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Research Article

Asymptotic Stability of Global Solutions to Non-isentropic Navier–Stokes Equations

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This paper studies the asymptotic stability of global solutions of the three-dimensional nonisentropic compressible Navier–Stokes equations, where the initial data satisfy the "well-prepared" initial conditions, and the velocity field and temperature satisfy the Dirichlet boundary condition and convective boundary condition, respectively, based on the incompressible limit of global solutions. With the uniform estimates with respect to both the Mach number ε and time t, we prove the exponentially asymptotic stability for global solutions of both the compressible Navier–Stokes equations and its limiting incompressible equations.

1. Introduction

In recent years, hydrodynamic equations have received considerable attention, in which Navier–Stokes equations [1] describing the motion of viscous fluids are the basic mathematical models of the hydrodynamic equations. Navier–Stokes equations have attracted much more attention due to its important theoretical and applied value in physics and mathematics.

The motions of highly subsonic viscous fluids are described by the following nondimensionalized Navier–Stokes equations:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{di} \mathrm{vS} + \frac{1}{\varepsilon^2}\nabla p = 0, \tag{2}$$

$$(\rho e)_t + \operatorname{div}(\rho \mathrm{ue}) + \mathrm{pdi}\,\mathrm{vu} - \operatorname{div}(\kappa \nabla \Gamma) = \varepsilon^2 S \cdot D(u), \tag{3}$$

The equations represent the conservation of mass, momentum, and energy, respectively. In addition, $\rho, u = (u^1, u^2, u^3), p, e, \Gamma$ indicate density, velocity, pressure, internal energy, and temperature, respectively. The matrix $S = 2\mu D(u) + \zeta \mathrm{di}\,\mathrm{vuI}$ is the viscous stress tensor, with $2D(u) = \nabla u + \nabla u^t$ (t denotes the transpose operation); the viscous coefficients μ, ζ are constants with $\mu > 0, \mu + 3\zeta/2 > 0$; ε is the Mach number; κ is the heat conductivity coefficient. In this paper, we consider the Navier–Stokes equations in a bounded domain $\Omega \in R^3$ and assume that the fluids being studied are perfect gases:

$$e = C_V \Gamma,\ p = R\rho\Gamma, \tag{4}$$

where $C_V > 0$ represents the specific heat capacity at constant volume, and R denotes the universal gas constant, and $\gamma = 1 + R/C_V$ is the ratio of specific heat capacities at constant pressure and constant volume.

As we know, Mach number is an important physical quantity to describe the compressibility of fluid. From the physical point of view, when the Mach number is less than approximately 0.3, the motions of compressible fluid are similar to incompressible fluid when Mach number vanishes. Strictly verifying this process from a mathematical point of view is the incompressible limit problem. However, the rigorous justification of the incompressible limit problem of Navier–Stokes equations poses challenging problems mathematically since singular phenomena usually occur in this process, as shown in [2–12].

In addition, many results on the incompressible limit of the nonisentropic Navier–Stokes equations have been achieved. For the local solutions, Dou et al. [13] studied the incompressible limit of local strong solutions to Navier–Stokes equations with positive thermal conductivity coefficient and the slip boundary condition. For the global solutions, Ren and Ou [14] proved the incompressible limit of global solutions to nonisentropic Navier–Stokes equations in the 3-D bounded domain where the velocity field and temperature satisfy Dirichlet boundary condition and convective boundary condition, respectively. The research results of the incompressible limit of nonisentropic Navier–Stokes equations describing perfect gas can be found in [15–20].

The stability conclusion of the hydrodynamic equations has important theoretical support for engineering technology and other fields, so it has attracted the attention of many mathematicians. In particular, the stability of solutions can be effectively demonstrated through the use of the incompressible limit, which lends greater significance to research.

Through the in-depth research of Navier–Stokes equations in the hydrodynamic equations, we find that the stability of solutions of Navier–Stokes equations has received much more attention, as shown in [21–25]. For the stability of solutions to nonisentropic Navier–Stokes equations, Xie and Li [26] showed that the equilibrium solutions of Navier–Stokes–Poisson equations are nonlinearly stable with $4/3 < \gamma < 2$; its equilibrium solutions are unstable with $\gamma = 4/3$, where γ is the adiabatic coefficient.

Through analyzing the current research status at home and abroad, it is found that in practical applications, exploring the stability of the solution of the hydrodynamic equations using the incompressible limit of the solution also has important research value. We also found that previous research on the Navier–Stokes equation system is still incomplete. Therefore, this article mainly studies the exponentially asymptotic stability problem of the Navier–Stokes equation system.

Currently, some conclusions about the stability of solutions of Navier–Stokes equations have been proved based on the incompressible limit. For nonisentropic Navier–Stokes equations with "well-prepared" initial data in the 3-D bounded domain Ω, Ren and Ou [27] proved the incompressible limit of the global strong solutions when the following boundary condition holds:

$$u \cdot n = 0, \tau \cdot (D(u) \cdot n) + \alpha u \cdot \tau = 0, \frac{\partial \theta}{\partial n} + \beta\theta = 0 \text{ on } \partial\Omega \times (0, T), \tag{5}$$

and the exponentially asymptotic stability of the global solutions based on the conclusion of incompressible limit is proved. Note that when the initial condition is the same as [27], the boundary conditions are different as follows:

$$u = 0, \frac{\partial \theta}{\partial n} + \beta\theta = 0, \text{on } \partial\Omega \times (0, T), \tag{6}$$

the exponentially asymptotic stability of solutions of three-dimensional Navier–Stokes equations has not been studied.

