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Research Article

Hosoya, Schultz, and Gutman Polynomials of Generalized Petersen Graphs $P(n,1)$ and $P(n,2)$

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The graph theory has wide important applications in various other types of sciences. In chemical graph theory, we have many topological polynomials for a graph G through which we can compute many topological indices. Topological indices are numerical values and descriptors which are used to quantify the physiochemical properties and bioactivities of the chemical graph. In this paper, we compute Hosoya polynomial, hyper-Wiener index, Tratch–Stankevitch–Zefirov index, Harary index, Schultz polynomial, Gutman polynomial, Schultz index, and Gutman index of generalized Petersen graphs $P(n,1)$ and $P(n,2)$.

1. Introduction

A graph G is an ordered pair $G(V;E)$ consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ such that each edge connects a pair of vertices. In the mathematical field of graph theory, the distance between two vertices u,v in a graph is the number of edges in a shortest path connecting them and denoted by $d(u,v)$, and the diameter of a graph is the length of the shortest path between the most distanced vertices denoted by $D(G)$, which is defined as $D(G) = \mathrm{Max}_{u\in V}\{d(u,v)\colon v \in V\}$.

The chemical compounds can be represented by using the mathematical tools of graph theory. Atoms are represented by vertices and bonds by edges. In this case, we obtain a molecular graph [1, 2].

A topological index is a numerical value which describes and explains important information about the chemical structure. These topological descriptors provide information of a chemical compound based on the arrangement of its atoms and their bonds. There are hundreds of topological indices that have found some applications in theoretical chemistry; especially in QSPR/QSAR research, some of them are based on distance and the others are based on degrees. Topological information of a molecule comprises the position and sometimes the type of the atoms defined in relation to the bonds that connect them. Such topological descriptors correlate with certain compound properties and activities [3–5].

In chemistry, discovery of the drugs commonly relies on the topological descriptors. If we compute topological indices of drug molecular structures, medical and pharmaceutical researchers will be able to understand their therapeutic properties which can compensate for the shortcomings of medicine and chemical experiments [6].

The Wiener index of a graph is defined as $W(G) = 1/2\sum_{u\in V}\sum_{v\in V} d(u,v)$, and it was put forward in 1947 by the American chemist Wiener [7]. At first, Wiener index was used for predicting the boiling of paraffin.

The hyper-Wiener index was introduced in 1993 by Randić [8] and defined as $WW(G) = 1/2(W(G) + 1/2\sum_{u\in V}\sum_{v\in V} d^2(u,v))$ where $d^2(u,v) = (d(u,v))^2$ and has been extensively studied with computational techniques through edge and vertex contributions [9–11]. Klein's original definition of the hyper-Wiener index was applicable just to trees, and therefore the hyper-Wiener index was later defined for any connected graph G [12].

The Harary index of a graph G was introduced by Plavšić et al. in 1993 [13] and by Ivanciuc et al. [14] independently and defined as $\mathrm{Ha}(G) = 1/2\sum_{u\in V}\sum_{v\in V} 1/d(u,v)$, and this index is also a useful topological index in chemical graph theory and has received much attention during the past decades.

Tratch et al. introduced Tratch–Stankevitch–Zefirov (TSZ) index as expanded Wiener index in 1990 [15].

The Hosoya polynomial of graph G was introduced by Hosoya in 1988 [16] as a counting polynomial and defined as $H(G,x) = 1/2\sum_{u\in V}\sum_{v\in V} x^{d(u,v)}$. At first, it was named Wiener polynomial, but today, it is called the Hosoya polynomial. The main advantage of the Hosoya polynomial is that it contains a wealth of information about distance-based graph invariants. Hosoya polynomial actually counts the number of distances of path of different lengths in a molecular graph, and through Hosoya polynomial, we can find many indices that depend on the distance.

The relationship between Hosoya polynomial, hyper-Wiener index, Tratch–Stankevitch–Zefirov index, and Harary index is given as follows [17]:

$$W(G) = \frac{\partial H(G,x)}{\partial x}\Big|_{x=1}, \mathrm{WW}(G) = \frac{1}{2}\frac{\partial^2 xH(G,x)}{\partial x^2}\Big|_{x=1},$$
$$\mathrm{Ha}(G) = \int_0^1 \frac{H(G,x)}{x}\mathrm{d}x, \mathrm{TSZ}(G) = \frac{1}{3!}\frac{\partial^3 x^2 H(G,x)}{\partial x^3}\Big|_{x=1}. \tag{1}$$

The Schultz index is a topological index that depends on distances between vertices in the graph, and it is widely studied and has a close relationship with that of Wiener index. Schultz index was introduced by Schultz in 1989 [18] and defined as follows: $Sc(G) = 1/2\sum_{u\in V}\sum_{v\in V}(d_u + d_v)\,d(u,v)$, and its original purpose was to give a technique for determining a molecular topological index to describe the structure of alkanes. The Schultz index has been shown to be a useful molecular descriptor in the design of molecules with desired properties [19].

Klavžar and Gutman in 1997 [20] defined the modified Schultz index, which is known as the Gutman index as a kind of a vertex-valency weighted sum of the distance between all pair of vertices in a graph. Gutman index is defined as follows: $\mathrm{Gut}(G) = 1/2\sum_{u\in V}\sum_{v\in V}(d_u \times d_v)d(u,v)$.

In chemical graph theory, there are two important polynomials for these structure descriptors which were introduced by Gutman in 2005 which are Schultz polynomial and Gutman polynomial of G.

The Schultz polynomial is defined as $\mathrm{Sc}(G,x) = 1/2\sum_{u\in V}\sum_{v\in V}(d_u + d_v)\,x^{d(u,v)}$.

The Gutman polynomial is defined as $\mathrm{Gut}(G,x) = 1/2\sum_{u\in V}\sum_{v\in V}(d_u \times d_v)\,x^{d(u,v)}$.

Schultz index is related to Schultz polynomial, and Gutman index is related to Gutman polynomial according to the following relationship: $\mathrm{Gut}(G) = \partial/\partial\ x\mathrm{Gut}(G,x)|_{x=1}$, $Sc(G) = \partial/\partial x\mathrm{Sc}(G,x)|_{x=1}$ [21].

In [22], we see Wiener index of generalized Petersen graph $P(n,k)$ for $k = 1,2,3$.

Schultz index was studied in [23–25]. Chen and Liu [26] studied the maximal and minimal Gutman index of unicyclic graphs. The Gutman index has been studied in [27].

Wiener index and Harary index have found many applications in chemistry and there are lots of papers dealing with these two indices [28, 29].

The Hosoya polynomial has been investigated on trees [30, 31], composite graphs [32, 33], and benzenoid graphs [34].

The Petersen graph is named after Julius Petersen, who in 1898 constructed it to be the smallest bridgeless cubic graph with no three-edge coloring. In 1950, Coxeter [35] introduced a family of graphs generalizing the Petersen graph.

