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Research Article

Approximate Solutions for Nash Differential Games

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In this paper, we are concerned with an open-loop Nash differential game. The necessary conditions for an open-loop Nash equilibrium solution are obtained, also the existence for the solution of the dynamical system of the differential game is studied. Picard method is used to find an approximate solution, and the uniform convergence is proved. Finally, we constructed figures for the analysis of the differential game. These results can be applied between economic and financial firms as well as industrial firms.

1. Introduction

Differential equations have a great importance in our life and many applications in physics and engineering fields [1]. Differential equations help us in describing all the phenomena in which there are rates of change and provide a description of the way this change works, for example, population growth, chemical reactions, launching rockets into space, the spread of diseases, and climate changes. The differential game is a direct application of a differential equation and game theory. The game theory is an important field in mathematics [2]. It has applications in almost all fields of social science, as well as in logic, system science, and computer science. It has an important role in Economics. John Forbes Nash is one of the mathematicians who made fundamental contributions to the game theory. Nash proposed a solution of a noncooperative game including two or more players in which every player is assumed to know the equilibrium strategies of the players. No one can change the strategy or move without the other players knowing that the players have the same strategies. In the game theory, differential games are a group of problems related to the modeling and analysis of conflict (competition) expressed as a dynamical system [3]. It means here a state variable evolves over time according to a differential equation. A state variable is one of the set of variables that are used to describe the mathematical "state" of a dynamical system. The state of a system describes enough about the system to determine its future behaviour in the absence of any external forces affecting the system. There are many applications of differential game in our life and it is an important case ([4–8]).

In [9], Hemeda introduced an integral iterative method (IIM) as a modification for PM to solve nonlinear integrodifferential and systems of nonlinear integrodifferential equations.

In [10], Joseph presented a duopolistic market problem in which two firms sell the same product competitively in a certain time. Each firm has its own market share and the strategy here is one for the two firms, that is, the advertising efforts. Kristina solved the differential game by using the maximum principle with a general inequality constraints theorem. She used a numerical method for finding the solutions. In this work, we used another method and a theorem for finding the necessary conditions for an open-loop Nash equilibrium differential game. We found the approximate solutions using the Picard method, constructed graphs for the solutions, and making comparisons to the two firms simultaneously. We illustrated those comparisons with each graph.

The rest of the paper is organized as the following: In Section 2, the dynamical system of the problem, its payoff functionals, and the necessary conditions for an open-loop

Nash equilibrium are presented. The aim of Section 3 is to find an approximate solution using the Picard method. Section 4 is preserved for the discussion of the results. Finally, Section 5 contains a conclusion.

2. Problem Formulation

In this section, first we explain the dynamical system of the problem, the payoff functionals, and the open-loop Nash equilibrium.

2.1. The Dynamical System. We consider two firms, Firm 1 and Firm 2. The two firms sell the same product in the market competitively [10], where $x(t)$ is the market share of Firm 1 at time t, $1 - x(t)$ is the market share of Firm 2 at time t. $u_1(t)$ and $u_2(t)$ are the controls which defined as follows: $u_1(t)$ is the advertising efforts of Firm 1 at time t, and $u_2(t)$ is the advertising efforts of Firm 2 at time t.

In Firm 1, its number of customers increases by its advertising efforts. The advertising efforts of Firm 2 take away the customers of Firm 1. Therefore, the system dynamics can be expressed as the following:

$$\begin{aligned} \frac{dx}{dt} &= u_1(t)(1 - x(t)) - u_2(t)x(t), \\ x(0) &= x_0, \\ t &\in [0, T], \end{aligned} \tag{1}$$

with constraints given by

$$0 \leq x(t) \leq 1. \tag{2}$$

As we are concerned with a differential game with two players, then we have the following definition.

Definition 1 (2-players differential game). In the differential game of two players on the time interval $[t_0, t_f]$, we have the following:

(1) A set of players $N = 1, 2$
(2) For each player $i \in N$, there is a vector of controls $u_i(t) \in U_i \in R^{n_i}$, where U_i is the set of admissible control values for player i
(3) A vector of state variables $x \in X \subset R^n$, where X is the set of admissible states
(4) A strategy set Ψ_i, where the strategy $\psi_i \in \Psi_i$ is a decision rule that defines the control $u_i(t) \in U_i$ as a function of the information available at time t

2.2. Payoff Functionals and Open-Loop Nash Equilibrium. We consider the state equation which describes the state of the game and the payoff functionals as the following:

$$\frac{dx}{dt} = f(x(t), u_1(t), u_2(t), t), \tag{3}$$

$$J_i(u_1(t), u_2(t)) = \int_{t_0}^{t_f} I_i(x(t), u_1(t), u_2(t), t)dt; \quad i = 1, 2. \tag{4}$$

Since the information structure is open-loop, then the equilibrium strategy of player i will be $u_i^*(t), t \in [t_0, t_f], i = 1, 2$.

For obtaining these strategies, we have Hamiltonian's function for each player i.

$H_i(\lambda_i, x, u_1, u_2, t) = I_i(x(t), u_1(t), u_2(t), t) + \lambda_i^T f(x, u_1, u_2, t)$; $t \in [t_0, t_f], i = 1, 2$, where λ_i is the costate vector for player i.

