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Research Article

Solution of Nonlinear Fredholm Integral Equations on Almost $\mathcal{Z}_{\perp}$-Contraction

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In this manuscript, we develop an orthogonal to basically $\mathcal{Z}$-contraction and demonstrate various fixed point theorems of nonlinear Fredholm integral equation solutions in such a contraction. By using these ideas of discovering the fixed point theorems, we can also build the application of the Fredholm integral equation.

1. Introduction

The fixed point theorem is one of the fascinating subjects in nonlinear functional analysis because of how widely it may be used. Recently, Kojasteh et al. [1] proposed a simulation function that integrates certain known fixed point findings. Following that, Karapinar and Hima Bindu [2] explored fixed point findings on the almost $\mathcal{Z}$-contraction. Roldán-López-de-Hierro et al. [3] somewhat adjusted their definition of simulation function and explored the presence and uniqueness of coincidence points of two nonlinear operators using this type of control function. Alharbi et al. [4] studied the existence and uniqueness of certain operators that establish a novel contractive condition by integrating the ideas of admissible function and simulation function in the setting of full b-metric spaces. Alqahtani et al. demonstrated that a fixed point exists and is unique for particular operators in [5]. These operators were thoroughly examined using simulation functions in ∇-symmetric quasi-metric spaces. Aydi et al. [6] provided a number of fixed locations for α-acceptable triangular contraction mappings. Vetro [7] identified a common fixed point and a point of a coincidence for two self-mappings and showed that such points are zeros of a given function defined in both metric space and partial metric space. Vetro [8] proposed the idea of ordered S-G-contraction by integrating the current notions of (F, Φ)-contraction and Z-contraction. Radenovic et al. [9] directly established certain common fixed point solutions for two and three mappings under weak contractive circumstances, and some of these results are enhanced by utilizing various control function parameters. Samet et al. [10] developed a novel idea of (α, ψ)-contractive type mappings, established a fixed point theorem for (α, ψ)-contractive type mappings in full metric spaces, and improved several earlier findings in the literature (see, also [11]). Gordji et al. [12], in particular, brought the Banach contraction principle to the situation of an orthogonal set (briefly, O-set). Eshaghi Gordji and Habibi [13] proved the existence and uniqueness of the fixed point of the Cauchy problem for the first-order differential equation in the setting of orthogonal metric space, and this

notion has been expanded in multiple ways by other writers (see, for example, [14, 15]). In the context of w-distances, Dhivya et al. [16] established a fixed point solution using nonlinear Fredholm integral equations. Sevinik-Adiguzel et al. [17] addressed the existence and uniqueness of solutions to a fixed point theorem of a specific form of nonlinear Volterra integral-dynamic equation on time scales. Kumari Panda et al. [18] achieved some fixed point findings and proposed a very easy solution for a Volterra integral problem utilizing the fixed point approach in the situation of dislocated extended b-metric space. Sahin [19] introduced a p-cyclic contraction mapping by combining the ideas of cyclic contraction mapping and p-contraction mapping and investigated the sufficient conditions for the existence of a solution to nonlinear Fredholm integral equations.

In this paper, we establish the fixed point theorems for a solution of nonlinear Fredholm integral equations on orthogonal almost $\mathscr{Z}$-contraction. Various examples are presented to illustrate our obtained results.

2. Preliminaries

Throughout the paper, let $\mathscr{N}_0 = \mathscr{N} \cup \{0\}$, where $\mathscr{N}$ represents the set of positive integers. Normally, $\mathscr{R}$ denotes the set of all real numbers. In addition, we denote the set of nonnegative reals $\mathscr{R}_0^+$: $[0, \infty)$.

Khojasteh et al. [1] introduced a new control function namely a simulation function as follows:

Definition 1 (see [1]). A simulation function $\mathfrak{C}$ mapping from $\mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ satisfies the following conditions:

($\mathfrak{C}$1) $\mathfrak{C}(\zeta, \varkappa) < \varkappa - \zeta \forall \zeta, \varkappa > 0$

($\mathfrak{C}$2) For each sequences $\{\zeta_\varsigma\}, \{\varkappa_\varsigma\}$ in $(0, \infty)$ such that $\lim_{\varsigma \longrightarrow \infty} \zeta_\varsigma = \lim_{\varsigma \longrightarrow \infty} \varkappa_\varsigma > 0$, then

$$\lim_{\varsigma \longrightarrow \infty} \sup \mathfrak{C}(\zeta_\varsigma, \varkappa_\varsigma) < 0. \tag{1}$$

We observe that

$$(\mathfrak{C}_1) \Rightarrow \mathfrak{C}(\zeta, \zeta) < 0 \quad \forall \zeta > 0. \tag{2}$$

Simulation function examples are given as follows.

Example 1. Let $\mathscr{Z}$ denote the collection of all simulation functions $\mathfrak{C}$: $\mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}$. A function $\mathfrak{C}(\zeta, \varkappa) = \digamma\varkappa - \zeta$, $\forall \varkappa, \zeta \in \mathbb{R}_0^+$, and $\digamma \in [0, 1)$ is a simulation function.

Example 2. A continuous function Φ on a self-map $\mathbb{R}_0^+$ satisfies $\Phi(\zeta) = 0 \Leftrightarrow \zeta = 0$ and a mapping $\mathfrak{C}_{\mathbb{R}}$: $\mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ defined by

$$\mathfrak{C}_{\mathbb{R}}(\varkappa, \zeta) = \varkappa - \Phi(\varkappa) - \zeta \quad \forall \varkappa, \zeta \in \mathbb{R}_0^+. \tag{3}$$

Then, $\mathfrak{C}_{\mathbb{R}}$ is a simulation function.

$\mathscr{Z}$-contraction introduced by Khojasteh et al. [1] as follows.

Definition 2 (see [1]). A self-map β defined on a metric space (Λ, ∇) is said to be $\mathscr{Z}$-contraction with respect to (shortly, w.r.to) $\mathfrak{C} \in \mathscr{Z}$ if

$$\mathfrak{C}(\nabla(\beta\mathfrak{b}, \beta\vartheta), \nabla(\mathfrak{b}, \vartheta)) \geq 0 \quad \forall \mathfrak{b}, \vartheta \in \Lambda. \tag{4}$$

Here, recall the idea of an orthogonal set (or O-set), some examples, and its properties.

Definition 3 (see [12]). A binary relation $\perp$ defined on a nonvoid set $\Lambda \times \Lambda$ satisfies the conditions

$$\exists \mathfrak{b}_0 \in \Lambda\colon \ (\forall \mathfrak{b} \in \Lambda, \mathfrak{b} \perp \mathfrak{b}_0) \text{ or } (\forall \mathfrak{b} \in \Lambda, \mathfrak{b}_0 \perp \mathfrak{b}). \tag{5}$$

Then, the set Λ is called an orthogonal set (shortly O-set), and it is denoted by $(\Lambda, \perp)$.

An example of O-set is given as follows.

Example 3. Let Λ be the set of all ABO blood types were the most common factor considered in human paternity testing. A man who has type AB blood could not be a father of a child with type O blood, because he would pass on either the A or the B allele to all of his offspring. Define a binary relation $\perp$ on Λ defined by $\vartheta \perp \mathfrak{b}$ if ϑ inherits types of blood to $\mathfrak{b}$. If a child $\mathfrak{b}_0$ is a type of blood O, then we get $\vartheta \perp \mathfrak{b}_0 \quad \forall \vartheta \in \Lambda$. Hence, (Λ, ∇) is an O-set, and $\mathfrak{b}_0$ is not have a unique blood type. In this case, a child $\mathfrak{b}_0$ may be blood type B, and we have $\mathfrak{b}_0 \perp \vartheta \quad \forall \vartheta \in \Lambda$.