Let n and k be positive integers, for $n \geq 3$ and $1 \leq k \leq n-1$. The generalized Petersen graph is a graph on $2n$ vertices with $V(P(n,k)) = \{v_i, u_i;\ 0 \leq i \leq n-1\}$ and edges $E(P(n,k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k};\ 0 \leq i \leq n-1\}$ where all subscripts are taken as modulo n [36].

2. Discussion and Results

In this paper, we compute Hosoya polynomial of generalized Petersen graph $P(n,k)$ for $k = 1,2$, and through Hosoya polynomial of generalized Petersen graph $P(n,k)$ for $k = 1,2$, we compute hyper-Wiener index, Tratch–Stankevitch–Zefirov index, and Harary index. Also, we compute Schultz polynomial of generalized Petersen graph $P(n,k)$ for $k = 1,2$ and through it compute Schultz index. Also, we compute Gutman polynomial of generalized Petersen graph $P(n,k)$ for $k = 1,2$ and through it compute Gutman index.

In this research, we will rely on tables, so that each table contains two columns in the first column, we will find the possible distance between different pairs of vertices of generalized Petersen graph and in the second column will show the number of pairs.

Theorem 1. *For $n = 2m$, where m is a positive integer and $n \geq 6$, we have*

$$
\begin{aligned}
H(P(n,1),x) &= 3nx^{1} + 4n\sum_{i=2}^{(n-2/2)} x^{i} + 3nx^{(n-1/2)} + nx^{(n+2/2)},\\
\mathrm{WW}(P(n,1)) &= \frac{n^{4}+6n^{3}+14n^{2}}{12},\\
\mathrm{TSZ}(P(n,1)) &= \frac{n^{5}+12n^{4}+56n^{3}+120n^{2}}{96},\\
\mathrm{Ha}(P(n,1)) &= 3n + \frac{2n}{n+2} + 4n\sum_{i=2}^{(n-2/2)} \frac{1}{i} + 6,\\
\mathrm{Sc}(P(n,1),x) &= 18nx^{1} + 24n\sum_{i=1}^{(n-2/2)} x^{i} + 18nx^{(n-2/2)} + 6nx^{(n+2/2)},\\
\mathrm{Gut}(P(n,1),x) &= 27nx^{1} + 36n\sum_{i=1}^{(n-2/2)} x^{i} + 27nx^{(n+2/2)} + 9nx^{(n+2/2)},\\
\mathrm{Sc}(P(n,1)) &= 3n^{2}(n+2),\\
\mathrm{Gut}(P(n,1)) &= 9n^{2}\left(\frac{1}{2}n+1\right).
\end{aligned}
\tag{2}
$$

Proof. As we see from Figure 1, the distance sequence is computed for generalized Petersen graph $P(n,1)$ for $n = 2m$ and presented in Table 1.

From Table 1, we have

$$
\begin{aligned}
H(P(n,1),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V} x^{d(u,v)}\\
&= 3nx^{1} + 4n\sum_{i=2}^{n-2/2} x^{i} + 3nx^{n/2} + nx^{(n+2/2)},\\
\mathrm{WW}(P(n,1)) &= \frac{1}{2}\frac{\partial^{2}xH(P(n,1),x)}{\partial x^{2}}\bigg|_{x=1}\\
&= \frac{1}{2}\frac{\partial^{2}}{\partial x^{2}}\left(3nx^{2} + 4n\sum_{i=2}^{n-2/2} x^{i+1} + 3nx^{(n+2/2)} + nx^{(n+4/2)}\right)\bigg|_{x=1}\\
&= \frac{n^{4}+6n^{3}+14n^{2}}{12},\\
\mathrm{TSZ}(P(n,1)) &= \frac{1}{3!}\frac{\partial^{3}x^{2}H(P(n,1),x)}{\partial x^{3}}\bigg|_{x=1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{6}\frac{\partial^3}{\partial x^3}\left(3nx^3 + 4n\sum_{i=2}^{n-2/2} x^{i+2} + 3nx^{(n+4/2)} + nx^{(n+6/2)}\right)\Bigg|_{x=1} \\
&= \frac{n^5 + 12n^4 + 56n^3 + 120n^2}{96}, \\
\mathrm{Ha}(P(n,1)) &= \int_0^1 \frac{H(P(n,1),x)}{x}\mathrm{d}x \\
&= \int_0^1 \left(3n + 4n\sum_{i=2}^{\frac{n-2}{2}} x^{i-1} + 3nx^{(n-2/2)} + nx^{n/2}\right)\mathrm{d}x \\
&= 3n + \frac{2n}{n+2} + 4n\sum_{i=2}^{n-2/2}\frac{1}{i} + 6, \\
\mathrm{Sc}(P(n,1),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}\left(d_u + d_v\right)x^{d(u,v)} \\
&= 6(3n)x^1 + 6(4n)\sum_{i=1}^{n-2/2} x^i + 6(3n)x^{n/2} + 6nx^{(n+2/2)} \\
&= 18nx^1 + 24n\sum_{i=1}^{n-2/2} x^i + 18nx^{n/2} + 6nx^{(n+2/2)}, \\
\mathrm{Gut}(P(n,1),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}\left(d_u \times d_v\right)x^{d(u,v)} \\
&= 9(3n)x^1 + 9(4n)\sum_{i=1}^{n-2/2} x^i + 9(3n)x^{n/2} + 9nx^{(n+2/2)} \\
&= 27nx^1 + 36n\sum_{i=1}^{n-1/2} x^i + 27nx^{n/2} + 9nx^{(n+2/2)}, \\
\mathrm{Sc}(P(n,1)) &= \frac{\partial}{\partial x}\mathrm{Sc}(P(n,1),x)|_{x=1} \\
&= 18n + 24n\left(2 + 3 + 4 + \ldots + \frac{n-2}{2}\right) + 18n\left(\frac{n}{2}\right) + 6n\left(\frac{n+2}{2}\right) \\
&= 3n^2(n+2), \\
\mathrm{Gut}(P(n,1)) &= \frac{\partial}{\partial x}\mathrm{Gut}(P(n,1),x)|_{x=1} \\
&= 27n + 36n\left(2 + 3 + 4 + \ldots + \frac{n-2}{2}\right) + 27n\left(\frac{n}{2}\right) + 9n\left(\frac{n+2}{2}\right) \\
&= 9n^2\left(\frac{1}{2}n + 1\right).
\end{aligned}
\tag{3}
$$

□

Theorem 2. *For $n = 2m + 1$ and $n \geq 5$, we have*

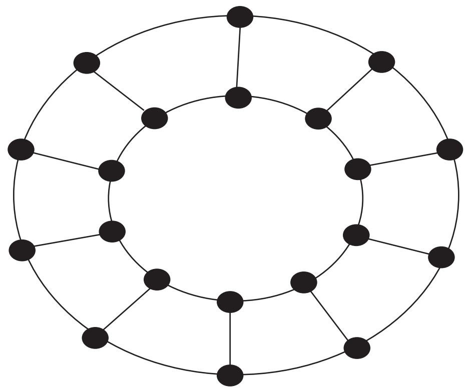

Figure 1: Generalized Petersen graph $P(10,1)$.