Definition 2. In the 2-players differential game given by Definition 1 of a duration $[t_0, t_f]$, we say that player i's information structure is open-loop; if at time t, the only information available to player i is the initial state of the game x_0; hence, his strategy set can be written as $\Psi_i(t) = x_0, t \in [t_0, t_f]$.

Definition 3. If $J_1(u_1(t), u_2(t))$ and $J_2(u_1(t), u_2(t))$ are cost functionals for players 1, 2, respectively, then an ordered pair control (u_1^*, u_2^*) is a Nash equilibrium strategy if for each $i = 1, 2$, we have

$$J_i(u_1^*, u_2^*) \leq J_i(u_1, u_2^*). \tag{5}$$

For simplicity, the Nash equilibrium concept means that if one player tries to change his strategy from his own side, he cannot improve his own optimization criterion.

Now, we can define the payoff functionals of the two firms of our problem (1) and (2) as the following:

$$\begin{aligned} J_1 &= \int_0^T e^{-r_1 t}[\phi_1 x(t) - c_1 u_1(t)]dt, \\ J_2 &= \int_0^T e^{-r_2 t}[\phi_2(1 - x(t)) - c_2 u_2(t)]dt, \end{aligned} \tag{6}$$

such that $[t_0, t_f] = [0, T]$, r_i is the interest rate of Firm i, ϕ_i is the fractional revenue potential of Firm i, and $c_i(s)$ is the advertising cost function of the two firms.

Assuming that $c_i(s) = k_i s^2/2$; where k_i is a positive constant and

$$\begin{aligned} &f(x(t), u_1(t), u_2(t), t) = u_1(t)(1 - x(t)) - u_2(t)x(t), \\ &I_1(x(t), u_1(t), u_2(t), t) = \phi_1 x(t) - c_1(u_1(t)), \\ &I_2(x(t), u_1(t), u_2(t), t) = \phi_2(1 - x(t)) - c_2(u_2(t)). \end{aligned} \tag{7}$$

The Hamiltonian function of player 1 is

$$H_1(\lambda_1, x, u_1, u_2, t) = \phi_1 x(t) - \frac{k_1 u_1^2}{2} + \lambda_1 [u_1(t)(1 - x(t)) - u_2(t)x(t)]. \quad (8)$$

The Hamiltonian function of player 2 is

$$H_2(\lambda_2, x, u_1, u_2, t) = \phi_2 (1 - x(t)) - \frac{k_2 u_2^2}{2} + \lambda_2 [u_1(t)(1 - x(t)) - u_2(t)x(t)]. \quad (9)$$

Theorem 1. *Let $f(x(t), u_1(t), u_2(t), t)$ and $I_i(x(t), u_1(t), u_2(t), t)$, $i = 1, 2$ be continuously differentiable on R^n i.e., $f(x(t), u_1(t), u_2(t), t)$: $R^n \times R^s \times [0, T] \longrightarrow R$, $f \in C^1, s = \Sigma_{j=1}^N s_j, i \neq j$, $I_i(x(t), u_1(t), u_2(t), t)$: $R^n \times R^s \times [0, T] \longrightarrow R, I_i \in C^1, i = 1, 2$. If $u_i^*(t), t_0 \leq t \leq t_f$ is an open-loop Nash-equilibrium solution and $x^*(t), t_0 \leq t \leq t_f$ be the corresponding state trajectory, then there exists 2 costate vectors λ_i: $[t_0, t_f] \longrightarrow R^n$, and 2-Hamiltonian functions $H_i(\lambda_i, x, u_1, u_2, t) = I_i(x(t), u_1(t), u_2(t), t) + \lambda_i^T f(x, u_1, u_2, t)$ such that the following conditions are satisfied:*

$$\begin{aligned} &\frac{dx^*}{dt} = f(x^*(t), u_1^*(t), u_2^*(t), t), \quad x^*(0) = x_0, \\ &\frac{d\lambda_i(t)}{dt} = -\frac{\partial H_i(\lambda_i, x^*, u_1^*, u_2^*, t)}{\partial x}, \quad i = 1, 2, \\ &\frac{\partial H_1(\lambda_1, x^*, u_1^*, u_2^*, t)}{\partial u_1} = 0, \\ &\frac{\partial H_2(\lambda_2, x^*, u_1^*, u_2^*, t)}{\partial u_2} = 0, \end{aligned} \quad (10)$$

with boundary conditions,

$$\begin{aligned} &x^*(0) = x_0, \\ &\lambda_i(t_f) = 0, \\ &i = 1, 2, \end{aligned} \quad (11)$$

where $u_i^*, i = 1, 2$, is an open-loop Nash equilibrium strategy, and x^* is the corresponding equilibrium state trajectory.

The proof of the thereom is proved in [11].

3. The Approximate Solution by Using the Picard Method

The purpose of this section is to find the existence and convergence of the solution for the problem.

3.1. Existence and Convergence of the Solution. Now, we study the existence of the solution for the problem (10)–(11) (see [12, 13]) and we apply the Picard method for finding an approximate solution and studying the convergence of this solution.