Definition 4 (see [12]). A sequence $\{\mathfrak{b}_\varsigma\}$ defined on a nonvoid O-set $(\Lambda, \perp)$ is said to be an orthogonal sequence (briefly, O-sequence) if

$$\left(\forall \varsigma \in \mathbb{N}, \mathfrak{b}_\varsigma \perp \mathfrak{b}_{\varsigma+1}\right) \text{ or } \left(\forall \varsigma \in \mathbb{N}, \mathfrak{b}_{\varsigma+1} \perp \mathfrak{b}_\varsigma\right). \tag{6}$$

Definition 5 (see [13]). A metric space $(\Lambda, \perp, \nabla)$ is said to be an orthogonal metric space (shortly, OMS) if $(\Lambda, \perp)$ is an O-set and (Λ, ∇) is a metric space (shortly, MS).

Definition 6 (see [13]). In OMS $(\Lambda, \perp, \nabla)$, a self-map β is orthogonal continuous (or $\perp$-continuous) at $\mathfrak{b} \in \Lambda$ if for each O-sequence $\{\mathfrak{b}_\varsigma\}$ in Λ with $\mathfrak{b}_\varsigma \longrightarrow \mathfrak{b}$ as $\varsigma \longrightarrow \infty$, then we have $\beta(\mathfrak{b}_\varsigma) \longrightarrow \beta(\mathfrak{b})$ as $\varsigma \longrightarrow \infty$. Also, β is said to be $\perp$-continuous on Λ if β is $\perp$-continuous in each $\mathfrak{b} \in \Lambda$.

Definition 7 (see [13])

(1) Let $(\Lambda, \perp, \nabla)$ be an OMS, then the O-sequence $\{\mathfrak{b}_\varsigma\}$ in Λ converges to $\mathfrak{b}$ in Λ, iff $\beta(\mathfrak{b}_\varsigma, \mathfrak{b}) \longrightarrow 0$ as $\varsigma \longrightarrow \infty$. We denote by $\mathfrak{b}_\varsigma \longrightarrow \mathfrak{b}$.

(2) Let $(\Lambda, \perp, \nabla)$ be an OMS. We say that the O-sequence $\{\mathfrak{b}_\varsigma\}$ in Λ is a Cauchy O-sequence if and only if $\beta(\mathfrak{b}_\varsigma) \longrightarrow \beta(\mathfrak{b})$ as $\varsigma \longrightarrow \infty$.

(3) OMS $(\Lambda, \perp, \nabla)$ is an orthogonal complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

Definition 8 (see [13]). A self-map β on O-set Λ is said to be $\perp$-preserving if $\beta\mathfrak{b}\perp\beta\vartheta$ whenever $\mathfrak{b}\perp\vartheta$.

The following are the main results of [1].

Theorem 1. *Every $\mathscr{Z}$-contraction on a complete MS has a unique fixed point.*

In the next section, we use the following lemma in Radenovic et al. [9].

Lemma 1 (see [9]). *Let $\{\wp_\varsigma\}$ be an O-sequence in a MS (Λ, ∇) such that $\nabla(\wp_{\varsigma+1}, \wp_\varsigma)$ is nonincreasing and that*

$$\lim_{\varsigma \longrightarrow \infty} \nabla\left(\wp_{\varsigma+1}, \wp_\varsigma\right) = 0. \tag{7}$$

If $\{\wp_{2\varsigma}\}$ is not a Cauchy O-sequence, then there exist a $\nabla > 0$ and two strictly increasing O-sequences $\{\eta_\digamma\}$ and $\{\varsigma_\digamma\}$ of positive integers such that the following O-sequences tend to ∇ when $\digamma \longrightarrow \infty$: $\nabla(\wp_{2\eta_\digamma}, \wp_{2\varsigma_\digamma})$, $\nabla(\wp_{2\eta_\digamma}, \wp_{2\varsigma_{\digamma+1}})$, $\nabla(\wp_{2\eta_{\digamma-1}}, \wp_{2\varsigma_\digamma})$, $\nabla(\wp_{2\eta_{\digamma-1}}, \wp_{2\varsigma_{\digamma+1}})$, $\nabla(\wp_{2\eta_{\digamma+1}}, \wp_{2\varsigma_{\digamma+1}})$.

Definition 9. A self-map β is defined on a nonvoid O-set Λ and a map ∂: $\Lambda \times \Lambda \longrightarrow [0, \infty)$. β is said to be orthogonal extended ∂-admissible (shortly, extended $\partial_\perp$-admissible) if $\forall \mathfrak{b}, \vartheta \in \Lambda$ with $\mathfrak{b}\perp\vartheta$,

$$\partial(\mathfrak{b}, \beta\mathfrak{b}) \geq 1 \Rightarrow \partial\left(\beta\mathfrak{b}, \beta^{1+\wp}\mathfrak{b}\right) \geq 1, \quad \forall \wp \in \mathbb{N}. \tag{8}$$

If we put $\wp = 1$ in (8), we say that β is called orthogonal ∂-admissible (shortly, $\partial_\perp$-admissible).

Furthermore, if β is extended $\partial_\perp$-admissible, then we get

$$\partial(\beta^\varsigma\mathfrak{b}, \beta^\eta\mathfrak{b}) \geq 1 \quad \forall \eta, \varsigma \in \mathbb{N} \text{ with } \eta > \varsigma. \tag{9}$$

Set $\eta = \varsigma + \wp > \varsigma$, by (9), we get $\partial(\beta^{\varsigma-1}\mathfrak{b}, \beta^\varsigma\mathfrak{b}) \geq 1$ and $\partial(\beta^\varsigma\mathfrak{b}, \beta^{\varsigma+\wp}\mathfrak{b}) \geq 1$.

3. Main Results

Definition 10. A self-map β on an OMS $(\Lambda, \perp, \nabla)$ is said to be orthogonal $\mathscr{Z}$-contraction (shortly, $\mathscr{Z}_\perp$−contraction) with respect to (shortly, w.r.to) $\mathfrak{C} \in \mathscr{Z}$ if, $\forall \mathfrak{b}, \vartheta \in \Lambda$ with $\mathfrak{b}\perp\vartheta$, such that the following condition holds:

$$\mathfrak{C}(\nabla(\beta\mathfrak{b}, \beta\vartheta), \nabla(\mathfrak{b}, \vartheta)) \geq 0. \tag{10}$$

