}$, we have

$$\left\| |T| x_n | \wedge u \right\| = \left\| |T x_n| \wedge u \right\|. \tag{17}$$

In other words, $Tx_n \xrightarrow{un} 0$ if and only if $|T|(x_n) \xrightarrow{un} 0$. Thus, the proof is complete.

(2) By using Proposition 2.7 of [12] $\mathrm{Sol}(T(U)) = \mathrm{Sol}(|T|(U))$, where U is the closed unit ball of E. Therefore, T is u-L-weakly compact if and only if $|T|$ is u-L-weakly compact. Now, without loss of generality we assume that $|T|$ is u-L-weakly compact. Let $\{x_n\}$ be a disjoint sequence in U. Since $|T|$ is a lattice homomorphism, for $n \neq m$ we have

$$\| T |x_n| \wedge \| T |x_m| = |T||x_n| \wedge |T||x_m| = |T|\left(|x_n| \wedge |x_m|\right) = 0. \tag{18}$$

Therefore, $\{|T|x_n\}$ is a disjoint sequence in $\mathrm{Sol}(T(U))$. Since $|T|$ is u-L-weakly compact, $\{|T|x_n\}$ is un-convergent to zero. Thus, $|T|$ is a u-M-weakly compact operator. It follows from previous part that T is also u-M-weakly compact.

In the following theorem, we present a characterization of Banach lattices with order continuous norm. □

Theorem 6. *Let E be a Banach lattice. The following statements are equivalent.*

(1) E has order continuous norm

(2) $I: E \longrightarrow E$ is u-L-weakly compact

(3) $I: E \longrightarrow E$ is u-M-weakly compact

As a result of Theorem 1 and Corollary 1, we have the following lattice approximation property.

Corollary 2. *Let E be a Banach lattice with order continuous norm. Then, for each $w \in E^+$ and for each $\epsilon > 0$, there exists some $u \in E^+$ such that*

$$\left\| (|x| - u)^+ \wedge w \right\| < \epsilon \tag{19}$$

holds for all $x \in E$ with $\|x\| \leq 1$.

3. Conclusion

The classification of M-weakly compact and L-weakly compact operators have been extended to unbounded case as u-M-weakly and u-L-weakly compact operators, respectively. The classification of u-M-weakly and u-L-weakly compact operators are similar to M-weakly compact and L-weakly compact operators, respectively, in some properties, but in the some others they can be different. That is, a compact operator need not to be u-L- or u-M-weakly compact, and on the other hand, u-M- and u-L-weakly compact operators does not have the duality properties. The u-L- and u-M-weakly compact operators have a lattice approximation properties such as L- and M-weakly compact operators, and they satisfy the domination problem. An operator T is u-M-weakly compact if and only if its modulus is u-M-weakly compact and if T or its modulus is u-L-weakly compact then both of them are u-L- and u-M-weakly compact operators.

Data Availability

No data were used to support this study.

Disclosure

A preprint version of this article is available on arXiv (https://arxiv.org/abs/2109.07184).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] V. G. Troitsky, "Measures of non-compactness of operators on Banach lattices," *Positivity*, vol. 8, no. 2, pp. 165–178, 2004.

[2] Y. Deng, M. O'Brien, and V. G. Troitsky, "Unbounded norm convergence in Banach lattices," *Positivity*, vol. 21, no. 3, pp. 963–974, 2017.

[3] M. Kandić, M. A. A. Marabeh, and V. G. Troitsky, "Unbounded norm topology in Banach lattices," *Journal of Mathematical Analysis and Applications*, vol. 451, no. 1, pp. 259–279, 2017.

[4] P. Meyer-Nieberg, "Über klassen schwach kompakter operatoren in banachverbänden," *Mathematische Zeitschrift*, vol. 138, no. 2, pp. 145–159, 1974.

[5] F. Afkir, K. Bouras, A. Elbour, and S. El Filali, "Weak compactness of almost L-weakly and almost M-weakly compact operators," *Quaestiones Mathematicae*, vol. 44, no. 9, pp. 1145–1154, 2021.

[6] B. Akay and Ö. Gök, "On the lattice properties of almost L-weakly and almost M-weakly compact operators," *Journal of Function Spaces*, vol. 2021, Article ID 1755373, 5 pages, 2021.

[7] K. Bouras, D. Lhaimer, and M. Moussa, "On the class of almost L-weakly and almost M-weakly compact operators," *Positivity*, vol. 22, no. 5, pp. 1433–1443, 2018.

[8] A. Elbour, F. Afkir, and M. Sabiri, "Some properties of almost L-weakly and almost M-weakly compact operators," *Positivity*, vol. 24, no. 1, pp. 141–149, 2020.

[9] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Vol. 119, Springer Science & Business Media, Berlin, Germany, 2006.

[10] P. Meyer-Nieberg, *Banach Lattices*, Springer Science & Business Media, Berlin, Germany, 2012.

[11] N. Gao, V. G. Troitsky, and F. Xanthos, "Uo-convergence and its applications to Cesàro means in Banach lattices," *Israel Journal of Mathematics*, vol. 220, no. 2, pp. 649–689, 2017.

[12] K. Haghnezhad Azar and R. Alavizadeh, "On the modulus of disjointness-preserving operators and b-AM-compact operators on Banach lattices," *Annals of Functional Analysis*, vol. 9, no. 1, pp. 101–110, 2018.