Hindawi
Journal of Mathematics
Volume 2023, Article ID 7408231, 5 pages
https://doi.org/10.1155/2023/7408231

Research Article

A New Sequence Which Converges Faster towards the Euler–Mascheroni Constant

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Received 15 October 2022; Revised 1 February 2023; Accepted 30 May 2023; Published 17 August 2023

Academic Editor: Mohammad W. Alomari

The aim of this paper is to propose a new sequence that approximates the Euler–Mascheroni constant which converges faster towards its limit and to establish new inequalities for this constant.

1. Introduction

It is well known that the following sequence:

$$\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n, \quad n \geq 1, \tag{1}$$

is convergent to a limit denoted by $\gamma = 0.5772\ldots$, known as Euler–Mascheroni constant. Many authors have obtained different estimations for $\gamma_n - \gamma$; for example, the following inequalities increase better:

$$\begin{aligned} &\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2(n-1)}, n \geq 2, \\ &\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, n \geq 1, \\ &\frac{1}{2n+2/5} < \gamma_n - \gamma < \frac{1}{2n+1/3}, n \geq 1. \end{aligned} \tag{2}$$

The convergence of the sequence γ_n to γ is very slow [1–3].

DeTemple [4] modified the logarithmic term of γ_n and showed that the following sequence:

$$R_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right), \tag{3}$$

converges to γ with the rate of convergence n^{-2} since

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \quad n \geq 1. \tag{4}$$

Negoi [5] modified the logarithmic term of γ_n and showed that the following sequence:

$$T_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right), \tag{5}$$

is strictly increasing and convergent to γ with the rate of convergence n^{-3}. Moreover, he proved that

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}, \quad n \geq 1. \tag{6}$$

Chen and Mortici [6] proved that for all integers $n \geq 1$, we have

$$\frac{1}{48(n+a)^3} \leq \gamma - T_n < \frac{1}{48(n+b)^3}, \tag{7}$$

with the best possible constants as

$$\begin{aligned} a &= \frac{1}{\sqrt[3]{48[1-\gamma+\ln(37/24)]}} - 1 = 0.27380525\ldots, \\ b &= \frac{83}{360} = 0.23055555\ldots. \end{aligned} \tag{8}$$

A convergence result to γ with the rates of convergence n^{-4}, respectively, n^{-6} was obtained by Mortici [7]:

$$\lim_{n\longrightarrow\infty} n^4\left(\nu_n-\gamma\right)=-\frac{37}{5760},$$
$$\lim_{n\longrightarrow\infty} n^6\left(\mu_n-\gamma\right)=-\frac{74381}{29030400}, \tag{9}$$

where $\nu_n=1+1/2+\ldots+1/n-\ln(n^2+n+7/24/n+1/2)$, and

$$\mu_n=1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln\frac{n^3+3/2n^2+227/240n+107/480}{n^2+n+97/240}. \tag{10}$$

Yang [8] proved that

$$\nu_n-\gamma=O\left(\frac{1}{n^5}\right), \tag{11}$$

where $\nu_n=1+1/2+\ldots+1/n-\ln(n+1/2+1/24n-1/48n^2+23/5760n^3)$.

Now, we define the sequence $S_n=1+1/2+\ldots+1/n-\ln(n+1/2+1/24n-1/48n^2)$, for $n\geq 1$, and we prove that for all integers $n\geq 1$, we have

$$\frac{23}{5760(n+\alpha)^4}\leq S_n-\gamma<\frac{23}{5760(n+\beta)^4}, \tag{12}$$

with the best possible constants $\alpha=\sqrt[4]{23/5760[1-\gamma-\ln(73/48)]}-1=0.0315\ldots,\beta=-7/46=-0.1521\ldots$.

2. The Main Result

Starting from the sequences R_n and T_n, we consider the family of sequences

$$S_n(a,b,c)=1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln\left(n+a+\frac{b}{n}+\frac{c}{n^2}\right), \tag{13}$$

for $a,b,c\in\mathbf{R}, n\geq 1$, and

$$L_n(a,b,c)=S_n(a,b,c)-\gamma=1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln\left(n+a+\frac{b}{n}+\frac{c}{n^2}\right)-\gamma, \tag{14}$$

which converge to zero.