Definition 11. Let ∂: $\Lambda \times \Lambda \longrightarrow [0, \infty)$ be a function and a self-map β on OMS $(\Lambda, \perp, \nabla)$ is called an orthogonal almost $\mathscr{Z}$-contraction (shortly, almost $\mathscr{Z}_\perp$−contraction) w. r. to $\mathfrak{C} \in \mathscr{Z}$ if $\exists \mathfrak{C} \in \mathscr{Z}$, $\varpi \in \mathscr{G}$, and $\mathscr{L} \geq 0$ such that $\forall \mathfrak{b}, \vartheta \in \Lambda$ with $\mathfrak{b}\perp\vartheta$

$$\partial(\mathfrak{b}, \vartheta) \geq 1 \Rightarrow \mathfrak{C}(\nabla(\beta\mathfrak{b}, \beta\vartheta), \Omega(\mathfrak{b}, \vartheta)) \geq 0, \tag{11}$$

where

$$\Omega(\mathfrak{b}, \vartheta) = \varpi(\mathscr{E}(\mathfrak{b}, \vartheta))\mathscr{E}(\mathfrak{b}, \vartheta) + \mathscr{L}\mathscr{N}(\mathfrak{b}, \vartheta), \tag{12}$$

with

$$\begin{aligned} \mathscr{E}(\mathfrak{b}, \vartheta) &= \nabla(\mathfrak{b}, \vartheta) + |\nabla(\mathfrak{b}, \beta\mathfrak{b}) - \nabla(\vartheta, \beta\vartheta)| \text{ and} \\ \mathscr{N}(\mathfrak{b}, \vartheta) &= \min\{\nabla(\mathfrak{b}, \beta\mathfrak{b}), \nabla(\vartheta, \beta\vartheta), \nabla(\mathfrak{b}, \beta\vartheta), \nabla(\vartheta, \beta\mathfrak{b})\}. \end{aligned} \tag{13}$$

The following is our first main result:

Theorem 2. *Let β be a self-map on O-complete MS $(\Lambda, \perp, \nabla)$ with an orthogonal element $\mathfrak{b}_0$ satisfying the following conditions:*

(i) β is $\perp$-preserving

(ii) β is an almost-$\mathscr{Z}_\perp$-contraction

(iii) β is an extended $\partial_\perp$-admissible pair

(iv) $\exists \mathfrak{b}_0 \in \Lambda$ such that $\partial(\mathfrak{b}_0, \beta\mathfrak{b}_0) \geq 1$

(v) either

(a) β is $\perp$-continuous or

(b) if there exists O-sequence $\{\mathfrak{b}_\varsigma\}$ in Λ such that $\partial(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}) \geq 1 \quad \forall \varsigma$, then there is a subsequence $\{\mathfrak{b}_\varsigma(\digamma)\}$ of $\{\mathfrak{b}_\varsigma\}$ as $\varsigma \longrightarrow \infty$ such that $\partial(\mathfrak{b}_\varsigma(\digamma), \mathfrak{b}) \geq 1, \quad \forall \digamma$

Then, β has a fixed point.

Proof. Consider $(\Lambda, \perp)$ is an O-set, there exists

$$\mathfrak{b}_0 \in \Lambda\text{: } \forall \mathfrak{b} \in \Lambda, \mathfrak{b}\perp\mathfrak{b}_0 \text{ (or) } \forall \mathfrak{b} \in \Lambda, \mathfrak{b}_0\perp\mathfrak{b}. \tag{14}$$

It follows that $\mathfrak{b}_0\perp\beta\mathfrak{b}_0$ or $\beta\mathfrak{b}_0\perp\mathfrak{b}_0$. Let

$$\begin{aligned} \mathfrak{b}_1 &= \beta\mathfrak{b}_0, \\ \mathfrak{b}_2 &= \beta\mathfrak{b}_1, \\ \mathfrak{b}_3 &= \beta\mathfrak{b}_2 \cdots \mathfrak{b}_\varsigma \\ &= \beta\mathfrak{b}_{\varsigma-1}, \\ \mathfrak{b}_{\varsigma+1} &= \beta\mathfrak{b}_\varsigma \quad \forall \varsigma \in \mathbb{N}. \end{aligned} \tag{15}$$

Since (*iv*), there is a starting the initial point $\mathfrak{b}_0 \in \Lambda$ such that $\partial(\mathfrak{b}_0, \beta\mathfrak{b}_0) \geq 1$. We construct an O-sequence $\{\mathfrak{b}_\varsigma\}_{\varsigma \in \mathbb{N}_0}$ in Λ by $\mathfrak{b}_{\varsigma+1} = \beta\mathfrak{b}_\varsigma \forall \varsigma \geq 0$. Now, we arises two cases:

(i) If $\exists \varsigma \in \mathbb{N} \cup \{0\}$ such that $\mathfrak{b}_\varsigma = \mathfrak{b}_{\varsigma+1}$, then we have $\beta\mathfrak{b}_\varsigma = \beta_\varsigma$. It is clearly $\mathfrak{b}_\varsigma$ is a fixed point of β (shortly, fix $\{\beta\}$). Therefore, the proof is finished.

(ii) If $\mathfrak{b}_\varsigma \neq \mathfrak{b}_{\varsigma+1}$, for any $\varsigma \in \mathbb{N} \cup \{0\}$, then we have $\nabla(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_\varsigma) > 0$, for each $\varsigma \in \mathbb{N}$. On the contrary, where $\mathfrak{b}_{\varsigma_0} = \mathfrak{b}_{\varsigma_0+1}$ for $\varsigma_0 \in \mathbb{N}$, we desired that $\mathfrak{b}_0$ is the required fixed point, i.e., $\mathfrak{b}_{\varsigma_0+1} = \beta\mathfrak{b}_{\varsigma_0} = \mathfrak{b}_{\varsigma_0}$.

Since β is $\perp$-preserving, we have

$$\mathfrak{b}_\varsigma\perp\mathfrak{b}_{\varsigma+1} \text{ (or) } \mathfrak{b}_{\varsigma+1}\perp\mathfrak{b}_\varsigma. \tag{16}$$

This implies that $\{\mathfrak{b}_\varsigma\}$ is an *O*-sequence.

Now, from (iii) and (iv), we desired that

$$\partial(\mathfrak{b}_0, \beta\mathfrak{b}_0) = \partial(\mathfrak{b}_0, \mathfrak{b}_1) \geq 1 \Rightarrow \partial(\beta\mathfrak{b}_0, \beta\mathfrak{b}_1) = \partial(\mathfrak{b}_1, \mathfrak{b}_2) \geq 1. \tag{17}$$

Continuing in this way, we get $\partial(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}) \geq 1$ for all $\varsigma \in \mathbb{N}_0$. Furthermore, by regarding (9), we derive that

$$\partial\left(\mathfrak{b}_\varsigma, \mathfrak{b}_\eta\right) \geq 1 \text{ for all } \varsigma, \eta \in \mathbb{N}_0 \text{ with } \eta > \varsigma. \tag{18}$$

Firstly, we want to prove that $\nabla(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1})$ is decreasing. On contrary, suppose that $\nabla(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}) < \nabla(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2})$. Since (iv), we find that

$$\begin{aligned} 0 &\leq \mathcal{C}\left(\nabla\left(\beta\mathfrak{b}_\varsigma, \beta\mathfrak{b}_{\varsigma+1}\right), \Omega\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right)\right) \\ &= \mathcal{C}\left(\nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right), \Omega\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right)\right) \\ &< \Omega\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) - \nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right), \end{aligned} \tag{19}$$

which implies that

$$\begin{aligned} \nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right) \leq \Omega\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) &= \varpi\left(\mathscr{E}\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right)\right)\mathscr{E}\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) \\ &\quad + \mathscr{L}\mathscr{N}\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right), \end{aligned} \tag{20}$$

where

$$\begin{aligned} \mathscr{N}\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+1}\right) &= \min\left\{\nabla\left(\mathfrak{b}_\varsigma, \beta\mathfrak{b}_\varsigma\right), \nabla\left(\mathfrak{b}_{\varsigma+1}, \beta\mathfrak{b}_{\varsigma+1}\right), \nabla\left(\mathfrak{b}_\varsigma, \beta\mathfrak{b}_{\varsigma+1}\right), \nabla\left(\mathfrak{b}_{\varsigma+1}, \beta\mathfrak{b}_\varsigma\right)\right\} \\ &= \min\left\{\nabla\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right), \nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right), \nabla\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+2}\right), \nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+1}\right)\right\} \\ &= 0 \text{ and} \\ \mathscr{E}\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) &= \nabla\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) + \left|\nabla\left(\mathfrak{b}_\varsigma, \beta\mathfrak{b}_\varsigma\right) - \nabla\left(\mathfrak{b}_{\varsigma+1}, \beta\mathfrak{b}_{\varsigma+1}\right)\right| \\ &= \nabla\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) + \left|\nabla\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) - \nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right)\right| \\ &= \nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right). \end{aligned} \tag{21}$$