In this paper, we consider the low Mach number fluid which can be regarded as a perturbation near the incompressible fluid, where the density and temperature are usually set to be constants. Therefore, we set the density and temperature variations by σ and θ, respectively, as follows:

$$\rho = 1 + \varepsilon\sigma, \Gamma = 1 + \varepsilon\theta. \tag{7}$$

In this way, nondimensional systems (1)–(4) can be rewritten as follows:

$$\sigma_t + \operatorname{div}(\sigma u) + \frac{1}{\varepsilon}\operatorname{divu} = 0, \tag{8}$$

$$\rho(u_t + u \cdot \nabla u) + \frac{R}{\varepsilon}(\nabla\sigma + \nabla\theta) + R\nabla(\sigma\theta) = 2\mu \operatorname{div}(D(u)) + \zeta\nabla \operatorname{divu}, \tag{9}$$

$$C_V \rho(\theta_t + u \cdot \nabla\theta) + R(\rho\theta + \sigma)\operatorname{divu} + \frac{R}{\varepsilon}\operatorname{divu} = \kappa\Delta\theta + \varepsilon\left(2\mu|D(u)|^2 + \zeta(\operatorname{divu})^2\right). \tag{10}$$

We impose that u satisfies Dirichlet boundary conditions and θ satisfies convective boundary conditions

$$u = 0, \frac{\partial \theta}{\partial n} + \beta\theta = 0 \text{ on } \partial\Omega \times (0, T), \tag{11}$$

and the initial conditions

$$(\sigma, u, \theta)|_{t=0} = (\sigma_0, u_0, \theta_0)(x) \text{ in } \Omega, \tag{12}$$

where β is a positive constant and n is the unit outer normal vector to $\partial\Omega$.

This paper will discuss the proof process of stability conclusions for the global solutions of the 3D nonisentropic compressible Navier–Stokes equations in a bounded domain. In particular, the local existence of the solution (ρ, u, θ) has been established by Ren and Ou [14], and based on the uniform energy estimate in [14], we establish the exponentially asymptotic stability for global solutions of both the compressible Navier–Stokes equations and their limiting incompressible equations with the help of classical theory on the Stokes problem.

2. Preliminaries

The nondimensional nonisentropic compressible Navier–Stokes equations (8)–(10) have been given, and we now give some commonly used relations and inequalities.

During the energy estimates, the following relationship is regularly used:

$$\Delta u = \nabla \mathrm{divu} - \mathrm{curlcurlu} = 2\mathrm{div}(D(u)) - \nabla \mathrm{divu}. \quad (13)$$

Lemma 1. *(see [14]). Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ and outward normal n, then there exists a constant $C > 0$ independent of u, such that*

$$\|u\|_{H^s(\Omega)} \le C\left(\|\mathrm{divu}\|_{H^{s-1}(\Omega)} + \|\mathrm{curlu}\|_{H^{s-1}(\Omega)} + \|u\cdot n\|_{H^{s-1/2}(\partial\Omega)} + \|u\|_{H^{s-1}(\Omega)}\right), \quad (14)$$

for any $u \in H^s(\Omega), s \ge 1$.

Lemma 2. *(see [27]). Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ and outward normal n, then there exists a constant $C > 0$ independent of u, such that*

$$\|u\|_{H^s(\Omega)} \le C\left(\|\mathrm{divu}\|_{H^{s-1}(\Omega)} + \|\mathrm{curlu}\|_{H^{s-1}(\Omega)} + \|u\times n\|_{H^{s-1/2}(\partial\Omega)} + \|u\|_{H^{s-1}(\Omega)}\right), \quad (15)$$

for any $u \in H^s(\Omega), s \ge 1$.

Remark 3. The general forms of the conclusions of Lemmas 1 and 2 in [14, 27] are that there exists a constant $C > 0$ independent of u, such that

$$\begin{aligned}\|u\|_{W^{s,p}(\Omega)} &\le C\left(\|\mathrm{divu}\|_{W^{s-1,p}(\Omega)} + \|\mathrm{curlu}\|_{W^{s-1,p}(\Omega)} + \|u\cdot n\|_{W^{s-1/p,p}(\partial\Omega)} + \|u\|_{W^{s-1,p}(\Omega)}\right),\\ \|u\|_{W^{s,p}(\Omega)} &\le C\left(\|\mathrm{divu}\|_{W^{s-1,p}(\Omega)} + \|\mathrm{curlu}\|_{W^{s-1,p}(\Omega)} + \|u\times n\|_{W^{s-1/p,p}(\partial\Omega)} + \|u\|_{W^{s-1,p}(\Omega)}\right),\end{aligned} \quad (16)$$

for any $u \in W^{s,p}(\Omega)$.

Lemma 4. *(Poincaré's inequality). Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ and outward normal n, then there exists a constant $C > 0$ independent of u, such that*

$$\|u\|_{L^2(\Omega)} \le C\left(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}\right), \quad (17)$$

for any $u \in W^{s,p}(\Omega)$.

Lemma 5 *(see [28]). Let Ω be nonaxially symmetric, and u satisfy $u\cdot n|_{\partial\Omega} = 0$ where n is outward normal of Ω, then there exists a constant $C > 0$ independent of u, such that*

$$\|\nabla u\|_{L^2(\Omega)} \le C\|D(u)\|_{L^2(\Omega)}, \quad (18)$$

for any $u \in H^1(\Omega)$.

Remark 6. In this paper, the notations $\|\cdot\|_{H^k}$ for $k \ge 0$ present the norm of Sobolev space $H^k(\Omega)$, where $H^0(\Omega) = L^2(\Omega)$. In addition, we assume that $C_\eta, C_\delta, C_\vartheta$ are generic positive constants depending on generic small positive constants η, δ, ϑ, respectively. The notations C, C_i for $i = 1, 2, \ldots$ below present the positive constants which depend only on Ω, μ, λ and κ but not on the time t and Mach number ε. For simplicity, we denote the partial derivatives $\partial/\partial x_i$ by ∂_i, $\partial^2/\partial x_i\partial x_j$ by ∂_{ij}, and so on.