Consider the following system after reducing the necessary conditions of an open-loop Nash differential game as the following:

$$\begin{aligned} &\frac{dx}{dt} = f_1(x, \lambda_1, \lambda_2, t), \\ &\frac{d\lambda_1}{dt} = f_2(x, \lambda_1, \lambda_2, t) + \phi_1, \\ &\frac{d\lambda_2}{dt} = f_3(x, \lambda_1, \lambda_2, t) - \phi_2, \\ &x(0) = x_0, \\ &\lambda_1(T) = 0, \\ &\lambda_2(T) = 0. \end{aligned} \quad (12)$$

Assume these assumptions for our problem.

(1) $f_i(x(t), \lambda_1(t), \lambda_2(t), t)$: $R^n \times R^n \times R^n \times [0, T] \longrightarrow R$ are continuous and there are positive constants M_i s.t $|f_i| \leq M_i, i = 1, 2, 3$.

(2) f_i satisfies Lipschitz condition with Lipschitz constants $L_i, 0 < L_i < 1, i = 1, 2, 3$ such that

$$\begin{aligned} &|f_1(x, \lambda_1, \lambda_2, t) - f_1(y, \lambda_1, \lambda_2, t)| \leq L_1 |x - y|, \\ &|f_2(x, \lambda_1, \lambda_2, t) - f_2(x, p_1, \lambda_2, t)| \leq L_2 |\lambda_1 - p_1|, \\ &|f_3(x, \lambda_1, \lambda_2, t) - f_3(x, \lambda_1, p_2, t)| \leq L_3 |\lambda_2 - p_2|. \end{aligned} \quad (13)$$

We prove the existence of the solution of system (12), then we have to integrate the differential equation in (12), we get

$$x(t) = x_0 + \int_0^t f_1(x, \lambda_1, \lambda_2, t) dt, \quad (14)$$

$$\lambda_1(t) = \phi_1(t - T) + \int_T^t f_2(x, \lambda_1, \lambda_2, t) dt, \quad (15)$$

$$\lambda_2(t) = -\phi_1(t - T) + \int_T^t f_3(x, \lambda_1, \lambda_2, t) dt. \quad (16)$$

By differentiating the integral equations (14)–(16) with respect to t, we have

$$\frac{dx}{dt} = f_1(x,\lambda_1,\lambda_2,t),$$
$$\frac{d\lambda_1}{dt} = f_2(x,\lambda_1,\lambda_2,t) + \phi_1, \quad (17)$$
$$\frac{d\lambda_2}{dt} = f_3(x,\lambda_1,\lambda_2,t) - \phi_2.$$

Substituting $t = 0$ in (14), $t = T$ in (15) and (16), we obtain

$$\begin{aligned} x(0) &= x_0 + \int_0^0 f_1(x,\lambda_1,\lambda_2 dt, t) = x_0,\\ \lambda_1(T) &= \phi_1(T-T) + \int_T^T f_2(x,\lambda_1,\lambda_2,t)dt = 0,\\ \lambda_2(T) &= -\phi_1(T-T) + \int_T^T f_3(x,\lambda_1,\lambda_2 dt,t) = 0. \end{aligned} \quad (18)$$

Hence, the existence is proved from the equivalence between the system (12) and the integral equations ((14)–(16)). Therefore, there is a solution for our system.

Now, we apply the Picard method to find the solutions of the integral equations (14)–(16). The solution is constructed by the sequences,

$$\begin{aligned} x_n(t) &= x_0 + \int_0^t f_1(x_{n-1},(\lambda_1)_{n-1},(\lambda_2)_{n-1},t)dt,\\ x(0) &= x_0,\\ (\lambda_1(t))_n &= \phi_1(t-T) + \int_T^t f_2(x_{n-1},(\lambda_1)_{n-1},(\lambda_2)_{n-1},t)dt,\\ (\lambda_1)_0 &= 0,\\ (\lambda_2(t))_n &= -\phi_2(t-T) + \int_T^t f_3(x_{n-1},(\lambda_1)_{n-1},(\lambda_2)_{n-1},t)dt,\\ (\lambda_2)_0 &= 0, n = 1,2,3,\ldots. \end{aligned} \quad (19)$$

$x_n(t)$, $(\lambda_1(t))_n$, $(\lambda_2(t))_n$ can be written as the following:

$$\begin{aligned} x_n(t) &= x_0 + \sum_{j=1}^{n}(x_j - x_{j-1}),\\ (\lambda_1(t))_n &= \phi_1(t-T) + \sum_{j=1}^{n}((\lambda_1)_j - (\lambda_1)_{j-1}),\\ (\lambda_2(t))_n &= -\phi_2(t-T) + \sum_{j=1}^{n}((\lambda_2)_j - (\lambda_2)_{j-1}). \end{aligned} \quad (20)$$

If x_n, $(\lambda_1)_n$, and $(\lambda_2)_n$ are convergent, then the infinite series $\sum(x_j - x_{j-1}), \sum((\lambda_1)_j - (\lambda_1)_{j-1})$ and $\sum((\lambda_2)_j - (\lambda_2)_{j-1})$ are convergent, and the solution will be x, λ_1, and λ_2, where

$$\begin{aligned} x(t) &= \lim_{n\longrightarrow\infty} x_n,\\ \lambda_1(t) &= \lim_{n\longrightarrow\infty}(\lambda_1)_n,\\ \lambda_2(t) &= \lim_{n\longrightarrow\infty}(\lambda_2)_n. \end{aligned} \quad (21)$$