Hence, inequality (19) turns into

$$\begin{aligned} \nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right) &\leq \varpi\left(\mathscr{E}\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right)\right)\mathscr{E}\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) \\ &= \varpi\left(\nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right)\right)\nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right) \\ &< \nabla\left(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}\right). \end{aligned} \tag{22}$$

a contradiction from which we deduce that $\nabla(\mathfrak{b}_{\varsigma+1}, \mathfrak{b}_{\varsigma+2}) < \nabla(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1})$, for all ς since the O-sequence $\left\{\nabla(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1})\right\}$ is decreasing. Next, we prove that $\lim_{\varsigma \longrightarrow \infty} \nabla(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}) = 0$. Since $\left\{\nabla(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1})\right\}$ is decreasing and bounded below, from which it converges to Υ in $\mathbb{R}_0^+$

$$\lim_{\varsigma \longrightarrow \infty} \nabla\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) = \Upsilon. \tag{23}$$

It is evident that

$$\lim_{\varsigma \longrightarrow \infty} \mathscr{E}\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) = \Upsilon. \tag{24}$$

We assume that $\Upsilon = 0$. On contrary, suppose that $\Upsilon \neq 0$. Then, from (22) and $(\mathcal{C}_2)$, putting limit as $\varsigma \longrightarrow \infty$. Therefore,

$$\lim_{\varsigma \longrightarrow \infty} \varpi\left(\mathscr{E}\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right)\right) = 1 \Rightarrow \lim_{\varsigma \longrightarrow \infty} \mathscr{E}\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) = 0. \tag{25}$$

Consequently, $\Upsilon = 0$ and also

$$\Upsilon = \lim_{\varsigma \longrightarrow \infty} \nabla\left(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}\right) = 0. \tag{26}$$

As follows, we prove that O-sequence $\left\{\mathfrak{b}_\varsigma\right\}$ is a Cauchy O-sequence. Contrary, we assume $\left\{\mathfrak{b}_\varsigma\right\}$ is not a Cauchy O-sequence, then there exists a positive number ϵ and two O-sequences $\left\{\mathfrak{b}_{\varsigma_\mathsf{F}}\right\}, \left\{\mathfrak{b}_{\eta_\mathsf{F}}\right\}; \varsigma_\mathsf{F} > \eta_\mathsf{F} > \mathsf{F}$ such that

$$\begin{aligned} \nabla\left(\mathfrak{b}_{\eta_\mathsf{F}}, \mathfrak{b}_{\varsigma_\mathsf{F}}\right) &\geq \epsilon, \\ \nabla\left(\mathfrak{b}_{\eta_\mathsf{F}}, \mathfrak{b}_{\varsigma_\mathsf{F}-1}\right) &< \epsilon. \end{aligned} \tag{27}$$

Now take $\mathfrak{b} = \mathfrak{b}_{\eta_\mathsf{F}-1}$ and $\vartheta = \mathfrak{b}_{\varsigma_\mathsf{F}-1}$ in (11), we have

$$\partial\left(\mathfrak{b}_{\eta_\mathsf{F}-1}, \mathfrak{b}_{\varsigma_\mathsf{F}-1}\right) \geq 1 \text{ for all } k, \tag{28}$$

implies

$$0 \leq \mathcal{C}\left(\nabla\left(\beta \mathfrak{b}_{\eta_F-1}, \beta \mathfrak{b}_{\varsigma_F-1}\right), \Omega\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right)\right) < \Omega\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right) - \nabla\left(\beta \mathfrak{b}_{\eta_F-1}, \beta \mathfrak{b}_{\varsigma_F-1}\right), \tag{29}$$

where

$$N\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right) = \min\left\{\nabla\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\eta_F}\right), \nabla\left(\mathfrak{b}_{\varsigma_F-1}, \mathfrak{b}_{\varsigma_F}\right), \nabla\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F}\right), \nabla\left(\mathfrak{b}_{\varsigma_F-1}, \mathfrak{b}_{\eta_F}\right),\right\} \text{ and}$$
$$\epsilon\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right) = \nabla\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right) + \left|\nabla\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\eta_F}\right) - \nabla\left(\mathfrak{b}_{\varsigma_F-1}, \mathfrak{b}_{\varsigma_F}\right)\right|. \tag{30}$$

Due to Lemma 1, we have

$$\begin{aligned}\lim_{F \longrightarrow \infty} \nabla\left(\mathfrak{b}_{\eta_F}, \mathfrak{b}_{\varsigma_F}\right) &= \lim_{F \longrightarrow \infty} \nabla\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right) \\ &= \lim_{F \longrightarrow \infty} \nabla\left(\mathfrak{b}_{\eta_F}, \mathfrak{b}_{\varsigma_F-1}\right) \\ &= \lim_{F \longrightarrow \infty} \nabla\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F}\right) = \epsilon.\end{aligned} \tag{31}$$

Since

$$\mathscr{E}\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right) = \nabla\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right) + \left|\nabla\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\eta_F}\right) - \nabla\left(\mathfrak{b}_{\varsigma_F-1}, \mathfrak{b}_{\varsigma_F}\right)\right|. \tag{32}$$

Using (26) and (31), we have

$$\lim_{\varsigma \longrightarrow \infty} \mathscr{E}\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right) = \epsilon. \tag{33}$$

Let $\zeta_{\varsigma_F} = \nabla(\mathfrak{b}_{\eta_F}, \mathfrak{b}_{\varsigma_F})$ and $\varkappa_{\varsigma_F} = \Omega(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1})$, we have $\lim_{F \longrightarrow \infty} \varkappa_{\varsigma_F} = \lim_{F \longrightarrow \infty} \zeta_{\varsigma_F} = \epsilon$ and letting $F \longrightarrow \infty$ in (29)

$$\begin{aligned}0 &\leq \lim_{F \longrightarrow \infty} \sup \mathcal{C}\left(\nabla\left(\mathfrak{b}_{\eta_F}, \mathfrak{b}_{\varsigma_F}\right), \Omega\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right)\right) \\ &= \lim_{F \longrightarrow \infty} \sup \mathcal{C}\left(\nabla\left(\mathfrak{b}_{\eta_F}, \mathfrak{b}_{\varsigma_F}\right), \Omega\left(\mathfrak{b}_{\eta_F-1}, \mathfrak{b}_{\varsigma_F-1}\right)\right) \\ &= \lim_{F \longrightarrow \infty} \sup \mathcal{C}\left(\zeta_{\varsigma_F}, \varkappa_{\varsigma_F}\right).\end{aligned} \tag{34}$$

Then by (31) and (34) and keeping $(\mathcal{C}_2)$ in mind, we have

$$0 \leq \lim_{F \longrightarrow \infty} \sup \mathcal{C}\left(\zeta_{\varsigma_F}, \varkappa_{\varsigma_F}\right) < \lim_{F \longrightarrow \infty} \sup\left[\varkappa_{\varsigma_F} - \zeta_{\varsigma_F}\right] \longrightarrow [\epsilon - \epsilon] = 0, \tag{35}$$

a contradiction. Hence, proved our result and $\{\mathfrak{b}_\varsigma\}$ is a Cauchy O-sequence.

