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Research Article

Solvability of the System of Extended Nonlinear Mixed Variational-Like Inequalities and Proximal Dynamical System

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In this article, our goal is to study the system of extended nonlinear mixed variational-like inequalities (in short, SENMVLI) with a nonconvex functional in the setting of real Hilbert spaces and discuss the existence of solution of our considered problem. We propose a three-step iterative algorithm to calculate the approximate solutions of SENMVLI and investigate the convergence analysis as well as stability analysis of the proposed algorithm. Furthermore, we also study the proximal dynamical system for SENMVLI and prove that the trajectory of the solution of the extended proximal dynamical system converges globally exponentially to a unique solution of SENMVLI. Our suggested iterative algorithm and results have become the significant improvement, enhancement, and generalization of many previously known results in the literature.

1. Introduction

In prior 1960*s*, the concept of variational inequality originated by Hartmann and Stampacchia [1] has appeared as a fruitful and methodical mechanism to study a wide range of applications in economics, finance, pure and applied sciences, and optimization, see, e.g. [2–5]. Using novel and regenerated techniques, several extensions and generalizations of variational inequalities have been explored and developed in recent years. The functional, pivotal, and applicable generalizations of variational inequalities are variational-like inequalities and mixed variational-like inequalities which have significant applications in nonconvex optimizations and mathematical programming problems. For details, we refer to [6–8] and references therein.

In classical variational inequality theory, ones have been failing to exploit the projection method and its modified forms to analyze the existence of solutions of mixed variational-like inequalities involving the nonlinear term. To vanquish this flaw, it is assumed that the nonlinear term involving the mixed variational-like inequalities is a proper, convex, and lower-semicontinuous functional. It is well-known that the subdifferential of a proper, convex, and lower-semicontinuous functional is a maximal monotone operator. This characterization enables to define the resolvent operator associated with the maximal monotone operator. The resolvent operator technique is used to establish the equivalence between the mixed variational-like inequalities and fixed point problems. Such type of methods is called the operators splitting methods. For recent development of the subject, we refer to [9–12]. Noor [13, 14] has used the resolvent operator technique to propose and study some two-step forward-backward splitting methods. It has been noticed that the convergence of such type of splitting algorithms needs relatively relaxed strong monotonicity, which is a weaker constraint than cocoercivity. Glowinski and Tallec [15] and many authors have suggested and analyzed some three-step forward-backward splitting methods for solving various classes of variational inequalities by using the Lagrangian multipliers and auxiliary principle techniques. They have shown that three-step splitting methods are numerically more efficient and handy as compared with one-step and two-step splitting methods. They have studied the convergence of these splitting methods under the assumption that the underlying operators are monotone and Lipschitz continuous. For the convergence analysis of iterative-type splitting methods and their applications, we refer to [16–19] and references therein.

The dynamical system has appeared as a feasible substitute for solving variational inequalities with a specific interest on optimization problems. Dupuis and Nagurney [20], Friesz et al. [21], Noor [22], and many authors introduced and studied many projected dynamical systems associated with variational inequalities. In these dynamical systems, discontinuity appears due to the discontinuity of the projection operator which occurs on the right side of the ordinary differential equation. The novel importance of projected dynamical systems is that the set of stationary points of the projected dynamical systems is the set of solutions of the associated variational inequalities and all those problems which can be studied in the structure of variational inequalities. Since proximal dynamical systems are generalization of projected dynamical systems, therefore clearly the results enhance to global stability of modified projected dynamical systems. Moreover, a vast category of optimization problems can be considered as special cases of mixed variational inequalities (variational-like inequalities) and, therefore, can be solved by using the proximal dynamical systems.

Inspired and motivated by the research works mentioned above, in this article, we introduce and study a system of extended nonlinear mixed variational-like inequalities in real Hilbert space and discuss the existence of solution of our problem. Next, we propose and analyze a new three-step iterative scheme for solving the system of extended nonlinear mixed variational-like inequalities. The convergence and stability analysis for the system of extended nonlinear mixed variational-like inequalities are established. We also study proximal dynamical system associated with the system of extended nonlinear mixed variational-like inequalities. Finally, we show that the trajectory of the solution of extended nonlinear mixed variational-like proximal dynamical system converges globally exponentially to a unique solution of system of extended nonlinear mixed variational-like inequalities.