3. Main Results

Next, we introduce the conclusion of global existence and incompressible limit to the initial-boundary value problem (8)–(12), and see [14] for the proof.

Theorem 7. *(Global existence and incompressible limit [14]) Let $\varepsilon \in (0, \bar{\varepsilon}]$ for some constant $\bar{\varepsilon} \in (0, 1]$ and Ω be nonaxially symmetric initial datum $(\sigma_0, u_0, \theta_0)$ that satisfies the following conditions:*

$$(\sigma_0, u_0, \theta_0) \in H^2(\Omega), (\sigma_t(0), u_t(0), \theta_t(0)) \in L^2(\Omega). \quad (19)$$

Moreover, we assume that $\int_\Omega \sigma_0 \mathrm{d}x = 0$ and

$$\left\|\sigma_0, u_0, \theta_0\right\|_{H^2}^2 + \left\|\sigma_t(0), u_t(0), \theta_t(0)\right\|_{L^2}^2 \le \nu, \quad (20)$$

where ν is a sufficiently small positive constant. Then, there exists a unique global solution $(\sigma^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ to the initial-boundary value problem (8)–(12) in $\Omega \times R^+$ satisfying

$$\begin{aligned}(\sigma, u, \theta) &\in C\left(R^+, H^2\right), (\sigma_t, u_t, \theta_t) \in C\left(R^+, L^2\right),\\ (u, \theta) &\in L^2\left(R^+; H^3\right), (u_t, \theta_t) \in L^2\left(R^+; H^1\right),\end{aligned} \quad (21)$$

where $R^+ = [0, +\infty)$. Furthermore, the following uniform estimate in $\varepsilon \in (0, \bar{\varepsilon}]$ holds:

$$\|u^{\varepsilon}\|_{H^1}^2(t) + \|(\sigma^{\varepsilon}, \theta^{\varepsilon})\|_{H^2}^2(t) + \|(\sigma_t^{\varepsilon}, u_t^{\varepsilon}, \theta_t^{\varepsilon})\|_{L^2}^2(t) \le C\nu, \forall t \in R^+, \tag{22}$$

for some positive constant C. Thus $u \longrightarrow v$ *strongly in* $C(R^+_{\text{loc}}; H^s)$ *as* $\varepsilon \longrightarrow 0$ *for any* $0 \le s < 2$. *There exists a function* $\pi(x,t)$ *such that* (v, π) *is the unique global solution for the following initial-boundary value problem of isentropic incompressible Navier–Stokes equations:*

$$\begin{aligned} \operatorname{div} v &= 0, \text{in } \Omega \times (0, +\infty), \\ v_t + v \cdot \nabla v + \nabla \pi &= \mu \nabla v, in\, \Omega \times (0, +\infty), \\ v &= 0, \text{on } \partial\Omega \times [0, +\infty), \\ v|_{t=0} &= v_0, \text{in } \Omega, \end{aligned} \tag{23}$$

where v_0 *is the strong limit of* $\{u^{\varepsilon}(x,0)\}$ *in* H^s *as* $\varepsilon \longrightarrow 0$.

The main theorem of this paper is presented as below. Let $(\sigma^i, u^i, \theta^i)$ $(i = 1, 2)$ be two solutions of the problem (8)–(12) corresponding to two different initial data $(\sigma_0^i, u_0^i, \theta_0^i)$ $(i = 1, 2)$. In order to state the theorem precisely, we introduce the following notations (see [14]):

$$\begin{aligned} \Phi^i(t) :=& (M_1 + M_2)\left\| \sqrt{\rho}\, u^i, \sqrt{R}\,\sigma^i, \sqrt{C_V \rho}\,\theta^i \right\|_{L^2}^2 + \frac{M_1}{2}\left\| \left(\sqrt{\rho}\operatorname{div} u^i, \sqrt{R}\,\nabla\sigma^i, \sqrt{C_V\rho}\,\nabla\theta^i \right) \right\|_{L^2}^2 \\ &+ \frac{M_3}{2}\left\| \left(\sqrt{\rho}\, u_t^i, \sqrt{R}\,\sigma_t^i, \sqrt{C_V\rho}\,\theta_t^i \right) \right\|_{L^2}^2 + \frac{1}{2}\left\| \sqrt{\rho}\operatorname{curl} u^i \right\|_{L^2}^2 \\ &+ \left\| \chi_0 \nabla^2 u^i \right\|_{L^2}^2 + \left\| \nabla^2\sigma^i, \nabla^2\theta^i \right\|_{L^2}^2 + \int_{\widetilde{\Omega}} J\chi^2 \widetilde{\rho} \left| D_{\xi\tau}\widetilde{u} \right|^2 \mathrm{d}y. \end{aligned} \tag{24}$$

$$\begin{aligned} \Psi^i(t) :=& \left\| \left(u^i, \theta^i \right) \right\|_{H^1}^2 + M_1(2\mu + \zeta)\left\| \nabla \operatorname{div} u^i \right\|_{L^2}^2 + \left\| \Delta\theta^i \right\|_{L^2}^2 + \left\| \left(u_t^i, \theta_t^i \right) \right\|_{H^1}^2 \\ &+ \left\| \sigma^i \right\|_{H^2}^2 + (2C_7 C_{18} + 1)\left\| u^i \right\|_{H^2}^2 + \left\| u^i \right\|_{H^3}^2 + \left\| \nabla^3\theta^i \right\|_{L^2}^2 + \frac{1}{\varepsilon}\left\| \left(\nabla\sigma^i, \nabla\theta^i \right) \right\|_{H^1}^2. \end{aligned} \tag{25}$$