By the O-complete MS $(\Lambda, \perp, \nabla)$, the O-sequence approaches to some point $\nu \in \Lambda$ as $\varsigma \longrightarrow \infty$.

$$\lim_{\varsigma \longrightarrow \infty} \nabla\left(\mathfrak{b}_\varsigma, \nu\right) = \lim_{\varsigma \longrightarrow \infty} \nabla\left(\mathfrak{b}_{\varsigma+1}, \nu\right) = 0. \tag{36}$$

Now, we will prove $\beta\nu = \nu$. Since β is $\perp$-continuous, suppose from (iiia), we have

$$\begin{aligned}\beta\nu &= \beta\left(\lim_{\varsigma \longrightarrow \infty} \mathfrak{b}_\varsigma\right) \\ &= \lim_{\varsigma \longrightarrow \infty} \beta\left(\mathfrak{b}_\varsigma\right) \\ &= \lim_{\varsigma \longrightarrow \infty} \mathfrak{b}_{\varsigma+1} \\ &= \nu.\end{aligned} \tag{37}$$

Suppose we have (vb) and use the method of reductio ad absurdum. On contrary, assume that $\beta\nu \neq \nu$, i.e., $\nabla(\nu, \beta\nu) > 0$. By (16), there exists a subsequence $\{\mathfrak{b}_{\varsigma_F}\}$ of $\{\mathfrak{b}_\varsigma\}$ such that $\partial(\mathfrak{b}_{\varsigma_F}, \nu) \geq 1 \quad \forall F$.

It implies that

$$0 \leq \mathcal{C}\left(\nabla\left(\beta \mathfrak{b}_{\varsigma_F-1}, \beta\nu\right), \Omega\left(\mathfrak{b}_{\varsigma_F-1}, \nu\right)\right) < \Omega\left(\mathfrak{b}_{\varsigma_F-1}, \nu\right) - \nabla\left(\beta \mathfrak{b}_{\varsigma_F-1}, \beta\nu\right), \tag{38}$$

where

$$\begin{aligned}
\mathbf{N}\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \nu\right) &= \min\left\{\nabla\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \nu\right), \nabla(\nu, \beta\nu), \nabla\left(\nu, \mathfrak{b}_{\varsigma_{\mathsf{F}}}\right), \nabla\left(\beta\nu, \mathfrak{b}_{\varsigma_{\mathsf{F}}-1}\right)\right\},\\
\lim_{k \longrightarrow \infty} \mathbf{N}\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}\right) &= \min\left\{0, \nabla\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \mathfrak{b}_{\varsigma_{\mathsf{F}}}\right), 0, \nabla(\nu, \beta\nu)\right\},\\
&= 0, \text{ and}\\
\epsilon\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \nu\right) &= \nabla\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \nu\right) + \left|\nabla\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \mathfrak{b}_{\varsigma_{\mathsf{F}}}\right) - \nabla(\nu, \beta\nu)\right|\\
\lim_{\varsigma \longrightarrow \infty} \epsilon\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \nu\right) &= 0 + |0 - \nabla(\nu, \beta\nu)|,\\
&= \nabla(\nu, \beta\nu).
\end{aligned} \tag{39}$$

By putting $\mathsf{F} \longrightarrow \infty$ in (38), with the previous, we have

$$\begin{aligned}
0 &\leq \lim_{\mathsf{F} \longrightarrow \infty} \sup \mathfrak{C}\left(\nabla\left(\beta\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \beta\nu\right), \Omega\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \nu\right)\right),\\
&< \lim_{\mathsf{F} \longrightarrow \infty} \sup \Omega\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \nu\right) - \nabla\left(\beta\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \beta\nu\right),\\
&< \lim_{\mathsf{F} \longrightarrow \infty} \sup \varpi\left(\mathscr{E}\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \nu\right)\right)\mathscr{E}\left(\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \nu\right) - \nabla\left(\beta\mathfrak{b}_{\varsigma_{\mathsf{F}}-1}, \beta\nu\right),\\
&< \nabla(\nu, \beta\nu) - \nabla(\nu, \beta\nu),\\
&= 0,
\end{aligned} \tag{40}$$

which is on contrary. Hence, β has a fixed point ν of β. □

Theorem 3. *In addition, to the hypothesis of Theorem 2, let us suppose (vi) $\forall \wp, \quad \mathfrak{q} \in S_\beta(\Lambda)$ we have $\partial(\wp, \mathfrak{q}) \geq 1$.*

Where the O-set $S_\beta(\Lambda) \subset \Lambda$ is of all fix $\{\beta\}$, then β has a unique fixed point.

Proof. Now, to prove the uniqueness part, suppose that β has two distinct fixed points, namely, $\wp, \mathfrak{q} \in S_\beta(\Lambda)$ with $\beta\wp = \wp \neq q = \beta q$. On account of (vi), we have $\partial(\wp, \mathfrak{q}) \geq 1$, which implies

$$0 \leq \mathfrak{C}(\nabla(\beta\wp, \beta\mathfrak{q}), \Omega(\wp, \mathfrak{q})) < \Omega(\wp, \mathfrak{q}) - \nabla(\beta\wp, \beta\mathfrak{q}), \tag{41}$$

where

$$\Omega(\wp, \mathfrak{q}) = \varpi(\mathscr{E}(\wp, \mathfrak{q}))\mathscr{E}(\wp, \mathfrak{q}) + \mathscr{L}\mathscr{N}(\wp, \mathfrak{q}), \tag{42}$$

with

$$\begin{aligned}
\mathscr{E}(\wp, \mathfrak{q}) &= \nabla(\wp, \mathfrak{q}) + |\nabla(\wp, \beta\wp) - \nabla(\mathfrak{q}, \beta\mathfrak{q})|\\
&= \nabla(\wp, \mathfrak{q}) \text{ and}\\
\mathscr{N}(\wp, \mathfrak{q}) &= \min\{\nabla(\wp, \beta\wp), \nabla(\mathfrak{q}, \beta\mathfrak{q}), \nabla(\wp, \beta\mathfrak{q}), \nabla(\mathfrak{q}, \beta\wp)\}\\
&= 0.
\end{aligned} \tag{43}$$

Hence, expression (41) turns into

$$0 \leq \mathfrak{C}(\nabla(\beta\wp, \beta\mathfrak{q}), \Omega(\wp, \mathfrak{q})) < \Omega(\wp, \mathfrak{q}) - \nabla(\beta\wp, \beta\mathfrak{q}) = 0, \tag{44}$$

a contradiction and hence the proof. □

Example 4. Let $\Lambda = [0, 1]$ with a metric $\nabla(\mathfrak{b}, \vartheta) = |\mathfrak{b} - \vartheta|$ for all $\mathfrak{b}, \vartheta \in \Lambda$ with $\mathfrak{b} \perp \vartheta$. Let $\mathfrak{C}(\zeta, \varkappa) = \varkappa/2 - \zeta$ and considering $\varpi: \mathbb{R}_0^+ \longrightarrow [0, 1), \varpi(\zeta) = 1/\zeta \quad \forall \zeta \geq 0$ and $\mathscr{L} \geq 0$. Let $\beta: \Lambda \longrightarrow \Lambda$ be defined by $\beta\mathfrak{b} = \mathfrak{b}^2 \quad \forall \mathfrak{b} \in [0, 1]$ and $\partial: \Lambda \times \Lambda \longrightarrow \mathbb{R}_0^+$ defined by

$$\partial(\mathfrak{b}, \vartheta) = \begin{cases} 1, & \text{if } \mathfrak{b}, \quad \vartheta \in [0, 1],\\ 0, & \text{otherwise.} \end{cases}$$