2. Preliminaries

Throughout this article, we assume that $\mathscr{H}$ is a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

Let us recall the following well-known concepts and results.

Definition 1 (see [9, 23]). Let $g, T\colon \mathscr{H} \longrightarrow \mathscr{H}$ and $\zeta\colon \mathscr{H} \times \mathscr{H} \longrightarrow \mathscr{H}$ be the single-valued mappings. Then,

(i) ζ is said to be τ-Lipschitz continuous if, there exists a constant $\tau > 0$ such that

$$\|\zeta(p,q)\| \le \tau\|p-q\|, \quad \forall p,q \in \mathscr{H}. \tag{1}$$

(ii) ζ is said to be δ-strongly monotone if, there exists a constant $\delta > 0$ such that

$$\langle \zeta(p,q), p-q\rangle \ge \delta\|p-q\|^2, \quad \forall p,q \in \mathscr{H}. \tag{2}$$

(iii) g is said to be μ_g-strongly monotone if, there exists a constant $\mu_g > 0$ such that

$$\langle g(p)-g(q), p-q\rangle \ge \mu_g\|p-q\|^2, \quad \forall p,q \in \mathscr{H}. \tag{3}$$

(iv) g is said to be λ_g-Lipschitz continuous if, there exists a constant $\lambda_g > 0$ such that

$$\|g(p)-g(q)\| \le \lambda_g\|p-q\|, \quad \forall p,q \in \mathscr{H}. \tag{4}$$

(v) T is said to be ζ-relaxed Lipschitz continuous if, there exists a constant $\alpha > 0$ such that

$$\langle T(p)-T(q), \zeta(p,q)\rangle \le -\alpha\|p-q\|^2, \quad \forall p,q \in \mathscr{H}. \tag{5}$$

(vi) T is said to be μ_T-strongly monotone with respect to g if, there exists a constant $\mu_T > 0$ such that

$$\begin{aligned}&\langle T(g(p)) - T(g(q)), g(p)-g(q)\rangle \\ &\quad\ge \mu_T\|p-q\|^2, \quad \forall p,q \in \mathscr{H}.\end{aligned} \tag{6}$$

Definition 2 (see [9]). For each $i = 1,2,\ldots,m$, let $h_i\colon \mathscr{H}_i \longrightarrow \mathscr{H}_i$ and $A_i\colon \prod_{i=1}^m \mathscr{H}_i \longrightarrow \mathscr{H}_i$ be the single-valued mappings. Then, A_i is said to be

(i) γ-Lipschitz continuous in the i^{th}-argument if, there exist a constant $\gamma > 0$ such that

$$\left\|A_i(p_1,p_2,\ldots,p_{i-1},p_i,p_{i+1},\ldots,p_m) - A_i(p_1,p_2,\ldots,p_{i-1},\widehat{p}_i,p_{i+1},\ldots,p_m)\right\|_i \le \gamma\left\|p_i-\widehat{p}_i\right\|_i, \quad \forall p_i,\widehat{p}_i \in \mathscr{H}_i. \tag{7}$$

(ii) ϱ-strongly monotone in the i^{th}-argument if, there exists a constant $\varrho > 0$ such that

$$\langle A_i(p_1,p_2,\ldots,p_{i-1},p_i,p_{i+1},\ldots,p_m) - A_i(p_1,p_2,\ldots,p_{i-1},\widehat{p}_i,p_{i+1},\ldots,p_m), p_i-\widehat{p}_i\rangle \ge \varrho\left\|p_i-\widehat{p}_i\right\|_i^2, \quad \forall p_i,\widehat{p}_i \in \mathscr{H}_i. \tag{8}$$