If the three series are converged, then the three sequences x_n, $(\lambda_1)_n$ and $(\lambda_2)_n$ will be converged, respectively, to $x(t), \lambda_1(t)$ and $\lambda_2(t)$. For proving the uniform convergence of x_n, $(\lambda_1)_n$ and $(\lambda_2)_n$, we have to consider the three associated series,

$$\begin{aligned} &\sum_{n=1}^{\infty}(x_n - x_{n-1}),\\ &\sum_{n=1}^{\infty}((\lambda_1)_n - (\lambda_1)_{n-1}),\\ &\sum_{n=1}^{\infty}((\lambda_2)_n - (\lambda_2)_{n-1}). \end{aligned} \quad (22)$$

For $n = 1$, we get

$$\begin{aligned} x_1 - x_0 &= \int_0^t f_1(x_0,(\lambda_1)_0,(\lambda_2)_0,t)dt,\\ |x_1 - x_0| &= \left|\int_0^t f_1(x_0,(\lambda_1)_0,(\lambda_2)_0,t)dt\right|,\\ |x_1 - x_0| &\le M_1 T,\\ (\lambda_1)_1 - (\lambda_1)_0 &= \phi_1(t-T) + \int_T^t f_2(x_0,(\lambda_1)_0,(\lambda_2)_0,t)dt,\\ |(\lambda_1)_1 - (\lambda_1)_0| &\le (\phi_1 + M_2)|t-T|,\\ |(\lambda_1)_1 - (\lambda_1)_0| &\le (\phi_1 + M_2)T,\\ (\lambda_2)_1 - (\lambda_2)_0 &= -\phi_2(t-T) + \int_T^t f_3(x_0,(\lambda_1)_0,(\lambda_2)_0,t)dt,\\ |(\lambda_2)_1 - (\lambda_2)_0| &\le (\phi_2 + M_3)|t-T|,\\ |(\lambda_2)_1 - (\lambda_2)_0| &\le (\phi_2 + M_3)T. \end{aligned} \quad (23)$$

Now, we shall get an estimation for $x_n - x_{n-1}$, $(\lambda_1)_n - (\lambda_1)_{n-1}$, and $(\lambda_2)_n - (\lambda_2)_{n-1}$

$$\begin{aligned}
&\left|x_n - x_{n-1}\right| = \left|\int_0^t f_1(x_{n-1},(\lambda_1)_{n-1},(\lambda_2)_{n-1},t)\mathrm{d}t - \int_0^t f_1(x_{n-2},(\lambda_1)_{n-2},(\lambda_2)_{n-2},t)\mathrm{d}t\right|,\\
&\left|x_n - x_{n-1}\right| \le L_1 T\left|x_{n-1} - x_{n-2}\right|,\\
&\left|(\lambda_1)_n - (\lambda_1)_{n-1}\right| = \left|\int_T^t f_2(x_{n-1},(\lambda_1)_{n-1},(\lambda_2)_{n-1},t)\mathrm{d}t - \int_T^t f_2(x_{n-2},(\lambda_1)_{n-2},(\lambda_2)_{n-2},t)\mathrm{d}t\right|,\\
&\left|(\lambda_1)_n - (\lambda_1)_{n-1}\right| \le L_2 T\left|(\lambda_1)_{n-1} - (\lambda_1)_{n-2}\right|,\\
&\left|(\lambda_2)_n - (\lambda_2)_{n-1}\right| = \left|\int_T^t f_3(x_{n-1},(\lambda_1)_{n-1},(\lambda_2)_{n-1},t)\mathrm{d}t - \int_T^t f_3(x_{n-2},(\lambda_1)_{n-2},(\lambda_2)_{n-2},t)\mathrm{d}t\right|,\\
&\left|(\lambda_2)_n - (\lambda_2)_{n-1}\right| \le L_3 T\left|(\lambda_2)_{n-1} - (\lambda_2)_{n-2}\right|.
\end{aligned} \tag{24}$$

By using the first estimation in (24) and putting $n = 2$, we get

$$\begin{aligned}
&\left|x_2 - x_1\right| \le L_1 T\left|x_1 - x_0\right|,\\
&\left|x_2 - x_1\right| \le L_1 M_1 T^2.
\end{aligned} \tag{25}$$

At $n = 3$, we have

$$\begin{aligned}
&\left|x_3 - x_2\right| \le L_1 T\left|x_2 - x_1\right|,\\
&\left|x_3 - x_2\right| \le L_1^2 M_1 T^3,
\end{aligned} \tag{26}$$

and so on

$$\left|x_n - x_{n-1}\right| \le L_1^{n-1} M_1 T^n. \tag{27}$$

By using the second estimation in (24), we have at $n = 2$

$$\begin{aligned}
&\left|(\lambda_1)_2 - (\lambda_1)_1\right| \le L_2 T\left|(\lambda_1)_1 - (\lambda_1)_0\right|,\\
&\left|(\lambda_1)_2 - (\lambda_1)_1\right| \le L_2(\phi_1 + M_2)T^2.
\end{aligned} \tag{28}$$