Table 1: The distance sequence for generalized Petersen graph $P(n,1)$ for $n=2m$.

$d(u,v)$	The number of pairs
1	$3n$
2	$4n$
3	$4n$
.	.
.	.
.	.
$n-2/2$	$4n$
$n/2$	$3n$
$n+2/2$	n

$$\begin{aligned}
H(P(n,1),x) &= 3nx^{1} + 4n\sum_{i=2}^{(n-1/2)} x^{i} + 2nx^{(n+1/2)},\\
\mathrm{WW}(P(n,1)) &= \frac{n^{4}+6n^{3}+11n^{2}-6n}{12},\\
\mathrm{TSZ}(P(n,1)) &= \frac{n^{5}+12n^{4}+50n^{3}+84n^{2}-51n}{96},\\
\mathrm{Ha}(P(n,1)) &= 3n + \frac{4n}{n+2} + 4n\sum_{i=2}^{(n-1/2)} \frac{1}{i},\\
\mathrm{Sc}(P(n,1),x) &= 18nx^{1} + 24n\sum_{i=1}^{(n-1/2)} x^{i} + 12nx^{(n+1/2)},\\
\mathrm{Gut}(P(n,1),x) &= 27nx^{1} + 36n\sum_{i=1}^{(n-1/2)} x^{i} + 18nx^{(n+1/2)},\\
\mathrm{Sc}(P(n,1)) &= 3n\left(n^{2}+2n-1\right),\\
\mathrm{Gut}(P(n,1)) &= 9n\left(\frac{1}{2}n^{2}+n-\frac{1}{2}\right).
\end{aligned} \tag{4}$$

Proof. As we see from Figure 2, the distance sequence is computed for generalized Petersen graph $P(n,1)$ for $n=2m+1$ and presented in Table 2.

From Table 2, we have

$$\begin{aligned}
H(P(n,1),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V} x^{d(u,v)}\\
&= 3nx^{1} + 4n\sum_{i=2}^{(n-1/2)} x^{i} + 2nx^{(n+1/2)},\\
\mathrm{WW}(P(n,1)) &= \frac{1}{2}\left.\frac{\partial^{2} xH(P(n,1),x)}{\partial x^{2}}\right|_{x=1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(3nx^2 + 4n\sum_{i=2}^{(n-1/2)} x^{i+1} + 2nx^{(n+3/2)}\right)\Bigg|_{x=1} \\
&= \frac{n^4 + 6n^3 + 11n^2 - 6n}{12}, \\
\mathrm{TSZ}(P(n,1)) &= \frac{1}{3!}\frac{\partial^3 x^2 H(P(n,1),x)}{\partial x^3}\Bigg|_{x=1} \\
&= \frac{1}{6}\frac{\partial^3}{\partial x^3}\left(3nx^3 + 4n\sum_{i=2}^{(n-1/2)} x^{i+2} + 2nx^{(n+5/2)}\right)\Bigg|_{x=1} \\
&= \frac{n^5 + 12n^4 + 50n^3 + 84n^2 - 51n}{96}, \\
\mathrm{Ha}(P(n,1)) &= \int_0^1 \frac{H(P(n,1),x)}{x}\mathrm{d}x \\
&= \int_0^1 \left(3n + 4n\sum_{i=2}^{(n-1/2)} x^{i-1} + 2nx^{(n-1/2)}\right)\mathrm{d}x \\
&= 3n + \frac{4n}{n+2} + 4n\sum_{i=2}^{(n-1/2)} \frac{1}{i}, \\
\mathrm{Sc}(P(n,1),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}\left(d_u + d_v\right)x^{d(u,v)} \\
&= 6(3n)x^1 + 6(4n)\sum_{i=1}^{(n-1/2)} x^i + 6(2n)x^{(n+1/2)} \\
&= 18nx^1 + 24n\sum_{i=1}^{(n-1/2)} x^i + 12nx^{(n+1/2)}, \\
\mathrm{Gut}(P(n,1),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}\left(d_u \times d_v\right)x^{d(u,v)} \\
&= 9(3n)x^1 + 9(4n)\sum_{i=1}^{(n-1/2)} x^i + 9(2n)x^{(n+1/2)} \\
&= 27nx^1 + 36n\sum_{i=1}^{(n-1/2)} x^i + 18nx^{(n+1/2)}, \\
\mathrm{Sc}(P(n,1)) &= \frac{\partial}{\partial x}\mathrm{Sc}(P(n,1),x)|_{x=1} \\
&= 18n + 24n\left(2 + 3 + 4 + \ldots + \frac{n-1}{2}\right) + 12n\left(\frac{n+1}{2}\right) \\
&= 3n\left(n^2 + 2n - 1\right), \\
\mathrm{Gut}(P(n,1)) &= \frac{\partial}{\partial x}\mathrm{Gut}(P(n,1),x)|_{x=1} \\
&= 27n + 36n\left(2 + 3 + 4 + \ldots + \frac{n-1}{2}\right) + 18n\left(\frac{n+1}{2}\right) \\
&= 9n\left(\frac{1}{2}n^2 + n - \frac{1}{2}\right).
\end{aligned}
\tag{5}
$$

□

Theorem 3. *For $n = 4m$ and $n \geq 24$, we have*

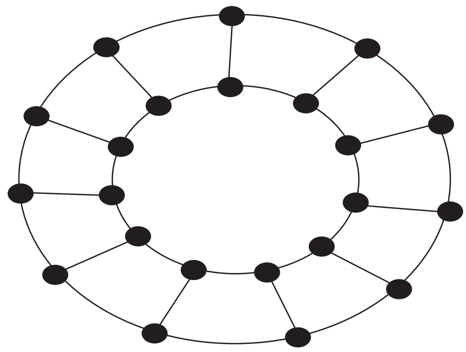

Figure 2: Generalized Petersen graph $P(11, 1)$.

Table 2: The distance sequence for generalized Petersen graph $P(n, 1)$ for $n = 2m + 1$.