(iii) μ_{g_i}-strongly monotone with respect to g_i in the i^{th}-argument if, there exists a constant $\mu_{g_i} > 0$ such that

$$\begin{aligned}&\langle A_i(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m) - A_i(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m), g_i(p_i) - g_i(\widehat{p}_i)\rangle \\ &\geq \mu_{g_i} \|g_i(p_i) - g_i(\widehat{p}_i)\|_i^2, \quad \forall p_i, \widehat{p}_i \in \mathscr{H}_i.\end{aligned} \tag{9}$$

Definition 3 (see [24]). A functional $f\colon \mathscr{H} \times \mathscr{H} \longrightarrow \mathbb{R}$ is said to be 0-diagonally quasi-concave (inshort, $0-\text{DQCV}$) in p if, for any finite set $\{p_1, p_2, \ldots, p_n\} \subset \mathscr{H}$ and for any $q = \sum_{i=1}^n t_i p_i$ with $t_i \geq 0$ and $\sum_{i=1}^n t_i = 1$,

$$\min_{1\leq i\leq n} f(p_i, q) \leq 0. \tag{10}$$

Definition 4 (see [24]). Let $\zeta\colon \mathscr{H} \times \mathscr{H} \longrightarrow \mathscr{H}$ be a mapping and $\psi\colon \mathscr{H} \longrightarrow \mathbb{R} \cup \{\infty\}$ be a proper functional. A vector $f^* \in H$ is called an ζ-subgradient of ψ at $p \in \text{do m}\psi$ if

$$\langle f^*, \zeta(q, p)\rangle \leq \psi(q) - \psi(p), \quad \forall q \in \mathscr{H}. \tag{11}$$

Each ψ can be associated with the following map $\partial_\zeta \psi$, called ζ-subdifferential of ψ at p, defined by

$$\partial_\zeta \psi(p) = \begin{cases} f^* \in \mathscr{H}\colon \langle f^*, \zeta(q, p)\rangle \leq \psi(q) - \psi(p), \forall q \in \mathscr{H}, & p \in \text{dom}\psi, \\ \varnothing, & p \notin \text{dom}\psi. \end{cases} \tag{12}$$

Definition 5 (see [12]). Let $\psi\colon \mathscr{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, ζ-subdifferential (may not be convex) functional, $\zeta\colon \mathscr{H} \times \mathscr{H} \longrightarrow \mathscr{H}$, $T\colon \mathscr{H} \longrightarrow \mathscr{H}$ be the mappings, and $I\colon \mathscr{H} \longrightarrow \mathscr{H}$ be an identity mapping. If for any given $z \in \mathscr{H}$ and $\rho > 0$, there exists a unique point $p \in \mathscr{H}$ satisfying

$$\langle (I - T)p - z, \zeta(q, p)\rangle \geq \rho\psi(p) - \rho\psi(q), \quad \forall q \in \mathscr{H}, \tag{13}$$

then the mapping $z \longmapsto p$, denoted by $\mathscr{J}_{\rho,T}^{\partial_\zeta \psi}(z)$, is said to be relaxed ζ-proximal operator of ψ. We have $z - (I - T)p \in \rho\partial_\zeta \psi(p)$, and it follows that

$$\mathscr{J}_{\rho,T}^{\partial_\zeta \psi}(z) = \left[(I - T) + \rho\partial_\zeta \psi\right]^{-1}(z). \tag{14}$$

Definition 6 (see [25]). Let $S, T\colon \mathscr{H} \longrightarrow \mathscr{H}$ be the single-valued mappings, $p_0 \in \mathscr{H}$ and

$$p_{n+1} = S(T, p_n), \tag{15}$$

defines an iterative sequence which yields a sequence of points $\{p_n\}$ in $\mathscr{H}$. Suppose that $Fix(T) = \{p \in \mathscr{H}\colon\ Tp = p\} \neq \varnothing$ and $\{p_n\}$ converges to a fixed point p^* of T. Let $\{u_n\} \subset \mathscr{H}$ and

$$\vartheta_n = \|u_{n+1} - S(T, u_n)\|. \tag{16}$$

If $\lim_{n\longrightarrow\infty} \vartheta_n = 0$, which implies that $u_n \longrightarrow p^*$, then the iterative sequence $\{p_n\}$ is said to be T-stable or stable with respect to T.