At $n = 3$

$$\left|(\lambda_1)_3 - (\lambda_1)_2\right| \le L_2^2(\phi_1 + M_2)T^3, \tag{29}$$

and so on

$$\left|(\lambda_1)_n - (\lambda_1)_{n-1}\right| \le L_2^{n-1}(\phi_1 + M_2)T^n. \tag{30}$$

By using the third estimation in (24), we get at $n = 2$

$$\begin{aligned}
&\left|(\lambda_2)_2 - (\lambda_2)_1\right| \le L_3 T\left|(\lambda_2)_1 - (\lambda_2)_0\right|,\\
&\left|(\lambda_2)_2 - (\lambda_2)_1\right| \le L_3(\phi_2 + M_3)T^2.
\end{aligned} \tag{31}$$

At $n = 3$,

$$\left|(\lambda_2)_3 - (\lambda_2)_2\right| \le L_3^2(\phi_2 + M_3)T^3, \tag{32}$$

and so on

$$\left|(\lambda_2)_n - (\lambda_2)_{n-1}\right| \le L_3^{n-1}(\phi_2 + M_3)T^n. \tag{33}$$

Since $L_i < 1$ such that $i = 1,2,3$ and $T < 1$, then $\sum_{n=1}^{\infty}(x_n - x_{n-1})$, $\sum_{n=1}^{\infty}((\lambda_1)_n - (\lambda_1)_{n-1})$ and $\sum_{n=1}^{\infty}((\lambda_2)_n - (\lambda_2)_{n-1})$ are uniformly convergent, and thus the sequences $x_n(t)$, $(\lambda_1(t))_n$ and $(\lambda_2(t))_n$ are uniformly convergent.

Now, we can apply the Picard method to obtain an approximate solution for our problem.

From Theorem 1, we get the system of the problem as the following:

$$\begin{aligned}
&\frac{dx}{dt} = u_1(1-x) - u_2 x,\\
&\frac{d\lambda_1}{dt} = -\lambda_1 u_1 - \lambda_1 u_2 + \phi_1,\\
&\frac{\lambda_2}{dt} = -\lambda_2 u_1 - \lambda_2 u_2 - \phi_2,\\
&u_1 = \frac{\lambda_1}{k_1}(1-x),\\
&u_2 = \frac{-\lambda_2}{k_2}x,\\
&x(0) = x_0,\\
&\lambda_1(T) = 0,\\
&\lambda_2(T) = 0.
\end{aligned} \tag{34}$$

After reducing these equations, we obtain

$$\begin{aligned}
&\frac{dx}{dt} = \frac{\lambda_1}{k_1}(1-x)^2 + \frac{\lambda_2}{k_2}x^2,\\
&\frac{d\lambda_1}{dt} = \frac{-\lambda_1^2}{k_1}(1-x) + \frac{\lambda_1\lambda_2}{k_2}x + \phi_1,\\
&\frac{d\lambda_2}{dt} = -\frac{\lambda_1\lambda_2}{k_1}(1-x) + \frac{-\lambda_2^2}{k_2}x - \phi_2,\\
&x(0) = x_0,\\
&\lambda_1(T) = 0,\\
&\lambda_2(T) = 0.
\end{aligned} \tag{35}$$

By integrating the differential equations in (35), we have the following system:

$$x(t) = x_0 + \int_0^t\left[\frac{\lambda_1}{k_1} - \frac{2\lambda_1}{k_1}x + \left(\frac{\lambda_1}{k_1} + \frac{\lambda_2}{k_2}\right)x^2\right]dt, \tag{36}$$

$$\lambda_1(t) = \int_T^t\left[\frac{-\lambda_1^2}{k_1} + \frac{\lambda_1^2}{k_1}x + \frac{\lambda_1\lambda_2}{k_2}x + \phi_1\right]dt, \tag{37}$$

$$\lambda_2(t) = \int_T^t \left[-\frac{\lambda_1 \lambda_2}{k_1} + \frac{\lambda_1 \lambda_2}{k_1} x + \frac{\lambda_2^2}{k_2} x - \phi_2 \right] dt. \quad (38)$$

By applying the Picard method to equations (36)–(38), we have

$$\begin{aligned}
x_n(t) &= x_0 + \int_0^t \left[\frac{(\lambda_1)_{n-1}}{k_1} - \frac{2(\lambda_1)_{n-1}}{k_1} x_{n-1} + \left(\frac{(\lambda_1)_{n-1}}{k_1} + \frac{(\lambda_2)_{n-1}}{k_2} \right) x_{n-1}^2 \right] dt,\\
(\lambda_1(t))_n &= \int_T^t \left[\frac{-(\lambda_1)_{n-1}^2}{k_1} + \frac{(\lambda_1)_{n-1}^2}{k_1} x_{n-1} + \frac{(\lambda_1)_{n-1}(\lambda_2)_{n-1}}{k_2} x_{n-1} + \phi_1 \right] dt,\\
(\lambda_2(t))_n &= \int_T^t \left[-\frac{(\lambda_1)_{n-1}(\lambda_2)_{n-1}}{k_1} + \frac{(\lambda_1)_{n-1}(\lambda_2)_{n-1}}{k_1} x_{n-1} + \frac{(\lambda_2)_{n-1}^2}{k_2} x_{n-1} - \phi_2 \right] dt,\\
x(0) &= x_0,\\
(\lambda_1)_0 &= 0,\\
(\lambda_2)_0 &= 0.
\end{aligned} \quad (39)$$