$d(u, v)$	The number of pairs
1	$3n$
2	$4n$
3	$4n$
.	.
.	.
.	.
$n - 1/2$	$4n$
$n + 1/2$	$2n$

$$\begin{aligned}
H(P(n,2),x) &= 3nx^1 + 6nx^2 + 7nx^3 + 7nx^4 + 8n\sum_{i=5}^{n-4/4} x^i \\
&\quad + \frac{15}{2}nx^{n/4} + 6nx^{n+4/4} + \frac{5}{2}nx^{n+8/4}, \\
\mathrm{WW}(P(n,2)) &= \frac{n^4 + 24n^3 + 212n^2 - 648n}{48}, \\
\mathrm{TSZ}(P(n,2)) &= \frac{n^5 + 40n^4 + 632n^3 + 4544n^2 - 20736n}{768}, \\
\mathrm{Ha}(P(n,2)) &= \frac{121}{12}n + \frac{24n}{n+4} + \frac{10n}{n+8} + 30 + 8n\sum_{i=5}^{n-4/4}\frac{1}{i}, \\
\mathrm{Sc}(P(n,2),x) &= 18nx^1 + 36nx^2 + 42nx^3 + 42nx^4 + 48n\sum_{i=5}^{n-4/4} x^i \\
&\quad + 45nx^{n/4} + 36nx^{n+4/4} + 15nx^{n+8/4}, \\
\mathrm{Gut}(P(n,2),x) &= 27nx^1 + 54nx^2 + 63nx^3 + 63nx^4 + 72n\sum_{i=5}^{n-4/4} x^i \\
&\quad + \frac{135}{2}nx^{n/4} + 54nx^{n+1/4} + \frac{45}{2}nx^{n+8/4}, \\
\mathrm{Sc}(P(n,2)) &= n\left(\frac{3}{2}n^2 + \frac{72}{4}n - 30\right), \\
\mathrm{Gut}(P(n,2)) &= n\left(\frac{9}{4}n^2 + 27n - 45\right).
\end{aligned} \tag{6}$$

Proof. As we see from Figure 3, the distance sequence is computed for generalized Petersen graph $P(n, 2)$ for $n = 4m$ and presented in Table 3.

From Table 3, we have

$$
\begin{aligned}
H(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V} x^{d(u,v)}\\
&= 3nx^{1} + 6nx^{2} + 7nx^{3} + 7nx^{4} + 8n\sum_{i=5}^{(n-4/4)} x^{i}\\
&\quad + \frac{15}{2}nx^{(n/4)} + 6nx^{(n+4/4)} + \frac{5}{2}nx^{(n+8/4)},\\
\mathrm{WW}(P(n,2)) &= \frac{1}{2}\frac{\partial^{2} xH(P(n,2),x)}{\partial x^{2}}\bigg|_{x=1}\\
&= \frac{1}{2}\frac{\partial^{2}}{\partial x^{2}}\left(3nx^{2} + 6nx^{3} + 7nx^{4} + 7nx^{5} + 8n\sum_{(i=6)}^{(n/4)} x^{i} + \frac{15}{2}nx^{((n+4)/4)}\right.\\
&\quad \left. + 6nx^{((n+8)/4)} + \frac{5}{2}nx^{((n+12)/4)}\right)\bigg|_{x=1}\\
&= \frac{n^{4} + 24n^{3} + 212n^{2} - 648n}{48},\\
\mathrm{TSZ}(P(n,2)) &= \frac{1}{3!}\left(\frac{\partial^{3} x^{2}H(P(n,2),x)}{\partial x^{3}}\right)\bigg|_{x=1}\\
&= \frac{1}{6}\frac{\partial^{3}}{\partial x^{3}}\left(3nx^{3} + 6nx^{4} + 7nx^{5} + 7nx^{6} + 8n\sum_{(i=7)}^{((n+4)/4)} x^{i} + \frac{15}{2}nx^{((n+8)/4)}\right.\\
&\quad \left. + 6nx^{((n+12)/4)} + \frac{5}{2}nx^{((n+16)/4)}\right)\bigg|_{x=1}\\
&= \frac{n^{5} + 40n^{4} + 632n^{3} + 4544n^{2} - 20736n}{768},\\
\mathrm{Ha}(P(n,2)) &= \int_{0}^{1}\frac{H(P(n,2),x)}{x}\mathrm{d}x\\
&= \int_{0}^{1}\left(3n + 6nx^{1} + 7nx^{2} + 7nx^{3} + 8n\sum_{i=4}^{(n-8/4)} x^{i} + \frac{15}{2}nx^{(n-4/4)}\right.\\
&\quad \left. + 6nx^{(n/4)} + \frac{5}{2}nx^{(n+4/4)}\right)\mathrm{d}x\\
&= \frac{121}{12}n + \frac{24n}{n+4} + \frac{10n}{n+8} + 30 + 8n\sum_{i=5}^{(n-4/4)}\frac{1}{i},\\
Sc(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}\left(d_{u} + d_{v}\right)x^{d(u,v)}\\
&= 3(6)nx^{1} + 6(6)nx^{2} + 7(6)nx^{3} + 7(6)nx^{4} + 8(6)n\sum_{i=5}^{n-4/4} x^{i}\\
&\quad + \frac{15}{2}(6)nx^{(n/4)} + 6(6)nx^{(n+4/4)} + \frac{5}{2}(6)nx^{(n+8/4)}\\
&= 18nx^{1} + 36nx^{2} + 42nx^{3} + 42nx^{4} + 48n\sum_{i=5}^{(n-4/4)} x^{i}\\
&\quad + 45nx^{(n/4)} + 36nx^{(n+4/4)} + 15nx^{(n+8/4)},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Gut}(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}(d_u\times d_v)\,x^{d(u,v)}\\
&= 3(9)nx^1 + 6(9)nx^2 + 7(9)nx^3 + 7(9)nx^4 + 8(9)n\sum_{i=5}^{(n/4)-1}x^i\\
&\quad + \frac{15}{2}(9)nx^{(n/4)} + 6(9)nx^{(n+4/4)} + \frac{5}{2}(9)nx^{(n+8/4)}.\\
&= 27nx^1 + 54nx^2 + 63nx^3 + 63nx^4 + 72n\sum_{i=5}^{(n-4/4)}x^i\\
&\quad + \frac{135}{2}nx^{(n/4)} + 54nx^{(n+4/4)} + \frac{45}{2}nx^{(n+8/4)},\\
\mathrm{Sc}(P(n,2)) &= \frac{\partial}{\partial x}\mathrm{Sc}(P(n,2),x)|_{x=1}\\
&= 18n + 36(2)n + 42(3)n + 42(4)n + 48n\left(5+6+7+\ldots+\frac{n-4}{4}\right)\\
&\quad + 45n\left(\frac{n}{4}\right) + 36n\left(\frac{n+4}{4}\right) + 15n\left(\frac{n+8}{4}\right)\\
&= n\left(\frac{3}{2}n^2 + \frac{72}{4}n - 30\right),\\
\mathrm{Gut}(P(n,2)) &= \frac{\partial}{\partial x}\mathrm{Gut}(P(n,2),x)|_{x=1}\\
&= 27n + 54(2)n + 63(3)n + 63(4)n + 72n\left(5+6+7+\ldots+\frac{n-4}{4}\right)\\
&\quad + \frac{135}{2}n\left(\frac{n}{4}\right) + 54n\left(\frac{n+4}{4}\right) + \frac{45}{2}n\left(\frac{n+8}{4}\right)\\
&= n\left(\frac{9}{2}n^2 + 27n - 45\right).
\end{aligned}
\tag{7}
$$