Theorem 1 (see [12]). *Let $\zeta\colon \mathscr{H} \times \mathscr{H} \longrightarrow \mathscr{H}$ be a δ-strongly monotone and τ-Lipschitz continuous mapping such that $\zeta(p, q) = -\zeta(q, p)$, for all $p, q \in \mathscr{H}$. Let $T\colon \mathscr{H} \longrightarrow \mathscr{H}$ be ζ-relaxed Lipschitz continuous mapping with constant α and $I\colon H \longrightarrow H$ be an identity mapping. Let $\psi\colon \mathscr{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, lower-semicontinuous, ζ-subdifferential functional which may not be convex, and for any $z, p \in \mathscr{H}$, the mapping $f(q, p) = \langle z - (I - R)p, \zeta(q, p)\rangle$ is 0-DQCV in q. Then, for any $\rho > 0$, and any $z \in \mathscr{H}$, there exists a unique $p \in \mathscr{H}$ such that $p = \mathscr{J}_{\rho,T}^{\partial_\zeta \psi}(z)$, and hence, the relaxed ζ-proximal operator $\mathscr{J}_{\rho,T}^{\partial_\zeta \psi}$ of ψ is well-defined and $(T/(\alpha + \delta))$-Lipschitz continuous, i.e.,*

$$\left\|\mathscr{J}_{\rho,T}^{\partial_\zeta \psi}(p) - \mathscr{J}_{\rho,T}^{\partial_\zeta \psi}(q)\right\| \leq \frac{\tau}{(\alpha + \delta)}\|p - q\|, \quad \forall p, q \in \mathscr{H}. \tag{17}$$

Lemma 1 (see [26]). *Let $\{\wp_n\}, \{\nu_n\}$, and $\{\varrho_n\}$ be nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that*

$$\wp_{n+1} \leq (1 - \varpi_n)\wp_n + \nu_n\varpi_n + \varrho_n, \quad \forall n \geq n_0, \tag{18}$$

where $\varpi \in [0, 1]$, $\sum_{n=0}^\infty \varpi_n = \infty$, $\lim_{n\longrightarrow\infty} \nu_n = 0$, $\sum_{n=0}^\infty \varrho_n < \infty$. Then, $\lim_{n\longrightarrow\infty} \wp_n = 0$.

3. Formulation of the Problem and Existence Result

For each $i \in \Lambda = \{1, 2, 3, \ldots, m\}$, let $\mathscr{H}_i$ be a real Hilbert space equipped with the norm $\|.\|_i$, and let $h_i, g_i, T_i\colon \mathscr{H}_i \longrightarrow \mathscr{H}_i$, $\zeta_i\colon \mathscr{H}_i \times \mathscr{H}_i \longrightarrow \mathscr{H}_i$, and $A_i\colon \prod_{j=1}^m \mathscr{H}_j \longrightarrow \mathscr{H}_i$ be the nonlinear single-valued mappings, respectively. Let $\psi_i\colon \mathscr{H}_i \times \mathscr{H}_i \longrightarrow \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $p_i \in \mathscr{H}_i$, $\psi_i(\cdot, p_i)$ is lower semicontinuous, ζ_i-subdifferential, proper functional on $\mathscr{H}_i \times \mathscr{H}_i$ (may not be convex) satisfying $h_i(\mathscr{H}_i) \cap \text{do m}(\partial\psi_{\zeta_i}(\cdot, p_i)) \neq \varnothing$, where $\partial\psi_{\zeta_i}(\cdot, p_i)$ is a ζ_i-subdifferential of $\psi_i(\cdot, p_i)$. We consider the following system of extended nonlinear mixed variational-like inequalities (in short, SENMVLI).