As shown in [14], the following energy inequality holds:

$$\frac{d}{dt}\Phi^i(t) + \frac{1}{2}\Psi^i(t) \le \frac{1}{2} C_{20}\Psi^i(t)\left[\Phi^i(t) + \left(\Phi^i(t) \right) \right], \quad \forall 0 \le t \le T, \varepsilon \in (0, \overline{\varepsilon}]. \tag{26}$$

Based on Theorem 1 and the previous inequalities, the exponentially asymptotic stability of global solutions of three-dimensional nonisentropic compressible Navier–Stokes equations is obtained. Next, we introduce three following notations:

$$\overline{\sigma} = \sigma^1 - \sigma^2, \overline{u} = u^1 - u^2, \overline{\theta} = \theta^1 - \theta^2. \tag{27}$$

The primary findings of this paper are presented as follows:

Theorem 8. *(Stability result) There exist positive constants C and* T^*, *such that*

$$\|(\overline{u}, \overline{\sigma}, \overline{\theta})\|_{L^2}^2(t) \le C\left\| \overline{u}_0, \overline{\sigma}_0, \overline{\theta}_0 \right\|_{L^2}^2 e^{-mt}, t \ge T^*, \tag{28}$$

$$\left\| v^1 - v^2 \right\|_{L^2}^2(t) \le C\left\| v_0^1 - v_0^2 \right\|_{L^2}^2 e^{-mt}, t \ge T^*, \tag{29}$$

where v^i *is the solution to the incompressible Navier–Stokes equation (23) with initial data* v_0^i $(i = 1, 2)$.

4. Stability Proof

In this paper, we will discuss the procedure of proving the stability conclusion of the global solution of three-dimensional nonisentropic compressible Navier–Stokes equations in bounded domain. By referring to the uniform energy estimates of Mach number and time in literature [14] and based on the relevant conclusions of Stokes problem, the nonisentropic compressible Navier–Stokes equations and the exponentially asymptotic stability of the corresponding limit incompressible Navier–Stokes equations are derived.

Through the initial-boundary problem (8)–(12), it follows that

$$\overline{\sigma}_t + \frac{1}{\varepsilon}\mathrm{div}\overline{u} = -\mathrm{div}\left(\overline{\sigma}u^1 + \sigma^2\overline{u}\right) = f_1, \tag{30}$$

$$\begin{aligned}
&\rho^2\overline{u}_t - (2\mu\mathrm{div}(D(\overline{u})) + \zeta\nabla\mathrm{div}\overline{u}) + \frac{R}{\varepsilon}(\nabla\overline{\sigma} + \nabla\overline{\theta})\\
&= R\nabla\left(\overline{\sigma}\theta^1 + \sigma^2\overline{\theta}\right) - \varepsilon\overline{\sigma}u_t^1 - \rho^2(\overline{u}\cdot\nabla)u^2 - \rho^2\left(u^1\cdot\nabla\right)\overline{u} - \varepsilon\overline{\sigma}\left(u^1\cdot\nabla\right)u^1\\
&= f_2,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&C_V\rho^2\overline{\theta}_t + \frac{R}{\varepsilon}\mathrm{div}\overline{u} - \kappa\Delta\overline{\theta}\\
&= -\varepsilon C_V\overline{\sigma}\sigma_t^1 - C_V\left[\rho^2(\overline{u}\cdot\nabla)\theta^2 + \rho^2\left(u^1\cdot\nabla\right)\overline{\theta} + \varepsilon\overline{\sigma}\left(u^1\cdot\nabla\right)\theta^1\right]\\
&\quad - \varepsilon\left[2\mu\nabla\overline{u}\left(\nabla u^1 + \nabla u^2\right) + \zeta\mathrm{div}\overline{u}\left(\mathrm{div}u^1 + \mathrm{div}u^2\right)\right]\\
&\quad - R\left(\rho^2\overline{\theta}\mathrm{div}u^2 + \rho^2\theta^1\mathrm{div}\overline{u}\right) + \varepsilon\overline{\sigma}\theta^1\mathrm{div}u^1 + \overline{\sigma}\mathrm{div}u^1 + \sigma^2\mathrm{div}\overline{u}\\
&= f_3.
\end{aligned} \tag{32}$$

Lemma 9. *The following inequality holds:*

$$\begin{aligned}
&\frac{d}{dt}\left\|\sqrt{\rho^2}\,\overline{u}, \sqrt{R}\,\overline{\sigma}, \sqrt{C_V\rho^2}\,\overline{\theta}\right\|_{L^2}^2 + C^\alpha\|\overline{u},\overline{\theta}\|_{H^1}^2\\
&\quad\le \eta\|\overline{\sigma}\|_{L^2}^2 + \left(\Psi^1(t) + \Psi^2(t)\right)\left[C_\eta\left(\|\overline{\sigma}\|_{L^2}^2 + \|\overline{\theta}\|_{H^1}^2\right) + C\|\overline{u}\|_{H^1}^2\right],
\end{aligned} \tag{33}$$

where C and C^α are positive constants.