Using a Maclaurin growth series, we get

$$\begin{aligned}&L_{n+1}(a,b,c)-L_n(a,b,c)\\&=\frac{1}{n^2}\left(a-\frac{1}{2}\right)+\frac{1}{n^3}\left(-a^2-a+2b+\frac{2}{3}\right)+\frac{1}{n^4}\left(a^3-3ab+\frac{3a^2}{2}+a-3b+3c-\frac{3}{4}\right)\\&\quad+\frac{1}{n^5}\left(-a^4-2a^3+4a^2b-2a^2-2b^2+6ab-4ac-a+4b-6c+\frac{4}{5}\right)+O\left(\frac{1}{n^6}\right).\end{aligned} \tag{15}$$

If $a-1/2=0, -a^2-a+2b+2/3=0, a^3-3ab+3a^2/2+a-3b+3c-3/4=0$, then $a=1/2, b=1/24, c=-1/48$ and so

$$-a^4-2a^3+4a^2b-2a^2-2b^2+6ab-4ac-a+4b-6c+\frac{4}{5}=-\frac{23}{1440}. \tag{16}$$

It results that

$$S_n = S_n\left(\frac{1}{2}, \frac{1}{24}, -\frac{1}{48}\right) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2}\right),$$
$$L_n = L_n\left(\frac{1}{2}, \frac{1}{24}, -\frac{1}{48}\right) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2}\right) - \gamma. \tag{17}$$

Thus, we get

$$L_{n+1} - L_n = -\frac{23}{1440n^5} + O\left(\frac{1}{n^6}\right). \tag{18}$$

By a standard result, if a sequence x_n converges to zero and there exists $\lim_{n\longrightarrow\infty} n^k (x_n - x_{n+1}) = l$, then $\lim_{n\longrightarrow\infty} n^{k-1} x_n = l/k - 1$ (see, e.g., [9]).

In our case of L_n, we have $\lim_{n\longrightarrow\infty} n^5 (L_n - L_{n+1}) = 23/1440$ and so

$$\lim_{n\longrightarrow\infty} n^4 L_n = \frac{l}{k-1} = \frac{23}{5760}. \tag{19}$$

Starting from this result and using an elementary sequence method and MATLAB software for computation, we obtain the following:

Theorem 1. *For every integer $n \geq 1$, we have*

$$\frac{23}{5760(n+\alpha)^4} \leq S_n - \gamma < \frac{23}{5760(n+\beta)^4}, \tag{20}$$

with the best possible constants $\alpha = \sqrt[4]{23/5760[1 - \gamma - \ln(73/48)]} - 1 = 0.0315\ldots$, $\beta = -7/46 = -0.1521\ldots$.

Proof. We define the sequence

$$a_n = S_n - \gamma - \frac{23}{5760(n+a)^4} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2}\right) - \gamma - \frac{23}{5760(n+a)^4}, \tag{21}$$

for $a > -1$ and so $a_{n+1} - a_n = f(n)$, where

$$f(n) = \frac{1}{n+1} - \ln\left(n + \frac{3}{2} + \frac{1}{24(n+1)} - \frac{1}{48(n+1)^2}\right) + \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2}\right) - \frac{23}{5760(n+a+1)^4} + \frac{23}{5760(n+a)^4}. \tag{22}$$

The derivative of function f is equal to

$$f'(n) = -\frac{1}{(n+1)^2} - \frac{48n^3 + 144n^2 + 142n + 48}{(n+1)\left(48n^3 + 168n^2 + 194n + 73\right)} + \frac{48n^3 - 2n + 2}{n\left(48n^3 + 24n^2 + 2n - 1\right)}$$
$$- \frac{23}{1440} \frac{(n+a+1)^5 - (n+a)^5}{(n+a)^5 (n+a+1)^5} \tag{23}$$
$$= \frac{P(n)}{1440n(n+1)^2\left(48n^3 + 168n^2 + 194n + 73\right)\left(48n^3 + 24n^2 + 2n - 1\right)(n+a)^5 (n+a+1)^5}.$$