For each $\rho_i > 0$, find $(p_1, p_2, \ldots, p_m) \in \prod_{i=1}^m \mathscr{H}_i$ such that $h_i(p_i) \in \mathrm{dom}(\partial_{\zeta_i}\psi_i(\cdot, p_i))$ and for all $q_i \in \mathscr{H}_i$

$$\begin{cases} \langle h_1(p_1) - (g_1(p_1) - \rho_1 A_1(p_1, p_2, \ldots, p_m)), \zeta_1(q_1, h_1(p_1))\rangle \geq \rho_1\psi_1(h_1(p_1), p_1) - \rho_1\psi_1(q_1, p_1), \\ \langle h_2(p_2) - (g_2(p_2) - \rho_2 A_2(p_1, p_2, \ldots, p_m)), \zeta_2(q_2, h_2(p_2))\rangle \geq \rho_2\psi_2(h_2(p_2), p_2) - \rho_2\psi_2(q_2, p_2), \\ \langle h_3(p_3) - (g_3(p_3) - \rho_3 A_3(p_1, p_2, \ldots, p_m)), \zeta_3(q_3, h_3(p_3))\rangle \geq \rho_3\psi_3(h_3(p_3), p_3) - \rho_3\psi_3(q_3, p_3), \\ \vdots \\ \langle h_m(p_m) - (g_m(p_m) - \rho_m A_m(p_1, p_2, \ldots, p_m)), \zeta_m(q_m, h_m(p_m))\rangle \geq \rho_m\psi_m(h_m(p_m), p_m) - \rho_m\psi_m(q_m, p_m). \end{cases} \tag{19}$$

Equivalently, for each $i \in \Lambda$, above system can be written as

$$\langle h_i(p_i) - (g_i(p_i) - \rho_i A_i(p_1, p_2, \ldots, p_m)), \zeta_i(q_i, h_i(p_i))\rangle \geq \rho_i\psi_i(h_i(p_i), p_i) - \rho_i\psi_i(q_i, p_i). \tag{20}$$

Some special cases of problem (19) are as follows:

(i) If $\rho_i = 1, \psi_i = 0$, $h_i, g_i = I_i$ (identity mappings) and $\psi_i(q_i, p_i) = \psi_i(q_i)$, for each $i \in \Lambda$, then problem (20) reduces to the problem of finding $(p_1, p_2, \ldots, p_m) \in \prod_{i=1}^m \mathscr{H}_i$ such that

$$\langle A_i(p_1, p_2, \ldots, p_m), \zeta_i(q_i, p_i)\rangle \geq \psi_i(p_i) - \psi_i(q_i). \tag{21}$$

Problem (21) was considered and studied by Balooee [27].

(ii) If $\rho_i = 1, \psi_i = 0$, $h_i, g_i = I_i$ (identity mappings), and $\zeta_i(q_i, h_i(p_i)) = q_i - p_i$, for each $i \in \Lambda$, then problem (19) reduces to the problem of finding $(p_1, p_2, \ldots, p_m) \in \prod_{i=1}^m \mathscr{H}_i$ such that

$$\begin{cases} \langle A_1(p_1, p_2, \ldots, p_m), q_1 - p_1\rangle \geq 0, \\ \langle A_2(p_1, p_2, \ldots, p_m), q_2 - p_2\rangle \geq 0, \\ \langle A_3(p_1, p_2, \ldots, p_m), q_3 - p_3\rangle \geq 0, \\ \vdots \\ \langle A_m(p_1, p_2, \ldots, p_m), q_m - p_m\rangle \geq 0. \end{cases} \tag{22}$$

Problem (22) was considered and studied by Tang et al. [28].

By taking suitable choices of the mappings $g_i, h_i, T_i, \psi_i, \zeta_i$ and the space $\mathscr{H}_i$, for each $i \in \Lambda$, in above problem (19), one can easily obtain the problems considered and studied in [9, 13, 14, 22, 29, 30] and references therein.