For $n = 1$, we have

$$\begin{aligned}
x_1(t) &= x_0 + \int_0^t \left[\frac{(\lambda_1)_0}{k_1} - \frac{2(\lambda_1)_0}{k_1} x_0 + \left(\frac{(\lambda_1)_0}{k_1} + \frac{(\lambda_2)_0}{k_2} \right) x_0^2 \right] dt,\\
(\lambda_1(t))_1 &= \int_T^t \left[\frac{-(\lambda_1)_0^2}{k_1} + \frac{(\lambda_1)_0^2}{k_1} x_0 + \frac{(\lambda_1)_0(\lambda_2)_0}{k_2} x_0 + \phi_1 \right] dt,\\
(\lambda_2(t))_1 &= \int_T^t \left[-\frac{(\lambda_1)_0(\lambda_1)_0}{k_1} + \frac{(\lambda_1)_0(\lambda_2)_0}{k_1} x_0 + \frac{(\lambda_2)_0^2}{k_2} x_0 - \phi_2 \right] dt,\\
x(0) &= x_0,\\
(\lambda_1)_0 &= 0,\\
(\lambda_2)_0 &= 0.
\end{aligned} \quad (40)$$

Therefore, the first approximation for x, λ_1 and λ_2 is the following:

$$\begin{aligned}
x_1(t) &= x_0,\\
(\lambda_1(t))_1 &= \int_T^t \phi_1 \mathrm{d}t = \phi_1(t - T),\\
(\lambda_2(t))_1 &= \int_T^t -\phi_2 \mathrm{d}t = \phi_2(t - T).
\end{aligned} \quad (41)$$

Hence, the first approximation of u_1, u_2, c_1, and c_2 is the following:

$$\begin{aligned}
(u_1)_1 &= \frac{\phi_1(t - T)}{k_1}(1 - x_0),\\
(u_2)_1 &= \frac{-\phi_2(t - T)}{k_2}(1 - x_0),\\
(c_1)_1 &= \frac{k_1}{2}\left(\frac{\phi_1(t - T)}{k_1}(1 - x_0) \right)^2,\\
(c_2)_1 &= \frac{k_2}{2}\left(\frac{-\phi_2(t - T)}{k_2}(1 - x_0) \right)^2.
\end{aligned} \quad (42)$$

For $n = 2$, we have

$$
\begin{aligned}
x_2(t) &= x_0 + \int_0^t \left[\frac{(\lambda_1)_1}{k_1} - \frac{2(\lambda_1)_1}{k_1} x_1 + \left(\frac{(\lambda_1)_1}{k_1} + \frac{(\lambda_2)_1}{k_2} \right) x_1^2 \right] \mathrm{d}t, \\
(\lambda_1(t))_2 &= \int_T^t \left[\frac{-(\lambda_1)_1^2}{k_1} + \frac{(\lambda_1)_1^2}{k_1} x_1 + \frac{(\lambda_1)_1 (\lambda_2)_1}{k_2} x_1 + \phi_1 \right] \mathrm{d}t, \\
(\lambda_2(t))_2 &= \int_T^t \left[-\frac{(\lambda_1)_1 (\lambda_2)_1}{k_1} + \frac{(\lambda_1)_1 (\lambda_2)_1}{k_1} x_1 + \frac{(\lambda_2)_1^2}{k_2} x_1 - \phi_2 \right] \mathrm{d}t.
\end{aligned}
\tag{43}
$$

Therefore, the second approximation for x, λ_1 and λ_2 is the following:

$$
\begin{aligned}
x_2(t) &= x_0 - \frac{t^2 \phi_1}{2k_1} + \frac{tT\phi_1}{k_1} + \frac{t^2 x_0^2 \phi_1}{2k_1} - \frac{tTx_0^2\phi_1}{k_1} - \frac{t^2 x_0^2 \phi_2}{2k_2} + \frac{tTx_0^2\phi_2}{k_2}, \\
(\lambda_1(t))_2 &= \frac{\phi_1\left((1/3)(t-T)^3 k_2(-1+x_0)\phi_1 + k_1\left((t-T)k_2 - (1/3)(t-T)^3 x_0\phi_2\right)\right)}{k_1 k_2}, \\
(\lambda_2(t))_2 &= \frac{\phi_2\left(-(1/3)(t-T)^3 k_2(-1+x_0)\phi_1 + k_1\left(-tk_2 + Tk_2 + (1/3)(t-T)^3 x_0\phi_2\right)\right)}{k_1 k_2}.
\end{aligned}
\tag{44}
$$