By using MATLAB software, we obtain that

$$\begin{aligned}
P(n) = {} & (1589760a + 241920)n^{12} + \left(10333440a^2 + 14342400a + 1137120\right)n^{11} \\
& + \left(30735360a^3 + 83220480a^2 + 50965440a + 1516320\right)n^{10} \\
& + \left(55376640a^4 + 228810240a^3 + 269606880a^2 + 95341920a - 1287772\right)n^9 \\
& + \left(66769920a^5 + 383063040a^4 + 678117120a^3 + 467686080a^2 + 102533840a - 6360352\right)n^8 \\
& + \left(55641600a^6 + 428198400a^5 + 1032797760a^4 + 1066366080a^3 + 478650840a^2 + 62880760a - 8530713\right)n^7 \\
& + \left(31795200a^7 + 329011200a^6 + 1039248000a^5 + 1451739840a^4 + 980762000a^3 + 296293800a^2 \right.\\
& \left. + 18865820a - 5977032\right)n^6 \\
& + \left(11923200a^8 + 172108800a^7 + 707716800a^6 + 1283446080a^5 + 1171727780a^4 + 541074920a^3 \right.\\
& \left. + 108066130a^2 + 474710a - 2314079\right)n^5 \\
& + \left(2649600a^9 + 58579200a^8 + 321004800a^7 + 749145600a^6 + 886320000a^5 + 554953240a^4 + 174603500a^3 \right.\\
& \left. + 21250670a^2 - 1030300a - 429847\right)n^4 \\
& + \left(264960a^{10} + 11692800a^9 + 91749600a^8 + 280598400a^7 + 426297600a^6 + 347175360a^5 \right.\\
& \left. + 149300655a^4 + 30130550a^3 + 1871940a^2 - 67735a - 2369\right)n^3 \\
& + \left(1036800a^{10} + 14616000a^9 + 62272800a^8 + 123667200a^7 + 130075200a^6 + 73483200a^5 \right.\\
& \left. + 20506310a^4 + 2180600a^3 + 94990a^2 + 55890a + 12857\right)n^2 \\
& + \left(943200a^{10} + 6818400a^9 + 18892800a^8 + 26251200a^7 + 19432800a^6 + 7250400a^5 \right.\\
& \left. + 1059595a^4 + 16790a^3 + 16790a^2 + 8395a + 1679\right)n \\
& + 210240a^{10} + 1051200a^9 + 2102400a^8 + 2102400a^7 + 1051200a^6 + 210240a^5.
\end{aligned} \tag{24}$$

If $a = -7/46$, then

$$\begin{aligned}
P(n) = {} & -\frac{18540960}{23}n^{11} - \frac{2338428480}{529}n^{10} - \frac{125681084924}{12167}n^9 - \frac{3728266432152}{279841}n^8 \\
& - \frac{65995802665829}{6436343}n^7 - \frac{7024378333351798}{148035889}n^6 - \frac{171393143233377447}{13619301788}n^5 - \frac{25414600725659899}{156621970562}n^4 \\
& + \frac{46998390955853163}{28818442583408}n^3 + \frac{1745314443004087749}{331412089709192}n^2 + \frac{1199096848882140907}{1325648358836768}n - \frac{4981367799868005}{662824179418384} < 0,
\end{aligned} \tag{25}$$

for all $n \geq 1$, and then f is strictly decreasing.

We have $\lim_{n \longrightarrow \infty} f(n) = 0$ and then it follows $f(n) > 0$ for all $n \geq 1$, such that $(a_n)_{n \geq 1}$ is strictly increasing. Since (a_n) converges to zero, it results that $a_n < 0$ for all $n \geq 1$, such that

$$S_n - \gamma < \frac{23}{5760(n - 7/46)^4}, \text{ for all } n \geq 1. \tag{26}$$

If $a = \sqrt[4]{23/5760[1 - \gamma - \ln(73/48)]} - 1 = 0.0315\ldots$, then $a_1 = S_1 - \gamma - 23/5760(1 + a)^4 = 0$, $P(n) > 0$ for all $n \geq 2$ and then f is strictly increasing on $[2, \infty)$.