□

Theorem 4. *For $n = 4m + 1$ and $n \geq 17$, we have*

$$
\begin{aligned}
H(P(n,2),x) &= 3nx^1 + 6nx^2 + 7nx^3 + 7nx^4 + 8n\sum_{i=5}^{(n+3/4)}x^i + 2nx^{(n+7/4)},\\
\mathrm{WW}(P(n,2)) &= \frac{n^4 + 24n^3 + 185n^2 - 1044n}{48},\\
\mathrm{TSZ}(P(n,2)) &= \frac{n^5 + 40n^4 + 578n^3 + 3584n^2 - 24939n}{768},\\
\mathrm{Ha}(P(n,2)) &= \frac{121}{12}n + \frac{8n}{n+7} + 8n\sum_{i=5}^{(n+3/4)}\frac{1}{i},\\
\mathrm{Sc}(P(n,2),x) &= 18nx^1 + 36nx^2 + 42nx^3 + 42nx^4 + 48n\sum_{i=5}^{(n+3/4)}x^i + 12nx^{(n+7/4)},\\
\mathrm{Gut}(P(n,2),x) &= 27nx^1 + 54nx^2 + 63nx^3 + 63nx^4 + 72n\sum_{i=5}^{(n+3/4)}x^i + 18nx^{(n+7/4)},\\
\mathrm{Sc}(P(n,2)) &= n\left(\frac{3}{2}n^2 + 18n - \frac{87}{2}\right),\\
\mathrm{Gut}(P(n,2)) &= n\left(\frac{9}{4}n^2 + 27n - \frac{261}{4}\right).
\end{aligned}
\tag{8}
$$

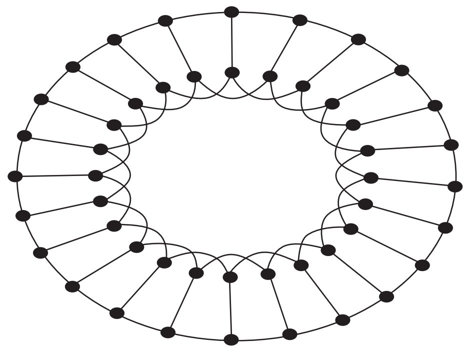

Figure 3: Generalized Petersen graph $P(24, 2)$.

Proof. As we see from Figure 4, the distance sequence is computed for generalized Petersen graph $P(n, 2)$ for $n = 4m + 1$ and presented in Table 4.

From Table 4, we have

Table 3: The distance sequence for generalized Petersen graph $P(n, 2)$ for $n = 4m$.

$d(u, v)$	The number of pairs
1	$3n$
2	$6n$
3	$7n$
4	$7n$
5	$8n$
6	$8n$
7	$8m$
.	.
.	.
.	.
$n-4/4$	$8n$
$n/4$	$15/2n$
$n+4/4$	$6n$
$n+8/4$	$5/2n$

$$
\begin{aligned}
H(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V} x^{d(u,v)} \\
&= 3nx^{1} + 6nx^{2} + 7nx^{3} + 7nx^{4} + 8n\sum_{i=5}^{(n+3/4)} x^{i} + 2nx^{(n+7/4)}, \\
\mathrm{WW}(P(n,2)) &= \frac{1}{2}\frac{\partial^{2} xH(P(n,2),x)}{\partial x^{2}}\bigg|_{x=1} \\
&= \frac{1}{2}\frac{\partial^{2}}{\partial x^{2}}\left(3nx^{2} + 6nx^{3} + 7nx^{4} + 7nx^{5} + 8n\sum_{i=6}^{(n+7/4)} x^{i} + 2nx^{(n+11/4)}\right)\bigg|_{x=1} \\
&= \frac{n^{4} + 24n^{3} + 185n^{2} - 1044n}{48}, \\
\mathrm{TSZ}(P(n,2)) &= \frac{1}{3!}\frac{\partial^{3} x^{2}H(P(n,2),x)}{\partial x^{3}}\bigg|_{x=1} \\
&= \frac{1}{6}\frac{\partial^{3}}{\partial x^{3}}\left(3nx^{3} + 6nx^{4} + 7nx^{5} + 7nx^{6} + 8n\sum_{i=7}^{(n+11/4)} x^{i} + 2nx^{(n+15/4)}\right)\bigg|_{x=1} \\
&= \frac{n^{5} + 40n^{4} + 578n^{3} + 3584n^{2} - 24939n}{768}, \\
\mathrm{Ha}(P(n,2)) &= \int_{0}^{1}\frac{H(P(n,2),x)}{x}\,\mathrm{d}x \\
&= \int_{0}^{1}\left(3n + 6nx^{1} + 7nx^{2} + 7nx^{3} + 8n\sum_{i=5}^{n-1/4} x^{i} + 2nx^{n+3/4}\right)\mathrm{d}x \\
&= \frac{121}{12} + \frac{8n}{n+7} + 8n\sum_{i=5}^{n+3/4}\frac{1}{i}, \\
\mathrm{Sc}(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}\left(d_u + d_v\right)x^{d(u,v)} \\
&= 3(6)nx^{1} + 6(6)nx^{2} + 7(6)nx^{3} + 7(6)nx^{4} + 8(6)n\sum_{i=5}^{n+3/4} x^{i} \\
&\quad + 2n(6)nx^{n+7/4}
\end{aligned}
$$

$$
\begin{aligned}
&= 18nx^{1} + 36nx^{2} + 42nx^{3} + 42nx^{4} + 48n\sum_{i=5}^{n+3/4} x^{i} + 12nx^{n+7/4},\\
\mathrm{Gut}(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}(d_u \times d_v)\, x^{d(u,v)}\\
&= 3(9)nx^{1} + 6(9)nx^{2} + 7(9)nx^{3} + 7(9)nx^{4} + 8(9)n\sum_{i=5}^{n+3/4} x^{i}\\
&\quad + 2(9)nx^{n+7/4}\\
&= 27nx^{1} + 54nx^{2} + 63nx^{3} + 63nx^{4} + 72n\sum_{i=5}^{n+3/4} x^{i} + 18nx^{(n+7/4)},\\
\mathrm{Sc}(P(n,2)) &= \frac{\partial}{\partial x}\mathrm{Sc}(P(n,2),x)|_{x=1}\\
&= 18n + 36(2)n + 42(3)n + 42(4)n + 48n\left(5+6+7+\ldots+\frac{n+3}{4}\right)\\
&\quad + 12n\left(\frac{n+7}{4}\right)\\
&= n\left(\frac{3}{2}n^{2} + 18n - \frac{87}{2}\right),\\
\mathrm{Gut}(P(n,2)) &= \frac{\partial}{\partial x}\mathrm{Gut}(P(n,2),x)|_{x=1}\\
&= 27n + 54(2)n + 63(3)n + 63(4)n + 72n\left(5+6+7+\ldots+\frac{n+3}{4}\right)\\
&\quad + 18n\left(\frac{n+7}{4}\right)\\
&= n\left(\frac{9}{2}n^{2} + 27n - \frac{261}{4}\right).
\end{aligned}
\tag{9}
$$