Hence, the second approximation of $u_1, u_2, c_1,$ and c_2 is the following:

$$
\begin{aligned}
(u_1)_2 &= \frac{\phi_1\left((1/3)(t-T)^3 k_2(-1+x_0)\phi_1 + k_1\left((t-T)k_2 - (1/3)(t-T)^3 x_0\phi_2\right)\right)}{k_1^2 k_2} \left(1 - \left(x_0 - \frac{t^2\phi_1}{2k_1} + \frac{tT\phi_1}{k_1} + \frac{t^2x_0^2\phi_1}{2k_1} - \frac{tTx_0^2\phi_1}{k_1} - \frac{t^2x_0^2\phi_2}{2k_2} + \frac{tTx_0^2\phi_2}{k_2}\right)\right), \\
(u_2)_2 &= \frac{\phi_2\left((-1/3)(t-T)^3 k_2(-1+x_0)\phi_1 + k_1\left(-tk_2 + Tk_2 + (1/3)(t-T)^3 x_0\phi_2\right)\right)}{k_1^2 k_2} \left(x_0 - \frac{t^2\phi_1}{2k_1} + \frac{tT\phi_1}{k_1} + \frac{t^2x_0^2\phi_1}{2k_1} - \frac{tTx_0^2\phi_1}{k_1} - \frac{t^2x_0^2\phi_2}{2k_2} + \frac{tTx_0^2\phi_2}{k_2}\right), \\
(c_1)_2 &= \left(\frac{\phi_1\left((1/3)(t-T)^3 k_2(-1+x_0)\phi_1 + k_1\left((t-T)k_2 - (1/3)(t-T)^3 x_0\phi_2\right)\right)}{k_1^2 k_2} \left(1 - \left(x_0 - \frac{t^2\phi_1}{2k_1} + \frac{tT\phi_1}{k_1} + \frac{t^2x_0^2\phi_1}{2k_1} - \frac{tTx_0^2\phi_1}{k_1} - \frac{t^2x_0^2\phi_2}{2k_2} + \frac{tTx_0^2\phi_2}{k_2}\right)\right)\right), \\
(c_2)_2 &= \frac{k_2}{2}\left(-\frac{\phi_2\left((-1/3)(t-T)^3 k_2(-1+x_0)\phi_1 + k_1\left(tk_2 + Tk_2 + (1/3)(t-T)^3 x_0\phi_2\right)\right)}{k_1 k_2^2} \left(x_0 - \frac{t^2\phi_1}{2k_1} + \frac{tT\phi_1}{k_1} + \frac{t^2x_0^2\phi_1}{2k_1} - \frac{tTx_0^2\phi_1}{k_1} - \frac{t^2x_0^2\phi_2}{2k_2} + \frac{tTx_0^2\phi_2}{k_2}\right)\right)^2.
\end{aligned}
\tag{45}
$$

By putting $x_0 = 0, \phi_1 = 0.1, \phi_2 = 0.3, k_1 = 0.25, k_2 = 0.5,$ and $T = 0.5$, then we have a comparison between the obtained approximate solutions of the two firms in our problem in the following figures:

Figure 1 indicates the approximated optimal solutions of the state for our problem. In our example x and $1 - x$ represented, respectively, the market share of Firm 1 and Firm 2 at time t, and from this figure, we found that the market share of Firm 1 is increasing and the market share of Firm 2 is decreasing with the time, and notice that Firm 2 has a greater market share than Firm 1 on the interval $t \in [0, 0.5]$.

In Figure 2, we show that the approximated optimal controls to the problem. In the problem, the controls u_1 and u_2 represented, respectively, the advertising efforts of Firm 1 and Firm 2 at time t, and from this figure, we noticed that the advertising efforts of Firm 1 is decreasing and the advertising efforts of Firm 2 increases until it reaches a certain time and then begins to decrease again. We found that Firm 1 has a greater advertising effort than Firm 2 on the interval $t \in [0, 0.5]$.

In Figure 3, we explore the approximated solutions of the advertising cost function to the problem. In the problem, c_1 and c_2 represented respectively the advertising cost function of Firm 1 and Firm 2 at time t, and from this figure we noticed that the advertising cost function of Firm 1 is decreasing and the advertising cost function of Firm 2 is

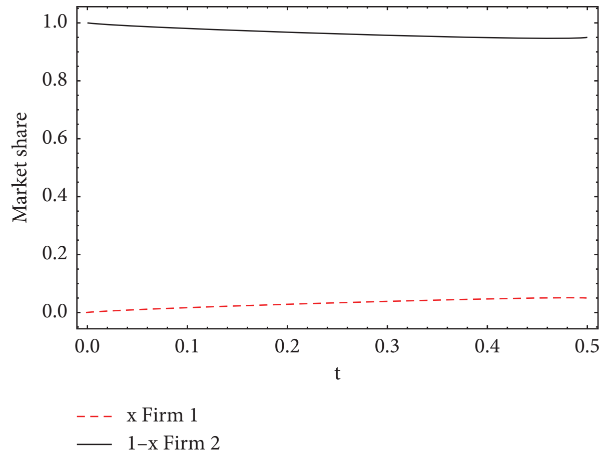

Figure 1: Market share of firm 1 and firm 2 at time t.