Proof. Note that by the boundary conditions (12), $\overline{u}$ and $\overline{\theta}$ satisfy the following boundary condition

$$\overline{u} = 0, \frac{\partial\overline{\theta}}{\partial n} + \beta\overline{\theta} = 0 \text{ in } \partial\Omega\times(0,T), \tag{34}$$

According to Lemma 1 and 2, $\mu + 3\zeta/2 > 0$ and the previous boundary condition, we have

$$\begin{aligned}
-\int_\Omega [2\mu\mathrm{div}(D(\overline{u})) + \zeta\nabla\mathrm{div}\overline{u}]\cdot\overline{u}\mathrm{d}x &= -\int_\Omega [\mu\Delta\overline{u} + (\mu+\zeta)\nabla\mathrm{div}\overline{u}]\cdot\overline{u}\mathrm{d}x\\
&= \int_\Omega \mu|\nabla\overline{u}|^2\mathrm{d}x - \int_{\partial\Omega}\mu\nabla\overline{u}\cdot\overline{u}\cdot\mathrm{n}\mathrm{d}S\\
&\quad + \int_\Omega(\mu+\zeta)(\mathrm{div}\overline{u})^2\mathrm{d}x - \int_{\partial\Omega}(\mu+\zeta)\mathrm{div}\overline{u}\cdot\overline{u}\cdot\mathrm{n}\mathrm{d}S\\
&= \int_\Omega\left[\mu|\nabla\overline{u}|^2 + (\mu+\zeta)(\mathrm{div}\overline{u})^2\right]dx\\
&\ge \gamma_1\|\overline{u}\|_{H^1}^2,\\
-\kappa\int_\Omega\Delta\overline{\theta}\cdot\overline{\theta}\mathrm{d}x &= \kappa\int_\Omega|\nabla\overline{\theta}|^2\mathrm{d}x - \int_{\partial\Omega}\overline{\theta}\nabla\overline{\theta}\cdot\mathrm{n}\mathrm{d}S\\
&= \kappa\int_\Omega|\nabla\overline{\theta}|^2\mathrm{d}x + \kappa\beta\int_{\partial\Omega}\overline{\theta}^2\mathrm{d}S\\
&\ge \gamma_2\|\overline{\theta}\|_{H^1}^2,
\end{aligned} \tag{35}$$

where γ_1 and γ_2 are positive constants.

We multiply both sides of equations (8)–(10) simultaneously by $R\overline{\sigma}$, $\overline{u}$ and $\overline{\theta}$, respectively, then summarizing the integrals of the resulting equations on Ω, and finally by integration by parts we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\sqrt{\rho^2}\,\overline{u}, \sqrt{R}\,\overline{\sigma}, \sqrt{C_V\rho^2}\,\overline{\theta}\right\|_{L^2}^2 + \gamma_1\|\overline{u}\|_{H^1}^2 + \gamma_2\|\overline{\theta}\|_{H^1}^2\\
&\quad\le\left|\int_\Omega f_1\cdot\overline{\sigma}\mathrm{d}x\right| + \left|\int_\Omega f_2\cdot\overline{u}\mathrm{d}x\right| + \left|\int_\Omega f_3\cdot\overline{\theta}\mathrm{d}x\right| = \sum_{i=1}^3 I_i,
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
&\int_\Omega \frac{R}{\varepsilon}(\nabla\overline{\sigma} + \nabla\overline{\theta})\cdot\overline{u}\mathrm{d}x \\
&= -\int_\Omega \frac{R}{\varepsilon}\mathrm{div}\overline{u}\cdot\overline{\sigma}\mathrm{d}x + \int_{\partial\Omega}\frac{R}{\varepsilon}\overline{\sigma}\cdot\overline{u}\cdot\mathrm{ndS} - \int_\Omega \frac{R}{\varepsilon}\mathrm{div}\overline{u}\cdot\overline{\theta}\mathrm{d}x + \int_{\partial\Omega}\frac{R}{\varepsilon}\overline{\theta}\cdot\overline{u}\cdot\mathrm{ndS} \\
&= -\int_\Omega \frac{R}{\varepsilon}\mathrm{div}\overline{u}\cdot\overline{\sigma}\mathrm{d}x - \int_\Omega \frac{R}{\varepsilon}\mathrm{div}\overline{u}\cdot\overline{\theta}\mathrm{d}x.
\end{aligned} \tag{37}$$

With the aid of Theorem 7 and the notations (24) and (25), we have

$$\begin{aligned}
I_1 &= \left|\int_\Omega \mathrm{div}\left(\overline{\sigma}u^1 + \sigma^2\overline{u}\right)\overline{\sigma}\mathrm{d}x\right| \\
&= \left|\int_\Omega \mathrm{div}\left(\overline{\sigma}u^1\right)\overline{\sigma}dx + \int_\Omega \mathrm{div}\left(\sigma^2\overline{u}\right)\overline{\sigma}\mathrm{d}x\right| \\
&= \left|\int_\Omega \left(\mathrm{div}\overline{u}\sigma^2 + \nabla\sigma^2\cdot\overline{u}\right)\overline{\sigma}\mathrm{d}x + \frac{1}{2}\int_\Omega \mathrm{divu}^1(\overline{\sigma})^2\mathrm{d}x\right| \\
&\le \eta\,\|\overline{\sigma}\|\|_{L^2}^2 + C_\eta\left(\|\overline{u}\|\|_{H^1}^2\left\|\sigma^2\right\|\|_{H^2}^2 + \|\overline{\sigma}\|\|_{L^2}^2\left\|u^1\right\|\|_{H^3}^2\right) \\
&\le \eta\,\|\overline{\sigma}\|\|_{L^2}^2 + C_\eta\left(\Psi^1(t) + \Psi^2(t)\right)\left(\|\overline{\sigma}\|\|_{L^2}^2 + \|\overline{u}\|\|_{H^1}^2\right),
\end{aligned} \tag{38}$$