□

Theorem 5. *For $n = 4m + 2$ and $n \geq 22$, we have*

$$
\begin{aligned}
H(P(n,2),x) &= 3nx^{1} + 6nx^{2} + 7nx^{3} + 7nx^{4} + 8n\sum_{i=5}^{(n-2/4)} x^{i}\\
&\quad + 7nx^{(n+2/)} + 4nx^{(n+6/4)} + nx^{(n+10/4)},\\
\mathrm{WW}(P(n,2)) &= \frac{n^{4} + 24n^{3} + 212n^{2} - 624n}{48},\\
\mathrm{TSZ}(P(n,2)) &= \frac{n^{5} + 40n^{4} + 632n^{3} + 4640n^{2} - 19824n}{768},\\
\mathrm{Ha}(P(n,2)) &= \frac{121}{12}n + \frac{14n}{n+2} + \frac{16n}{n+6} + \frac{4n}{n+10} + 8n\sum_{i=5}^{(n-2/4)}\frac{1}{i},\\
\mathrm{Sc}(P(n,2),x) &= 18nx^{1} + 36nx^{2} + 42nx^{3} + 42nx^{4} + 48n\sum_{i=5}^{(n-2/4)} x^{i}\\
&\quad + 42nx^{(n+2/4)} + 24nx^{(n+6/4)} + 15nx^{(n+10/4)},\\
\mathrm{Gut}(P(n,2),x) &= 27nx^{1} + 54nx^{2} + 63nx^{3} + 63nx^{4} + 72n\sum_{i=5}^{(n-2/4)} x^{i}\\
&\quad + 63nx^{(n+2/4)} + 36nx^{(n+6/4)} + 9nx^{(n+10/4)},
\end{aligned}
$$

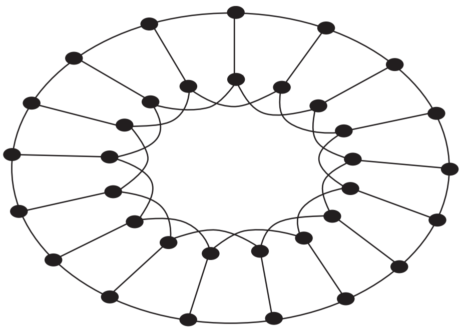

Figure 4: Generalized Petersen graph $P(17, 2)$.

Table 4: The distance sequence for generalized Petersen graph $P(n, 2)$ for $n = 4m + 1$.

$d(u, v)$	The number of pairs
1	$3n$
2	$6n$
3	$7n$
4	$7n$
5	$8n$
6	$8n$
7	$8m$
.	.
.	.
.	.
$n - 1/4$	$8n$
$n + 3/4$	$8n$
$n + 7/4$	$2n$

$$\begin{aligned}
\mathrm{Sc}(P(n,2)) &= n\left(\frac{3}{2}n^2 + 18n - 30\right),\\
\mathrm{Gut}(P(n,2)) &= n\left(\frac{9}{4}n^2 + 27n - 45\right).
\end{aligned} \tag{10}$$

Proof. As we see from Figure 5, the distance sequence is computed for generalized Petersen graph $P(n, 2)$ for $n = 4m + 2$ and presented in Table 5.

From Table 5, we have

$$\begin{aligned}
H(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V} x^{d(u,v)}\\
&= 3nx^1 + 6nx^2 + 7nx^3 + 7nx^4 + 8n\sum_{i=5}^{n-2/4} x^i\\
&\quad + 7nx^{(n+2/4)} + 4\mathrm{n}\mathrm{x}^{(n+6/4)} + \mathrm{n}\mathrm{x}^{(n+10/4)},\\
\mathrm{WW}(P(n,2)) &= \frac{1}{2}\frac{\partial^2 xH(P(n,2),x)}{\partial x^2}\bigg|_{x=1}\\
&= \frac{1}{2}\frac{\partial^2}{\partial x^2}\Bigg(3nx^2 + 6nx^3 + 7nx^4 + 7nx^5 + 8n\sum_{(i=6)}^{((n+2)/4)} x^i\\
&\quad + 7nx^{((n+6)/4)} + 4nx^{((n+10)/4)} + nx^{((n+14)/4)}\Bigg)\bigg|_{x=1}\\
&= \frac{n^4 + 24n^3 + 212n^2 - 624n}{48},\\
TSZ(P(n,2)) &= \frac{1}{3!}\left(\frac{\partial^3 x^2 H(P(n,2),x)}{\partial x^3}\right)\bigg|_{x=1}\\
&= \frac{1}{6}\frac{\partial^3}{\left(\partial x^3\right)}\Bigg(3nx^3 + 6nx^4 + 7nx^5 + 7nx^6 + 8n\sum_{i=7}^{n+6/4} x^i\\
&\quad + 7nx^{(n+10/4)} + 4nx^{(n+14/4)} + nx^{(n+18/4)}\Bigg)\bigg|_{x=1}\\
&= \frac{n^5 + 40n^4 + 632n^3 + 4640n^2 - 19824n}{768},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Ha}(P(n,2)) &= \int_0^1 \frac{H(P(n,2),x)}{x}\mathrm{d}x \\
&= \int_0^1 \Bigg(3n + 6nx^1 + 7nx^2 + 7nx^3 + 8n \sum_{i=5}^{(n-6/4)} x^i \\
&\quad +7nx^{(n-2/4)} + 4nx^{(n+2/4)} + nx^{(n+6/4)} \Bigg)\mathrm{d}x \\
&= \frac{121}{12}n + \frac{14n}{n+2} + \frac{16n}{n+6} + \frac{4n}{n+10} + 8n \sum_{i=5}^{(n-2/4)} \frac{1}{i}, \\
\mathrm{Sc}(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}\left(d_u + d_v\right)x^{d(u,v)} \\
&= 3(6)nx^1 + 6(6)\mathrm{n}\mathrm{x}^2 + 7(6)\mathrm{n}\mathrm{x}^3 + 7(6)\mathrm{n}\mathrm{x}^4 + 8(6)n \sum_{i=5}^{(n-2/4)} x^i \\
&\quad + 7(6)\mathrm{n}\mathrm{x}^{(n+2/4)} + 4(6)\mathrm{n}\mathrm{x}^{(n+6/4)} + (6)\mathrm{n}\mathrm{x}^{(n+10/4)} \\
&= 18\mathrm{n}\mathrm{x}^1 + 36\mathrm{n}\mathrm{x}^2 + 42\mathrm{n}\mathrm{x}^3 + 42\mathrm{n}\mathrm{x}^4 + 48n \sum_{i=5}^{(n-2/4)} x^i \\
&\quad + 42\mathrm{n}\mathrm{x}^{(n+2/4)} + 24\mathrm{n}\mathrm{x}^{(n+6/4)} + 6\mathrm{n}\mathrm{x}^{(n+10/4)}, \\
\mathrm{Gut}(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}\left(d_u \times d_v\right)x^{d(u,v)} \\
&= 3(9)\mathrm{n}\mathrm{x}^1 + 6(9)\mathrm{n}\mathrm{x}^2 + 7(9)\mathrm{n}\mathrm{x}^3 + 7(9)\mathrm{n}\mathrm{x}^4 + 8(9)n \sum_{i=5}^{(n-2/4)} x^i \\
&\quad + 7(9)nx^{(n+2/4)} + 4(9)nx^{(n+6/4)} + (9)nx^{(n+10/4)} \\
&= 27nx^1 + 54nx^2 + 63nx^3 + 63nx^4 + 72n \sum_{i=5}^{(n-2/4)} x^i \\
&\quad + 63nx^{(n+2/4)} + 36nx^{(n+6/4)} + 9nx^{(n+10/4)}, \\
Sc(P(n,2)) &= \frac{\partial}{\partial x} Sc(P(n,2),x)|_{x=1} \\
&= 18n + 36(2)n + 42(3)n + 42(4)n + 48n\left(5+6+7+\ldots+\frac{n-2}{4}\right) \\
&\quad + 42n\left(\frac{n+2}{4}\right) + 24n\left(\frac{n+6}{4}\right) + 6n\left(\frac{n+10}{4}\right) \\
&= n\left(\frac{3}{2}n^2 + 18n - 30\right), \\
\mathrm{Gut}(P(n,2)) &= \frac{\partial}{\partial x}\mathrm{Gut}(P(n,2),x)|_{x=1} \\
&= 27n + 54(2)n + 63(3)n + 63(4)n + 72n\left(5+6+7+\ldots+\frac{n-2}{4}\right) \\
&\quad + 63n\left(\frac{n+2}{4}\right) + 36n\left(\frac{n+6}{4}\right) + 9n\left(\frac{n+10}{4}\right) \\
&= n\left(\frac{9}{4}n^2 + 27n - 45\right).
\end{aligned}
\tag{11}
$$