Since $\partial(\mathfrak{b}, \vartheta) = 1, \mathfrak{b}, \vartheta \in [0, 1]$ implies

$$\begin{aligned}
\mathfrak{C}(\nabla(\beta\mathfrak{b}, \beta\vartheta), \Omega(\mathfrak{b}, \vartheta) &= \Omega(\mathfrak{b}, \vartheta) - \nabla(\beta\mathfrak{b}, \beta\vartheta))\\
&= \varpi(\mathscr{E}(\mathfrak{b}, \vartheta))\mathscr{E}(\mathfrak{b}, \vartheta) + \mathscr{L}\mathscr{N}(\mathfrak{b}, \vartheta) - \frac{1}{2}|\mathfrak{b} - \vartheta|\\
&= \frac{\mathscr{E}(\mathfrak{b}, \vartheta)}{\mathscr{E}(\mathfrak{b}, \vartheta)} + \mathscr{L}\mathscr{N}(\mathfrak{b}, \vartheta) - \frac{1}{2}|\mathfrak{b} - \vartheta|\\
&\leq 1 + \mathscr{L}\mathscr{N}(\mathfrak{b}, \vartheta) - \frac{1}{2}|\mathfrak{b} - \vartheta|\\
&\geq 0.
\end{aligned} \tag{45}$$

Therefore, β is almost $\mathscr{Z}_\perp$-contraction w. r. to $\mathfrak{C} \in \mathscr{Z}$. Hence, all hypothesis of Theorem 3 holds, and hence, β has a unique fixed point.

4. Immediately Consequence

The following will be a conclusion of our main results.

Theorem 4. *A self-map β on an O-complete MS $(\Lambda, \perp, \nabla)$ and a mapping $\partial: \Lambda \times \Lambda \longrightarrow \mathbb{R}_0^+$. Assume that $\exists \mathfrak{C} \in \mathscr{Z}, \varpi \in \mathscr{G}$ and $\mathscr{L} \geq 0$ such that $\forall \mathfrak{b}, \vartheta \in \Lambda$ with $\mathfrak{b} \perp \vartheta$*

$$\begin{aligned}
\partial(\mathfrak{b}, \vartheta) \geq 1 \Rightarrow \mathfrak{C}(\nabla(\beta\mathfrak{b}, \beta\vartheta), \quad & \varpi(\nabla(\mathfrak{b}, \vartheta))\nabla(\mathfrak{b}, \vartheta)\\
&+ \mathscr{L}\mathscr{N}(\mathfrak{b}, \vartheta)) \geq 0,
\end{aligned} \tag{46}$$

where

$$\mathcal{N}(\mathfrak{b},\vartheta)=\min\{\nabla(\mathfrak{b},\beta\mathfrak{b}),\nabla(\vartheta,\beta\vartheta),\nabla(\mathfrak{b},\beta\vartheta),\nabla(\vartheta,\beta\mathfrak{b})\}. \tag{47}$$

Furthermore, we suppose, $\forall\mathfrak{b},\vartheta\in\Lambda$ with $\mathfrak{b}\perp\vartheta$, that

(i) $\perp$-preserving

(ii) β is an extended $\partial_\perp$-admissible pair

(iii) $\exists\mathfrak{b}_0\in\Lambda$ such that $\partial(\mathfrak{b}_0,\beta\mathfrak{b}_0)\geq 1$

(iv) either

(iva) β is $\perp$-continuous, or

(ivb) if there exists O-sequence $\{\mathfrak{b}_\varsigma\}$ in Λ such that $\partial(\mathfrak{b}_\varsigma,\mathfrak{b}_{\varsigma+1})\geq 1\forall\varsigma$, then there is a subsequence $\{\mathfrak{b}_\varsigma(\mathsf{F})\}$ of $\{\mathfrak{b}_\varsigma\}$ as $\varsigma\longrightarrow\infty$ such that $\partial(\mathfrak{b}_\varsigma(\mathsf{F}),\mathfrak{b})\geq 1\forall\mathsf{F}$

(v) $\forall\wp,\quad \mathfrak{q}\in S_\beta(\Lambda)$ we have $\partial(\wp,\mathfrak{q})\geq 1$, where the O-set $S_\beta(\Lambda)\subset\Lambda$ is of all fix $\{\beta\}$.

Then, β has a fixed point.

Due to its similarity to the proof of Theorem 3, we omit this proof (and hence Theorem 2).

We omit the auxiliary function $\partial\colon\Lambda\times\Lambda\longrightarrow\mathbb{R}_0^+$ in Theorem 5, and we get the desirable result in the standard O-MSs.

Theorem 5. *A self-map β defined on an O-complete MS $(\Lambda,\perp,\nabla)$. Assume that $\exists\mathcal{C}\in\mathcal{Z},\varpi\in\mathcal{G}$ and $\mathcal{L}\geq 0$ such that $\forall\mathfrak{b},\vartheta\in\Lambda$ with $\mathfrak{b}\perp\vartheta$*

$$\mathcal{C}(\nabla(\beta\mathfrak{b},\beta\vartheta),\Omega(\mathfrak{b},\vartheta))\geq 0, \tag{48}$$

where $\Omega(\mathfrak{b},\vartheta)$, $\mathcal{E}(\mathfrak{b},\vartheta)$, and $\mathcal{N}(\mathfrak{b},\vartheta)$ are in the above section Theorem 2. Then, β has a unique fixed point.

Proof. Letting $\partial(\mathfrak{b},\vartheta)=1\quad\forall\mathfrak{b},\vartheta\in\Lambda$ with $\mathfrak{b}\perp\vartheta$, in Theorem 3. □

Let Φ be the family of all auxiliary functions and $\Phi\colon\mathbb{R}_0^+\longrightarrow\mathbb{R}_0^+$ be $\perp$-continuous with $\Phi(\zeta)=0$ iff $\zeta=0$.

Theorem 6. *A self-map β defined on an O-complete MS $(\Lambda,\perp,\nabla)$ and a mapping $\partial\colon\Lambda\times\Lambda\longrightarrow\mathbb{R}_0^+$. Assume that $\exists\Phi_1,\Phi_2\in\varphi$ with $\Phi_1(\zeta)<\zeta\leq\Phi_2(\zeta)\forall\zeta>0$, and $\varpi\in\mathcal{G}$ and $\mathcal{L}\geq 0$ such that $\forall\mathfrak{b},\vartheta\in\Lambda$ with $\mathfrak{b}\perp\vartheta$*

$$\partial(\mathfrak{b},\vartheta)\geq 1\Rightarrow\Phi_2(\nabla(\beta\mathfrak{b},\beta\vartheta))\leq\Phi_1(\Omega(\mathfrak{b},\vartheta)), \tag{49}$$

where $\Omega(\mathfrak{b},\vartheta)$, $\mathcal{E}(\mathfrak{b},\vartheta)$, and $\mathcal{N}(\mathfrak{b},\vartheta)$ are defined in the above section Theorem 2. Furthermore, suppose that $\forall\mathfrak{b},\vartheta\in\Lambda$ with $\mathfrak{b}\perp\vartheta$,