By applying the same approach, it follows that

$$\begin{aligned}
I_2 &= \left|\int_\Omega f_2\cdot\overline{u}\mathrm{d}x\right| \\
&= \left|\int_\Omega \left[R\nabla\left(\overline{\sigma}\theta^1 + \sigma^2\overline{\theta}\right) - \varepsilon\overline{\sigma}u_t^1 - \rho^2(\overline{u}\cdot\nabla)u^2\right]\cdot\overline{u}\mathrm{d}x\right| \\
&+ \left|\int_\Omega \left[-\rho^2\left(u^1\cdot\nabla\right)\overline{u} - \varepsilon\overline{\sigma}\left(u^1\cdot\nabla\right)u^1\right]\cdot\overline{u}\mathrm{d}x\right| \\
&\le \delta\|\overline{u}\|\|_{H^1}^2 + C_\delta\left(\Psi^1(t) + \Psi^2(t)\right)\left(\|\overline{\sigma}\|\|_{L^2}^2 + \|\overline{\theta}\|\|_{H^1}^2 + \|\overline{u}\|\|_{H^1}^2\right). \\
I_3 &= \left|\int_\Omega f_3\cdot\overline{\theta}\mathrm{d}x\right| \\
&\le \vartheta\|\overline{\theta}\|\|_{H^1}^2 + C_\vartheta\left(\Psi^1(t) + \Psi^2(t)\right)\left(\|\overline{\sigma}\|\|_{L^2}^2 + \|\overline{\theta}\|\|_{H^1}^2 + \|\overline{u}\|\|_{H^1}^2\right).
\end{aligned} \tag{39}$$

Choosing sufficiently small δ and ϑ, this lemma is proved. □

We now start to estimate $\|\overline{\sigma}\|\|_{L^2}^2$.Proof of the Theorem 8.

Lemma 10. *The following inequality holds:*

$$\lambda(t) \le C\lambda(0)\exp(C\nu), \tag{40}$$

where C is a positive constant and $\lambda(t) := = \|(\overline{u},\overline{\sigma},\overline{\theta})\|\|_{L^2}^2 + \int_0^t(\|\overline{\sigma}\|\|_{L^2}^2 + \|\overline{\theta}\|\|_{H^1}^2 + \|\overline{u}\|\|_{H^1}^2)\mathrm{d}s$.

Proof. Assuming that z exists, this constitutes the solution to the following Stokes problem:

$$\begin{aligned}
\Delta z - \nabla\omega &= 0, \text{in } \Omega\times(0,+\infty), \\
\mathrm{divz} &= \overline{\sigma}, \text{in } \Omega\times(0,+\infty), \\
z &= 0, \text{on } \partial\Omega\times(0,+\infty).
\end{aligned} \tag{41}$$

where $\int_\Omega\overline{\sigma}dx = 0$ is to be determined.

According to [29, 30], the solution z of the previous Stokes problem exists, and the following inequality holds:

$$\|z\|_{H^1}^2 \leq C\|\overline{\sigma}\|_{L^2}^2. \tag{42}$$

Multiplying (31) by z and then integrating on Ω, we obtain

$$\begin{aligned}
-\int_\Omega \frac{R}{\varepsilon}(\nabla\overline{\sigma})\cdot \mathrm{zdx} = \frac{R}{\varepsilon}\int_\Omega \overline{\sigma}\cdot \mathrm{di\,vzdx} = \frac{R}{\varepsilon}\|\overline{\sigma}\|\|_{L^2}^2 \\
= \int_\Omega \rho^2\overline{u}_t\cdot \mathrm{zdx} + \int_\Omega \frac{R}{\varepsilon}(\nabla\overline{\theta})\cdot \mathrm{zdx} - \int_\Omega \left(2\mu \mathrm{div}(D(\overline{u})) + \zeta\nabla \mathrm{div}\overline{u} + f_2\right)\cdot \mathrm{zdx} \\
= \sum_{i=1}^3 J_i.
\end{aligned} \tag{43}$$

A direct calculation shows that

$$J_1 = \frac{d}{dt}\int_\Omega \rho^2\overline{u}\cdot \mathrm{zdx} - \int_\Omega \rho^2\overline{u}\cdot z_t\mathrm{dx} - \int_\Omega \varepsilon\sigma_t^2\overline{u}\cdot \mathrm{zdx}. \tag{44}$$

Note that by (30), $\overline{\sigma}_t = -\mathrm{di\,vW}$, where $W = \overline{\sigma}u^1 + \sigma^2\overline{u} + 1/\varepsilon\overline{u}$ and

$$\|W\|_{L^2}^2 \leq C\left(\|\overline{\sigma}\|\|_{L^2}^2 + \|\overline{u}\|_{L^2}^2\right) + \frac{1}{\varepsilon}\|\overline{u}\|_{L^2}^2. \tag{45}$$

Applying the operator ∂_t to (41), we obtain

$$\begin{aligned}
\Delta z_t - \nabla\omega_t &= 0, \mathrm{in}\ \Omega\times(0,+\infty), \\
\mathrm{div}z_t &= \overline{\sigma}_t, \mathrm{in}\ \Omega\times(0,+\infty), \\
z_t &= 0, \mathrm{on}\partial\Omega\times(0,+\infty).
\end{aligned} \tag{46}$$

Assume that the solution of the Stokes problem is (U,P)

$$\begin{aligned}
\Delta U - \nabla P &= \rho^2\overline{u}, \mathrm{in}\ \Omega\times(0,+\infty), \\
\mathrm{di\,vU} &= 0, \mathrm{in}\ \Omega\times(0,+\infty), \\
U &= 0, \mathrm{on}\partial\Omega\times(0,+\infty).
\end{aligned} \tag{47}$$

Because of [30], one has

$$\|U\|\|_{H^2}^2 + \|\nabla P\|_{L^2}^2 \leq C\|\overline{u}\|\|_{L^2}^2. \tag{48}$$