□

Theorem 6. *For $n = 4m + 3$ and $n \geq 19$, we have*

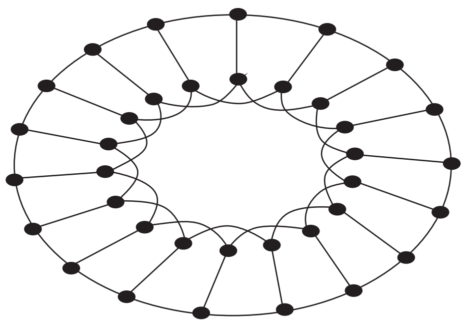

Figure 5: Generalized Petersen graph $P(18, 2)$.

Table 5: The distance sequence for generalized Petersen graph $P(n, 2)$ for $n = 4m + 2$.

$d(u, v)$	The number of pairs
1	$3n$
2	$6n$
3	$7n$
4	$7n$
5	$8n$
6	$8n$
7	$8m$
.	.
.	.
.	.
$n - 2/4$	$8n$
$n + 2/4$	$7n$
$n + 6/4$	$4n$
$n + 10/4$	n

$$\begin{aligned}
H(P(n,2),x) &= 3\mathrm{nx}^1 + 6\mathrm{nx}^2 + 7\mathrm{nx}^3 + 7\mathrm{nx}^4 + 8n \sum_{i=5}^{n+1/4} x^i \\
&\quad + 5\mathrm{nx}^{(n+5/4)} + nx^{(n+9/4)}, \\
\mathrm{WW}(P(n,2)) &= \frac{n^4 + 24n^3 + 197n^2 - 693n}{48}, \\
\mathrm{TSZ}(P(n,2)) &= \frac{n^5 + 40n^4 + 602n^3 + 4064n^2 - 22611n}{768}, \\
\mathrm{Ha}(P(n,2)) &= \frac{121}{12}n + \frac{20n}{n+5} + \frac{4n}{n+9} + 8n \sum_{i=5}^{n+1/4} \frac{1}{i}, \\
\mathrm{Sc}(P(n,2),x) &= 18\mathrm{nx}^1 + 36\mathrm{nx}^2 + 42\mathrm{nx}^3 + 42\mathrm{nx}^4 + 48n \sum_{i=5}^{n+1/4} x^i \\
&\quad + 30\mathrm{nx}^{(n+5/4)} + 6\mathrm{nx}^{(n+9/4)}, \\
\mathrm{Gut}(P(n,2),x) &= 27\mathrm{nx}^1 + 54\mathrm{nx}^2 + 63\mathrm{nx}^3 + 63\mathrm{nx}^4 + 72n \sum_{i=5}^{n+1/4} x^i \\
&\quad + 45\mathrm{nx}^{(n+5/4)} + 9\mathrm{nx}^{(n+9/4)}, \\
\mathrm{Sc}(P(n,2)) &= \frac{n}{2}\left(3n^2 + 9n - 75\right), \\
\mathrm{Gut}(P(n,2)) &= n\left(\frac{9}{4}n^2 + 27n - \frac{225}{4}\right).
\end{aligned} \tag{12}$$

Proof. As we see from Figure 6, the distance sequence is computed for generalized Petersen graph $P(n, 2)$ for $n = 4m + 3$ and presented in Table 6.