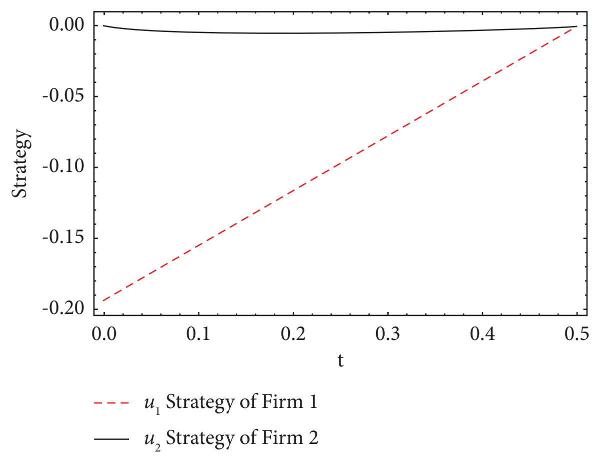

Figure 2: The advertising efforts of firm 1 and firm 2 at the time t.

marginally increasing. We found that Firm 1 has a greater advertising cost function than Firm 2 on the interval $t \in [0, 0.5]$.

In Figure 4, we present the approximated optimal solutions of the costate variables to the problem. The costate here represents the marginal value of the market share. In other words, the adjoint for this problem is the rate of change in the payoff for small changes in the market share. From this figure we see that in Firm 1, λ_1 increases while λ_2 decreases in Firm 2 on the time interval $t \in [0, 0.5]$.

4. Discussion

In this study, we discussed the competing between two firms in the market.

(1) Firm 2 is existing in the market and there is no competitor to it, and its market share is $1 - x(t)$. Firm 1 entered the market to compete this firm, and its market share is $x(t)$.

(2) Firm 1 started to make an advertising campaign by getting a loan. It earned profits and increased its sales.

(3) Firm 2 had no advertisements, while the number of customers of the new firm increased and so, its market share increased; therefore, the new firm controlled the market.

(4) Firm 2 started to lose and its sales decreased. It got a loan for doing advertisements because of its loss.

(5) The cost of Firm 1 was too much and after increasing its sales, the cost started to decrease and also the advertisements started to increase from the negative value. This means that debts decrease with time. After a specific time, the advertisements of the old firm started to increase and this increase is almost slight, so the cost seems constant.

(6) Finally, we used an iterative method (Picard method) for finding the solution and proved the convergence

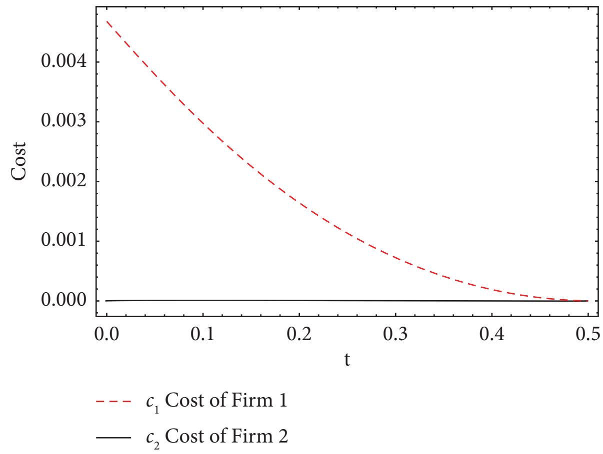

Figure 3: The advertising cost function of firm 1 and firm 2 at the time t.

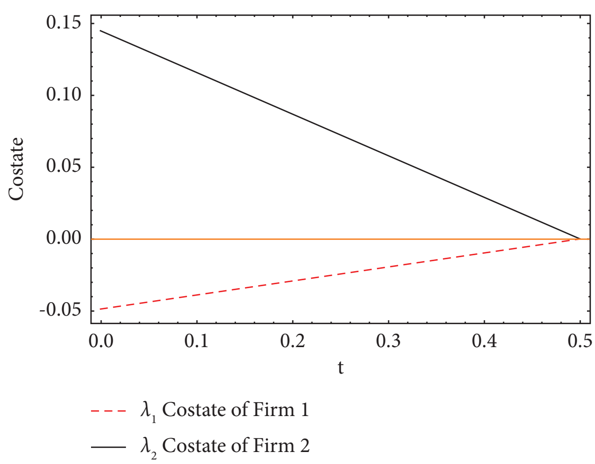

Figure 4: The costate variables of firm 1 and firm 2 and the time t.

of the solution. On the other hand, Kristina used a numerical algorithm for the solution (Forward-Backward sweep algorithm), but she did not determine the solution [10].

5. Conclusion

In this paper, we concerned with an open-loop Nash differential game. We proved that there is a solution for the problem, we studied the existence for the solution of the dynamical system of the differential game, and we used the Picard method for finding an approximate solution. Also, we studied the convergence for the Picard method to make sure that the approximate solution is uniformly convergent. Finally, we added the figures to compare between the behavior of the two firms in the problem with respect to market share, the advertising efforts, the advertising cost function, and the costate variables. We proved the convergence at $T < 1$, so we chose the interval [0, 0.5]. If we reach to $T = 1$, the solution will be divergent.

Data Availability

No new data were collected or generated for this article.

Conflicts of Interest

All authors declare that they have no conflicts of interest in this paper.

Authors' Contributions

A. A. Megahed proposed the idea of the manuscript and put the headline, H. F. A. Madkour wrote the manuscript text, A. A. Megahed, and H. F. A. Madkour studied all the analysis in the manuscript. Finally, A. A. Hemeda reviewed the manuscript.

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