Example 1. Let $\mathbb{R} = (-\infty, \infty), \mathscr{H}_i = [a, b]$. Let $G(p_1, p_2, \ldots, p_m)$ be a continuous real m-variable function with $G \in C^{(1)}(\mathscr{H}_i)$. Then, there exists an element $p_0 = (p_{0,1}, p_{0,2}, \ldots, p_{0,m}) \in \prod_{i=1}^m \mathscr{H}_i$ such that

$$G(p_{0,1}, p_{0,2}, \ldots, p_{0,m}) = \min_{(p_1, p_2, \ldots, p_m)} \in \prod_{i=1}^m \mathscr{H}_i G(p_1, p_2, \ldots, p_m). \tag{23}$$

This element p_0 must be a solution of the following system of variational inequalities:

$$\begin{cases} \langle \dfrac{\partial G}{\partial p_1}(p_1, p_2, \ldots, p_m), q_1 - p_1\rangle \geq 0, \\ \langle \dfrac{\partial G}{\partial p_2}(p_1, p_2, \ldots, p_m), q_2 - p_2\rangle \geq 0, \\ \langle \dfrac{\partial G}{\partial p_3}(p_1, p_2, \ldots, p_m), q_3 - p_3\rangle \geq 0, \\ \vdots \\ \langle \dfrac{\partial G}{\partial p_m}(p_1, p_2, \ldots, p_m), q_m - p_m\rangle \geq 0. \end{cases} \tag{24}$$

If fact, we have

$$\frac{\partial G}{\partial p_i}(p_1, p_2, \ldots, p_m) \begin{cases} = 0, p_{0,i} \in (a, b), \\ \geq 0, p_{0,i} = a, \\ \leq 0, p_{0,i} = b, \end{cases} \tag{25}$$

for all $i = 1, 2, \ldots, m$. Hence, p_0 must satisfy (24). In addition, the system of variational inequalities (24) is equivalent to

$$\langle \mathrm{grad}\, G(p), q - p\rangle \geq 0, \tag{26}$$

where $\mathrm{grad}\, G(x) = ((\partial G/\partial p_1), (\partial G/\partial p_2), \ldots, (\partial G/\partial p_m))$. This example is special case of a practical background of problem (19), where $A_i = (\partial G/\partial p_i), h_i, g_i = I_i, \rho_i = 1$ and $\psi_i = 0$, for all $i = 1, 2, \ldots, m$ and $\mathrm{grad}\, G(x) = A^*$.

The following lemma ensures the equivalence between the system of extended nonlinear mixed variational-like inequalities (19) and fixed point problem.

Lemma 2. *For each $i \in \Lambda$, let $(p_1, p_2, \ldots, p_m) \in \prod_{i=1}^{m} \mathscr{H}_i$ is a solution of the system of extended nonlinear mixed variational-like inequalities (19) if and only if $(p_1, p_2, \ldots, p_m)$ satisfies the following equation:*

$$h_i(p_i) = \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - (T_i(h_i(p_i)) + \rho_i A_i(p_1, p_2, \ldots, p_m))], \tag{27}$$

where $\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)} = [(I_i - T_i) + \rho_i \partial_{\zeta_i}\psi_i(.,p_i)]^{-1}$, T_i is ζ_i-relaxed Lipschitz continuous mapping with constant α_i, and I_i is the identity mapping on $\mathscr{H}_i$.

Proof. Assume that $(p_1, p_2, \ldots, p_m) \in \prod_{i=1}^{m} \mathscr{H}_i$ satisfies relation (27), i.e.,

$$h_i(p_i) = \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - (T_i(h_i(p_i)) + \rho_i A_i(p_1, p_2, \ldots, p_m))]. \tag{28}$$

Since $\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)} = [(I_i - T_i) + \rho_i \partial_{\zeta_i}(.,p_i)]^{-1}$, the above equality holds if and only if

$$\begin{aligned} &g_i(p_i) - (T_i(h_i(p_i)) + \rho_i A_i(p_1, p_2, \ldots, p_m)) \\ &\quad \in h_i(p_i) - T_i(h_i(p_i)) + \rho_i \partial_{\zeta_i}\psi_i(h_i(p_i), p_i). \end{aligned} \tag{29}$$

By using the definition of ζ_i-subdifferential of $\psi_i(.,p_i)$, the above relation holds if and only if

$$\begin{aligned} &\langle (g_i(p_i) - \rho_i A_i(p_1, p_2, \ldots, p_m)) - h_i(p_i), \zeta_i(q_i, h_i(p_i))\rangle \\ &\quad \leq \rho_i \psi_i(q_i, p_i) - \rho_i \psi_i(h_i(p_i), p_i). \end{aligned} \tag{30}$$