From Table 6, we have

$$H(P(n,2),x) = \frac{1}{2}\sum_{u\in V}\sum_{v\in V} x^{d(u,v)}$$
$$= 3\mathrm{nx}^1 + 6\mathrm{nx}^2 + 7\mathrm{nx}^3 + 7\mathrm{nx}^4 + 8n\sum_{i=5}^{n+1/4} x^i$$
$$+ 5nx^{(n+5/4)} + nx^{(n+9/4)},$$

$$\mathrm{WW}(P(n,2)) = \frac{1}{2}\frac{\partial^2 xH(P(n,2),x)}{\partial x^2}\Bigg|_{x=1}$$
$$= \frac{1}{2}\frac{\partial^2}{\partial x^2}\Bigg(3\mathrm{nx}^2 + 6\mathrm{nx}^3 + 7nx^4 + 7nx^5 + 8n\sum_{i=6}^{n+5/4} x^i$$
$$+ 5\mathrm{nx}^{(n+9/)} + \mathrm{nx}^{(n+13/)}\Bigg)\Bigg|_{x=1}$$
$$= \frac{n^4 + 24n^3 + 197n^2 - 693n}{48},$$

$$\mathrm{TSZ}(P(n,2)) = \frac{1}{3!}\frac{\partial^3 x^2 H(P(n,2),x)}{\partial x^3}\Bigg|_{x=1}$$
$$= \frac{1}{6}\frac{\partial^3}{\partial x^3}\Bigg(3\mathrm{nx}^3 + 6\mathrm{nx}^4 + 7\mathrm{nx}^5 + 7\mathrm{nx}^6 + 8n\sum_{i=7}^{n+9/4} x^i$$
$$+ 5\mathrm{nx}^{(n+13/4)} + \mathrm{nx}^{(n+17/4)}\Bigg)\Bigg|_{x=1}$$
$$= \frac{n^5 + 40n^4 + 602n^3 + 4064n^2 - 22611n}{768}$$

$$\mathrm{Ha}(P(n,2)) = \int_0^1 \frac{H(P(n,2),x)}{x}\mathrm{d}x$$
$$= \int_0^1 \Bigg(3n + 6\mathrm{nx}^1 + 7\mathrm{nx}^2 + 7\mathrm{nx}^3$$
$$+ 8n\sum_{i=5}^{\frac{n-3}{4}} x^i + 5\mathrm{nx}^{(n+1/4)} + nx^{(n+5/4)}\Bigg)\mathrm{d}x$$
$$= \frac{121}{12}n + \frac{20n}{n+5} + \frac{4n}{n+9} + 8n\sum_{i=5}^{n+1/4}\frac{1}{i},$$

$$\mathrm{Sc}(P(n,2),x) = \frac{1}{2}\sum_{u\in V}\sum_{v\in V}(d_u + d_v)x^{d(u,v)}$$
$$= 3(6)\mathrm{nx}^1 + 6(6)\mathrm{nx}^2 + 7(6)\mathrm{nx}^3 + 7(6)\mathrm{nx}^4 + 8(6)n\sum_{i=5}^{n+1/4} x^i$$
$$+ 5(6)\mathrm{nx}^{(n+5/4)} + (6)\mathrm{nx}^{(n+9/4)}$$
$$= 18\mathrm{nx}^1 + 36\mathrm{nx}^2 + 42\mathrm{nx}^3 + 42\mathrm{nx}^4 + 48n\sum_{i=5}^{n+1/4} x^i$$
$$+ 30\mathrm{nx}^{(n+5/4)} + 6\mathrm{nx}^{(n+9/4)},$$

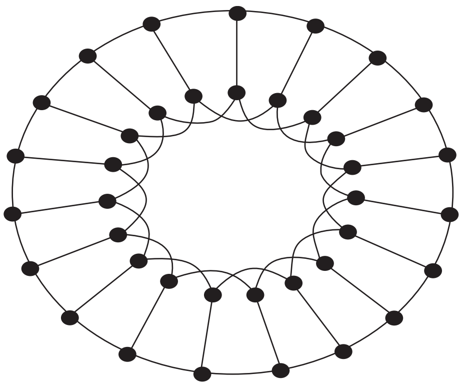

Figure 6: Generalized Petersen graph $P(19, 2)$.

Table 6: The distance sequence for generalized Petersen graph $P(n, 2)$ for $n = 4m + 3$.

$d(u, v)$	The number of pairs
1	$3n$
2	$6n$
3	$7n$
4	$7n$
5	$8n$
6	$8n$
7	$8m$
.	.
.	.
.	.
$n + 1/4$	$8n$
$n + 5/4$	$5n$
$n + 9/4$	n

$$\begin{aligned}
\mathrm{Gut}(P(n,2),x) &= \frac{1}{2}\sum_{u\in V}\sum_{v\in V}(d_u \times d_v)\, x^{d(u,v)} \\
&= 3(9)\mathrm{nx}^1 + 6(9)\mathrm{nx}^2 + 7(9)\mathrm{nx}^3 + 7(9)\mathrm{nx}^4 + 8(9)n\sum_{i=5}^{n+1/4} x^i \\
&\quad + 5(9)\mathrm{nx}^{(n+5/4)} + (9)\mathrm{nx}^{(n+9/4)} \\
&= 27\mathrm{nx}^1 + 54\mathrm{nx}^2 + 63\mathrm{nx}^3 + 63\mathrm{nx}^4 + 72n\sum_{i=5}^{n+1/4} x^i \\
&\quad + 45\mathrm{nx}^{(n+5/4)} + 9\mathrm{nx}^{(n+9/4)}, \\
\mathrm{Sc}(P(n,2)) &= \frac{\partial}{\partial x}\mathrm{Sc}(P(n,2),x)|_{x=1} \\
&= 18n + 36(2)n + 42(3)n + 42(4)n + 48n\left(5 + 6 + 7 + \ldots + \frac{n+1}{4}\right) \\
&\quad + 30n\left(\frac{n+5}{4}\right) + 6n\left(\frac{n+9}{4}\right) \\
&= \frac{n}{2}\left(3n^2 + 9n - 75\right), \\
\mathrm{Gut}(P(n,2)) &= \frac{\partial}{\partial x}\mathrm{Gut}(P(n,2),x)|_{x=1} \\
&= 27n + 54(2)n + 63(3)n + 63(4)n + 72n\left(5 + 6 + 7 + \ldots + \frac{n+1}{4}\right) \\
&\quad + 45n\left(\frac{n+5}{4}\right) + 9n\left(\frac{n+9}{4}\right) \\
&= n\left(\frac{9}{4}n^2 + 27n - \frac{225}{4}\right).
\end{aligned} \tag{13}$$

3. Conclusion

In this paper, we computed Hosoya polynomial, hyper-Wiener index, Tratch–Stankevitch–Zefirov index, Harary index, Schultz polynomial, Gutman polynomial, Schultz index, and Gutman index of generalized Petersen graph $P(n, k)$ for $k = 1, 2$. For $k = 1$, we discussed two cases of n, where $n = 2m$ and $n = 2m + 1$. For $k = 2$, we discussed four cases of n, where $n = 4m, 4m + 1, 4m + 2, 4m + 3$. It is possible to benefit from this study in computing some indices and topological polynomials for a generalized Petersen graph $P(n, k)$ for $k \geq 3$ depending on the same technique that we used in this paper. For each case of k, there are $2k$ discussion cases, and that depends on n mod $2k$, where n mod $2k = 0, 1, 2, \ldots, 2k - 1$. This study can be applied in chemistry because the computed indices that indicate chemical, physical, and biological properties of a chemical molecule structure have many important chemical applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally in the analysis and writing of the manuscript.

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