(i) $\perp$-preserving

(ii) β is an extended $\partial_\perp$-admissible pair

(iii) $\exists\mathfrak{b}_0\in\Lambda$ such that $\partial(\mathfrak{b}_0,\beta\mathfrak{b}_0)\geq 1$

(iv) either

(iva) β is $\perp$-continuous, or

(ivb) if there exists O-sequence $\{\mathfrak{b}_\varsigma\}$ in Λ such that $\partial(\mathfrak{b}_\varsigma,\mathfrak{b}_{\varsigma+1})\geq 1\forall\varsigma$, then there exists a subsequence $\{\mathfrak{b}_\varsigma(\mathsf{F})\}$ of $\{\mathfrak{b}_\varsigma\}$ as $\varsigma\longrightarrow\infty$ such that $\partial(\mathfrak{b}_\varsigma(\mathsf{F}),\mathfrak{b})\geq 1\forall\mathsf{F}$

(v) $\forall\wp,\quad\mathfrak{q}\in S_\beta(\Lambda)$ we have $\partial(\wp,\mathfrak{q})\geq 1$, where the O-set $S_\beta(\Lambda)\subset\Lambda$ is of all fix $\{\beta\}$.

Then, β has a fixed point.

Proof. Assume $\mathcal{C}(\zeta,\varkappa)=\Phi_1(\varkappa)-\Phi_2(\zeta)\quad\forall\zeta,\varkappa\geq 0$, where $\Phi_1(\zeta)<\zeta\leq\Phi_2(\zeta)\quad\forall\zeta>0$. It shows that $\mathcal{C}\in\mathcal{Z}$. As a conclusion, the required results are obtained as follows from Theorem 3. □

The following theorem demonstrates the fixed point in integral function.

Theorem 7. *A self-map β on an O-complete MS $(\Lambda,\perp,\nabla)$ and a mapping $\partial\colon\Lambda\times\Lambda\longrightarrow\mathbb{R}_0^+$. Assume that $\exists\Phi_1,\Phi_2\in\varphi$ with $\Phi_1(\zeta)<\zeta\leq\Phi_2(\zeta)\quad\forall\zeta>0$, $\varpi\in\mathcal{G}$, and $\mathcal{L}\geq 0$ such that $\forall\mathfrak{b},\quad\vartheta\in\Lambda$ with $\mathfrak{b}\perp\vartheta$*

$$\partial(\mathfrak{b},\vartheta)\geq 1\Rightarrow\nabla(\beta\mathfrak{b},\beta\vartheta)\leq\int_0^{\Omega(\mathfrak{b},\vartheta)}\mu(u)\mathrm{d}u, \tag{50}$$

where $\Omega(\mathfrak{b},\vartheta)$, $\mathcal{E}(\mathfrak{b},\vartheta)$, and $\mathcal{N}(\mathfrak{b},\vartheta)$ are defined in Theorem 2. Furthermore, assume that $\forall\mathfrak{b},\vartheta\in\Lambda$ with $\mathfrak{b}\perp\vartheta$,

(i) $\perp$-preserving

(ii) β is an extended- $\partial_\perp$-admissible pair

(iii) $\exists\mathfrak{b}_0\in\Lambda$ such that $\partial(\mathfrak{b}_0,\beta\mathfrak{b}_0)\geq 1$

(iv) either

(iva) β is $\perp$-continuous, or

(ivb) if there exists O-sequence $\{\mathfrak{b}_\varsigma\}$ in Λ such that $\partial(\mathfrak{b}_\varsigma,\mathfrak{b}_{\varsigma+1})\geq 1\quad\forall\varsigma$ and then there is a subsequence $\{\mathfrak{b}_\varsigma(\mathsf{F})\}$ of $\{\mathfrak{b}_\varsigma\}$ as $\varsigma\longrightarrow\infty$ such that $\partial(\mathfrak{b}_\varsigma(\mathsf{F}),\mathfrak{b})\geq 1\quad\forall\mathsf{F}$

(v) $\forall\wp,\quad\mathfrak{q}\in S_\beta(\Lambda)$ we have $\partial(\wp,\mathfrak{q})\geq 1$, where the O-set $S_\beta(\Lambda)\subset\Lambda$ is of all fix $\{\beta\}$.

Then, β has a fixed point.

Proof. Setting $\mathcal{C}(\zeta,\varkappa)=\varkappa-\int_0^\zeta\mu(u)\mathrm{d}u$ for all $\zeta,\varkappa\geq 0$.

It is clear that $\mathcal{C}\in\mathcal{Z}$. As a conclusion, the required results are obtained as follows from Theorem 3. □

The following theorem demonstrates the fixed point in $(\nabla-\Phi)$ -$\mathcal{Z}_\perp$-contraction.

Theorem 8. *A self-map β on an O-complete MS $(\Lambda,\perp,\nabla)$ and a function $\partial\colon\Lambda\times\Lambda\longrightarrow\mathbb{R}_0^+$. Let $\varpi\in\mathcal{G}$ and $\mathcal{L}\geq 0$. Suppose there exist a upper semicontinuous function $\Phi\colon\mathbb{R}_0^+\longrightarrow\mathbb{R}_0^+$ such that $\Phi(\zeta)<\zeta\forall\zeta>0$ and $\Phi(0)=0$. Suppose that $\forall\mathfrak{b},\vartheta\in\Lambda$ with $\mathfrak{b}\perp\vartheta$,*

$$\partial(\mathfrak{b},\vartheta)\geq 1\Rightarrow\nabla(\beta\mathfrak{b},\beta\vartheta)\leq\Phi(\Omega(\mathfrak{b},\vartheta)), \tag{51}$$

where $\Omega(\mathfrak{b},\vartheta)$, $\mathcal{E}(\mathfrak{b},\vartheta)$, and $\mathcal{N}(\mathfrak{b},\vartheta)$ are in the above section Theorem 2. Furthermore, suppose that $\forall\mathfrak{b},\vartheta\in\Lambda$ with $\mathfrak{b}\perp,\vartheta$,

(i) $\perp$-preserving

(ii) β *is an extended -* $\partial_{\perp}$*-admissible pair*

(iii) $\exists \mathfrak{b}_0 \in \Lambda$ *such that* $\partial(\mathfrak{b}_0, \beta\mathfrak{b}_0) \geq 1$

(iv) either

(iva) β *is* $\perp$*-continuous, or*

(ivb) if there exists O-sequence $\{\mathfrak{b}_\varsigma\}$ *in* Λ *such that* $\partial(\mathfrak{b}_\varsigma, \mathfrak{b}_{\varsigma+1}) \geq 1 \quad \forall \varsigma$*, then there is a subsequence* $\{\mathfrak{b}_\varsigma(\mathsf{F})\}$ *of* $\{\mathfrak{b}_\varsigma\}$ *as* $\varsigma \longrightarrow \infty$ *such that* $\partial(\mathfrak{b}_\varsigma(\mathsf{F}), \mathfrak{b}) \geq 1 \quad \forall \mathsf{F}$

(v) $\forall \wp, \quad \mathfrak{q} \in S_\beta(\Lambda)$ *we have* $\partial(\wp, \mathfrak{q}) \geq 1$*, where the O-set* $S_\beta(\Lambda) \subset \Lambda$ *is of all fix* $\{\beta\}$.

Then, β has a fixed point.

Proof. Set $\mathfrak{C}(\zeta, \varkappa) = \Phi(\varkappa) - \zeta, \quad \forall \zeta, \varkappa \geq 0$.