Since $\lim_{n \longrightarrow \infty} f(n) = 0$, it results that $f(n) < 0$ for all $n \geq 2$, such that $(a_n)_{n \geq 2}$ is strictly decreasing.

The sequence (a_n) converges to zero and then it results that $a_n > 0$ for all $n \geq 2$, such that

Table 1: Data related to the three sequences R_n, T_n and S_n.

n	R_n	T_n	S_n
10	0.5775929968...	0.5771962501...	0.5772160837...
11	0.5775303095...	0.5772009829...	0.5772159500...
12	0.5774820339...	0.5772042946...	0.5772158656...
13	0.5774440696...	0.5772066809...	0.5772158102...
14	0.5774136771...	0.5772084436...	0.5772157727...
15	0.5773889693...	0.5772097738...	0.5772157465...
16	0.5773686123...	0.5772107964...	0.5772157278...
17	0.5773516417...	0.5772115954...	0.5772157142...
18	0.5773373461...	0.5772122288...	0.5772157040...
19	0.5773251915...	0.5772127372...	0.5772156964...
20	0.5773147709...	0.5772131501...	0.5772156905...
30	0.5772604473...	0.5772149110...	0.5772156699...
40	0.5772410648...	0.5772153449...	0.5772156664...
50	0.5772320020...	0.5772155005...	0.5772156654...

$$\frac{23}{5760(n+a)^4} \leq S_n - \gamma, \text{ for all } n \geq 1. \tag{27}$$

□

Remark 2. Let us remark that, if $a > -7/46$, then $1589760a + 241920 > 0$ and then there exists $n_a \geq 1$ such that $P(n) > 0$ for all $n \geq n_a$ and then

$$\frac{23}{5760(n+a)^4} < S_n - \gamma < \frac{23}{5760(n-7/46)^4}, \tag{28}$$

for all $n \geq n_a$.

Remark 3. Returning to the sequences

$$\begin{aligned} R_n &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right), \\ T_n &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right), \\ S_n &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2}\right), \end{aligned} \tag{29}$$

with the rates of convergence n^{-2}, n^{-3}, and n^{-4}, respectively, I used MATLAB software for computing the terms R_n, T_n, and S_n, with the first 10 exact decimals, for several iterations $n \in \{10, 11, \ldots, 19, 20, 30, 40, 50\}$.

The data obtained are contained in Table 1, where we can see the faster convergence of the sequence S_n to $\gamma = 0.577215664901\ldots$ compared to R_n and T_n:

3. Conclusions

By modifying the logarithmic term of γ_n, we have constructed a new sequence that converges faster to the Euler–Mascheroni constant, with the convergence rate n^{-4}, compared to the sequence in [4] with convergence rate n^{-2} or those in [5, 6] with convergence rates n^{-3}.

Also, the idea of constructing the sequence S_n allows the construction of a new sequence with the convergence rate n^{-5}, starting from the family of sequences:

$$S_n^{(4)}(a,b,c,d) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln\left(n + a + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3}\right), \tag{30}$$

for $a, b, c, d \in \mathbf{R}, n \geq 1$.

More generally, starting from the family of sequences,

$$\begin{aligned} S_n^{(k)}(a_1, a_2, \ldots, a_k) &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} \\ &\quad - \ln\left(n + a_1 + \frac{a_2}{n} + \ldots + \frac{a_k}{n^{k-1}}\right), \end{aligned} \tag{31}$$

for $a_1, a_2, \ldots, a_k \in \mathbf{R}, k \geq 4, n \geq 1$, we find a sequence with a convergence rate n^{-k-1}.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

The APC was funded by "Dunarea de Jos" University of Galati, Romania.

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