From the boundary condition (12), W satisfies the following boundary condition:

$$W|_{\partial\Omega} = 0. \tag{49}$$

By utilizing integration by parts and the boundary condition provided above, we can obtain the following result:

$$\begin{aligned}
\left|\int_\Omega \rho^2\overline{u}\cdot z_t\mathrm{d}x\right| &= \left|\int_\Omega (\Delta U - \nabla P)\cdot z_t\mathrm{d}x\right| = \left|\int_\Omega U\cdot\Delta z_t + \mathrm{Pdi\,vz}_t\mathrm{d}x\right| \\
&= \left|\int_\Omega U\cdot\nabla\omega_t\mathrm{dx} + \int_\Omega P\overline{\sigma}_t\mathrm{d}x\right| \\
&= \left|\int_\Omega -\mathrm{di\,vU}\omega_t\mathrm{d}x + \int_{\partial\Omega} U\cdot\omega_t\cdot\mathrm{ndS} - \int_\Omega \mathrm{Pdi\,vWdx}\right| \\
&= \left|\int_\Omega \nabla P\cdot \mathrm{Wdx} - \int_{\partial\Omega} P\cdot W\cdot\mathrm{ndS}\right| \\
&\leq C\|\nabla P\|_{L^2}\|W\|_{L^2} \\
&\leq C\left(\|\overline{\sigma}\|\|_{L^2}^2 + \|\overline{u}\|\|_{L^2}^2\right) + \frac{1}{\varepsilon}\|\overline{u}\|\|_{L^2}^2.
\end{aligned} \tag{50}$$

In addition,

$$\left|\int_\Omega \varepsilon\sigma_t^2\overline{u}\cdot z\mathrm{dx}\right| \le C\varepsilon\left(\|\overline{u}\|_{L^2}^2 + \|z\|_{H^1}^2\right) \le C\varepsilon\left(\|\overline{u}\|_{H^1}^2 + \|\overline{\sigma}\|_{L^2}^2\right),$$
$$|J_2| \le \frac{R}{\varepsilon}\left(\eta\|z\|_{H^1}^2 + C_\eta\|\nabla\overline{\theta}\|_{L^2}^2\right) \le \frac{R}{\varepsilon}\left(\eta\|\overline{\sigma}\|_{L^2}^2 + C_\eta\|\nabla\overline{\theta}\|_{L^2}^2\right),$$
$$\begin{aligned} |J_3| &\|(2\mu\mathrm{div}(D(\overline{u})) + \zeta\nabla\mathrm{div}\overline{u})\|_{H^{-1}} + \|f_2\|_{H^{-1}}\|z\|_{H^1} \\ &\le C\left(\|2\mu\mathrm{div}(D(\overline{u})) + \zeta\nabla\mathrm{div}\overline{u}\|_{H^{-1}}^2 + \|f_2\|_{H^{-1}}^2 + \|\overline{\sigma}\|_{L^2}^2\right) \\ &\le C\left(\|\overline{\sigma}\|_{L^2}^2 + \|\overline{\theta}\|_{H^1}^2 + \|\overline{u}\|_{H^1}^2\right). \end{aligned} \tag{51}$$

According to Theorem 7, we have

$$\|f_2\|_{H^{-1}}^2 \le C\nu\left(\|\overline{\sigma}\|_{L^2}^2 + \|\overline{\theta}\|_{H^1}^2 + \|\overline{u}\|_{H^1}^2\right). \tag{52}$$

From the previous estimate, we obtain

$$\begin{aligned} &-\frac{d}{dt}\int_\Omega \rho^2\overline{u}\cdot z\mathrm{dx} + \frac{R}{\varepsilon}\|\overline{\sigma}\|_{L^2}^2 \\ &\le C\left(\|\overline{\sigma}\|\|_{L^2}^2 + \|\overline{\theta}\|\|_{H^1}^2 + \|\overline{u}\|\|_{H^1}^2\right) \\ &\quad + \frac{1}{\varepsilon}\|\overline{u}\|_{L^2}^2 + \frac{R}{\varepsilon}\left(\eta\|\overline{\sigma}\|\|_{L^2}^2 + C_\eta\|\nabla\overline{\theta}\|\|_{L^2}^2\right). \end{aligned} \tag{53}$$

This implies

$$\begin{aligned} &-\varepsilon\frac{d}{dt}\int_\Omega \rho^2\overline{u}\cdot z\mathrm{dx} + R\|\overline{\sigma}\|_{L^2}^2 \\ &\le C\varepsilon\left(\|\overline{\sigma}\|_{L^2}^2 + \|\overline{\theta}\|_{H^1}^2 + \|\overline{u}\|_{H^1}^2\right) \\ &\quad + \|\overline{u}\|_{L^2}^2 + R\left(\eta\|\overline{\sigma}\|_{L^2}^2 + C_\eta\|\nabla\overline{\theta}\|_{L^2}^2\right). \end{aligned} \tag{54}$$

Summarizing M× (33) and (54), we deduce this lemma.

$$\begin{aligned} &M\frac{d}{dt}\left\|\sqrt{\rho^2}\,\overline{u}, \sqrt{R}\,\overline{\sigma}, \sqrt{C_V\rho^2}\,\overline{\theta}\right\|_{L^2}^2 + MC^\alpha\|(\overline{u},\overline{\theta})\|_{H^1}^2 - \varepsilon\frac{d}{dt}\int_\Omega \rho^2\overline{u}\cdot z\mathrm{dx} + R\|\overline{\sigma}\|_{L^2}^2 \\ &\le \eta\|\overline{\sigma}\|_{L^2}^2 + C_\eta\left(\Psi^1(t) + \Psi^2(t)\right)\left(\|\overline{\sigma}\|_{L^2}^2 + \|\overline{\theta}\|_{H^1}^2 + \|\overline{u}\|_{H^1}^2\right) + \|\overline{u}\|_{L^2}^2 + C_\eta\|\nabla\overline{\theta}\|_{L^2}^2, \end{aligned} \tag{55}$$

where M is the undetermined coefficients. Integrating the previous inequality (55) on $(0,t)$, this implies