It shows that $\mathfrak{C} \in \mathscr{Z}$ (see e.g. [1–4]). Thus, the required results are obtained as follows from Theorem 3. □

5. Application to Fredholm Integral Equations

In this section, we give an application of a solution for the nonlinear Fredholm integral equation via almost $\mathscr{Z}_{\perp}$-contraction with OMS. Now, consider $\mathscr{C}[c, d]$, be the O-complete MS of continuous real-valued functions defined on [c, d], and a function

$$\mathfrak{b}(\zeta) = \zeta + \int_c^d H(\zeta, \mathfrak{b}(s))\mathrm{d}s, \tag{52}$$

where $\mathfrak{b} \in \mathscr{C}[c, d]$ with $c, d \in \mathbb{R}$ such that $c < d$ and H: $[c, d] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

Theorem 9. *Let* ψ: $\mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ *with* $\psi \in \Psi$ *such that* $\psi(\zeta) = 0$ *for all* $\zeta = 0$ *and* $\psi(\zeta) < \zeta \forall \zeta \geq 0$. *If*

$$\left|H(\zeta_1, w_1) - H(\zeta_2, w_2)\right| \leq \frac{|w_1| - |w_2|}{2(b-a)} - \frac{|\zeta_1| - |\zeta_2|}{b-a}. \tag{53}$$

$\forall \zeta_1, \zeta_2 \in [c, d]$ and for all $w_1, w_2 \in \mathbb{R}$, then (52) has a unique solution.

Proof. Set β: $\mathscr{C}[c, d] \longrightarrow \mathscr{C}[c, d]$ as

$$(\beta\mathfrak{b})(\zeta) = \zeta + \int_c^d H(\zeta, \mathfrak{b}(\varkappa))\mathrm{d}\varkappa. \tag{54}$$

$\forall \mathfrak{b} \in \Lambda$: $= \mathscr{C}[c, d]$ with metric

$$\nabla(\mathfrak{b}, \vartheta) = \sup_{\zeta \in [c, d]} |\mathfrak{b}(\zeta) - \vartheta(\zeta)|. \tag{55}$$

$\forall \mathfrak{b}, \vartheta \in \Lambda$ with $\mathfrak{b} \perp \vartheta$. It is clear that the MS $(\mathscr{C}[c, d], \perp, \nabla)$ is O-complete. Now, the function Ω: $\Lambda \times \Lambda \longrightarrow \mathbb{R}_0^+$ defined by

$$\Omega(\mathfrak{b}, \vartheta) = \begin{cases} \sup\limits_{\zeta \in (c, d)} |\mathfrak{b}(\zeta)| - \sup\limits_{\zeta \in (c, d)} \vartheta(\zeta), & \text{if } \mathfrak{b} \neq \vartheta, \\ 0, \quad \text{if } \mathfrak{b} = \vartheta, \end{cases} \tag{56}$$

for all $\mathfrak{b}, \vartheta \in \Lambda$. Clearly, Ω is orthogonal MS on Λ and a metric ∇. Now, we have seen that β satisfied almost $\mathscr{Z}_{\perp}$-contraction in (11). Let $\mathfrak{C}(\zeta, \varkappa) = 2\varkappa - -\zeta$ in (11) and suppose that $\mathfrak{b}, \vartheta \in \Lambda$ with $\mathfrak{b} \perp \vartheta$ and $\zeta_1, \zeta_2 \in [c, d]$. Thus, we get

$$\begin{aligned}
\left|\beta\mathfrak{b}(\zeta_1) - \beta\vartheta(\zeta_1)\right| &= \left|\left(\Phi(\zeta_1) + \int_c^d H(\zeta_1, \mathfrak{b}(\varkappa))\mathrm{d}\varkappa\right) - \left(\Phi(\zeta_2) + \int_c^d H(\zeta_2, y(\varkappa))\mathrm{d}\varkappa\right)\right| \\
&\leq \left(|\Phi(\zeta_1)| + \left|\int_c^d H(\zeta_1, \mathfrak{b}(\varkappa))\mathrm{d}\varkappa\right|\right) - \left(|\Phi(\zeta_2)| + \left|\int_c^d H(\zeta_2, y(\varkappa))\mathrm{d}\varkappa\right|\right) \\
&\leq |\Phi(\zeta_1)| - |\Phi(\zeta_2)| + \int_c^d \left|H(\zeta_1, \mathfrak{b}(\varkappa))\mathrm{d}\varkappa\right| - \int_c^d \left|H(\zeta_2, \mathfrak{b}(\varkappa))\mathrm{d}\varkappa\right| \\
&\leq |\zeta_1| - |\zeta_2| + \int_c^d \left(\left|H(\zeta_1, \mathfrak{b}(\varkappa))\mathrm{d}\varkappa\right| - \left|H(\zeta_1, \mathfrak{b}(\varkappa))\mathrm{d}\varkappa\right|\right)\mathrm{d}\varkappa \\
&\leq |\zeta_1| - |\zeta_2| + \int_c^d \left(\frac{(|x(s)| - |y(\varkappa)|)}{2(b-a)} - \frac{|\zeta_1| - |\zeta_2|}{b-a}\right)\mathrm{d}\varkappa \\
&\leq |\zeta_1| - |\zeta_2| + \frac{1}{b-a}\left[\int_c^d \frac{\Omega(\mathfrak{b}, \vartheta)}{2}\mathrm{d}\varkappa - \int_c^d \left(|\zeta_1| - |\zeta_2|\mathrm{d}\varkappa\right)\right] \\
&= \frac{\Omega(\mathfrak{b}, \vartheta)}{2} \\
&\leq \frac{\Omega(\mathfrak{b}, \vartheta)}{2}.
\end{aligned} \tag{57}$$

From this, we have

$$\sup_{\zeta \in [c,d]} |\beta\mathfrak{b}(\zeta)| + \sup_{\zeta \in [c,d]} |\beta\vartheta(\zeta)| \leq \frac{\Omega(\mathfrak{b},\vartheta)}{2}, \tag{58}$$

which implies that

$$\nabla(\beta\mathfrak{b},\beta\vartheta) \leq \frac{\Omega(\mathfrak{b},\vartheta)}{2}. \tag{59}$$

$\forall \mathfrak{b}, \quad \vartheta \in \Lambda$ with $\mathfrak{b} \neq \vartheta$. Therefore, we get

$$\begin{gathered} \frac{\Omega(\mathfrak{b},\vartheta)}{2} - \nabla(\beta\mathfrak{b},\beta\vartheta) \geq 0, \\ \mathfrak{C}(\nabla(\beta\mathfrak{b},\beta\vartheta)), \Omega(\mathfrak{b},\vartheta) \geq 0. \end{gathered} \tag{60}$$

$\forall \mathfrak{b}, \vartheta \in \Lambda, \mathfrak{b} \perp \vartheta$ with $\mathfrak{b} \neq \vartheta$. For $\mathfrak{b} = \vartheta$, it can be easy to verify that β satisfies almost $\mathscr{Z}_{\perp}$-contraction condition (11). Hence, β satisfies all conditions of Theorem 2, and it has a unique fixed point. Hence, we get solution for the nonlinear Fredholm integral (52) that has a unique fixed point. □

6. Conclusion

In this paper, we proved fixed point theorem of a solution to nonlinear Fredholm integral equations on an almost $\mathscr{Z}_{\perp}$-contraction.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors equally contributed to this paper and read and approved the final manuscript.

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