$$\begin{aligned} &M\left\|\left(\sqrt{\rho^2}\,\overline{u}, \sqrt{R}\,\overline{\sigma}, \sqrt{C_V\rho^2}\,\overline{\theta}\right)\right\|_{L^2}^2 + MC^\alpha\int_0^t\|(\overline{u},\overline{\theta})\|_{H^1}^2\mathrm{d}s + \frac{R}{2}\int_0^t\|\overline{\sigma}\|_{L^2}^2\mathrm{d}s, \\ &\le \varepsilon\int_\Omega \rho^2\overline{u}\cdot z\mathrm{d}x + C\left\|\sqrt{\rho^2}\,\overline{u}_0, \sqrt{R}\,\overline{\sigma}_0, \sqrt{C_V\rho^2}\,\overline{\theta}_0\right\|_{L^2}^2 \\ &\quad + C_\eta\int_0^t\left(\Psi^1(t) + \Psi^2(t)\right)\left(\|\overline{\sigma}\|_{L^2}^2 + \|\overline{\theta}\|_{H^1}^2 + \|\overline{u}\|_{H^1}^2\right)\mathrm{d}s \\ &\quad + C_\eta\int_0^t\|\nabla\overline{\theta}\|_{L^2}^2\mathrm{d}s + \int_0^t\|\overline{u}\|_{L^2}^2\mathrm{d}s, \end{aligned} \tag{56}$$

where

$$\left|\varepsilon\int_\Omega \rho^2\overline{u}\cdot z\mathrm{d}x\right| \le \varepsilon\left(\|\overline{u}\|_{L^2}^2 + \|z\|_{H^1}^2\right) \le \varepsilon\left(\|\overline{u}\|_{L^2}^2 + \|\overline{\sigma}\|_{L^2}^2\right). \tag{57}$$

Then, according to the two inequalities mentioned above, we get

$$(M-\varepsilon)\left\|\left(\sqrt{\rho^2}\,\overline{u},\sqrt{R}\,\overline{\sigma},\sqrt{C_V\rho^2}\,\overline{\theta}\right)\right\|_{L^2}^2+MC^\alpha\int_0^t\|(\overline{u},\overline{\theta})\|_{H^1}^2\mathrm{d}s+\frac{R}{2}\int_0^t\|\overline{\sigma}\|_{L^2}^2\mathrm{d}s,$$
$$\leq C\left\|\sqrt{\rho^2}\,\overline{u}_0,\sqrt{R}\,\overline{\sigma}_0,\sqrt{C_V\rho^2}\,\overline{\theta}_0\right\|_{L^2}^2$$
$$+C^\beta\int_0^t\left(\Psi^1(t)+\Psi^2(t)\right)\left(\|\overline{\sigma}\|_{L^2}^2+\|\overline{\theta}\|_{H^1}^2+\|\overline{u}\|_{H^1}^2\right)\mathrm{d}s$$
$$+C^\beta\int_0^t\|\nabla\overline{\theta}\|_{L^2}^2ds+\int_0^t\|\overline{u}\|_{L^2}^2\mathrm{d}s. \tag{58}$$

where constants η and ε are small enough.

Let $MC^\alpha>\max\{C^\beta+1,2\}$ and combined with 3.1 and 3.5, we have

$$\int_0^t\Psi^i(s)\mathrm{d}s\leq C\nu. \tag{59}$$

Clearly,

$$\lambda(t)\leq C\lambda(0)+C\int_0^t\left(\Psi^1(t)+\Psi^2(t)\right)\lambda(s)\mathrm{d}s. \tag{60}$$

Therefore, the lemma is proved by direct calculations and Gronwalls lemma. □

Proof. We suppose

$$\widehat{u}=e^{ht}\overline{u},\widehat{\theta}=e^{ht}\overline{\theta},\widehat{\sigma}=e^{ht}\overline{\sigma},$$
$$\widehat{f}_1=e^{ht}f_1,\widehat{f}_2=e^{ht}f_2,\widehat{f}_3=e^{ht}f_3, \tag{61}$$

where $h\in(0,1)$. In this way, we have

$$\widehat{\sigma}_t+\frac{1}{\varepsilon}\mathrm{div}\widehat{u}=\widehat{f}_1+h\widehat{\sigma},$$
$$\rho^2\widehat{u}_t-(2\mu\mathrm{div}(D(\widehat{u}))+\zeta\nabla\mathrm{div}\widehat{u})+\frac{R}{\varepsilon}(\nabla\widehat{\sigma}+\nabla\widehat{\theta})=\widehat{f}_2+h\rho^2\widehat{u},$$
$$C_V\rho^2\widehat{\theta}_t+\frac{R}{\varepsilon}\mathrm{div}\widehat{u}-\kappa\Delta\widehat{\theta}=\widehat{f}_3+C_V\rho^2\widehat{\theta}. \tag{62}$$

We estimate $\widehat{u}$, $\widehat{\theta}$ and $\widehat{\sigma}$ by repeating the steps done for $\overline{u}$, $\overline{\theta}$ and $\overline{\sigma}$ in Lemmas 9 and 10, and the conclusion is similar to (40), where

$$\|(\widehat{u},\widehat{\sigma},\widehat{\theta})\|_{L^2}^2+\int_0^t\left(\|\widehat{\sigma}\|_{L^2}^2+\|\widehat{\theta}\|_{H^1}^2+\|\widehat{u}\|_{H^1}^2\right)\mathrm{d}s\leq C\left\|\left(\widehat{u}_0,\widehat{\sigma}_0,\widehat{\theta}_0\right)\right\|_{L^2}^2\exp(C\nu). \tag{63}$$

Multiplying both sides of the inequality (63) by e^{-2ht} and choosing sufficiently large t, (28) is obtained. According to Theorem 8, (28) and (29) can be improved as $\varepsilon\longrightarrow 0$.

In this way, the theorem is proved. □

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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