Hence, we have

$$\begin{aligned} &\langle h_i(p_i) - (g_i(p_i) - \rho_i A_i(p_1, p_2, \ldots, p_m)), \zeta_i(q_i, h_i(p_i))\rangle \\ &\quad \geq \rho_i \psi_i(h_i(p_i), p_i) - \rho_i \psi_i(q_i, p_i), \end{aligned} \tag{31}$$

i.e., $(p_1, p_2, \ldots, p_m)$ is a solution of system of extended nonlinear mixed variational-like inequalities (19). □

In the next theorem, we discuss the existence and uniqueness of the solution of the SENMVLI (19).

Theorem 2. *For each $i \in \Lambda$, let $g_i, h_i, T_i: \mathscr{H}_i \longrightarrow \mathscr{H}_i$, $\zeta_i: \mathscr{H}_i \times \mathscr{H}_i \longrightarrow \mathscr{H}_i$, and $A_i: \prod_{i=1}^{m} \mathscr{H}_i \longrightarrow \mathscr{H}_i$ be the nonlinear single-valued mappings such that g_i is λ_{g_i}-Lipschitz continuous and μ_{g_i}-strongly monotone, h_i is λ_{h_i}-Lipschitz continuous and μ_{h_i}-strongly monotone such that $h_i(\mathscr{H}_i) = \mathscr{H}_i$, T_i is λ_{T_i}-Lipschitz continuous, relaxed α_i-Lipschitz continuous, and μ_{T_i}-strongly monotone with respect to h_i, ζ_i is τ_i-Lipschitz continuous, and ζ_i is δ_i-strongly monotone such that $\zeta_i(p_i, q_i) = -\zeta_i(q_i, p_i)$, for each $p_i, q_i \in \mathscr{H}_i$, A_i is $\lambda_{A_{ii}}$-Lipschitz continuous in the i^{th}-argument and $\nu_{i,j}$-Lipschitz continuous in the j^{th}-argument for each $j \in \Lambda, i \neq j$, and $\mu_{A_{ii}}$-strongly monotone in the i^{th}-argument with respect to g_i, respectively. Let $\psi_i: \mathscr{H}_i \times \mathscr{H}_i \longrightarrow \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $p_i \in \mathscr{H}_i$, $\psi_i(\cdot, p_i)$ is lower-semicontinuous, ζ_i-subdifferential, proper functional on $\mathscr{H}_i \times \mathscr{H}_i$ (may not be convex) satisfying $h_i(\mathscr{H}_i) \cap \mathrm{dom}(\partial\psi_{\zeta_i}(\cdot, p_i)) \neq \varnothing$, where $\partial\psi_{\zeta_i}(\cdot, p_i)$ is a ζ_i-subdifferential of $\psi_i(\cdot, p_i)$. Suppose that there exist constants $\rho_i > 0$, $\xi_i > 0$ such that for each $z_i \in \mathscr{H}_i$*

$$\left\| \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}(z_i) - \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,q_i)}(z_i) \right\| \leq \xi_i \|p_i - q_i\|, \tag{32}$$

and the following conditions are satisfied:

$$\begin{cases} \theta_i = \xi_i + \sqrt{\left(1 - 2\mu_{h_i} + \lambda_{h_i}^2\right)} + \sum_{k \in \Lambda, k \neq i} \dfrac{\tau_k \rho_k}{\alpha_k + \delta_k} \nu_{k,i} < 1, \\ \tau_i \sqrt{\left(\lambda_{g_i}^2 - 2\rho_i \mu_{A_{ii}} + \rho_i^2 \lambda_{A_{ii}}^2\right)} < \left[(1 - \theta_i)(\alpha_i + \delta_i) - \tau_i \lambda_{T_i} \lambda_{h_i}\right], \\ 2\mu_{h_i} < 1 + \lambda_{h_i}^2, \\ 2\rho_i \mu_{A_{ii}} < \lambda_{g_i}^2 + \rho_i^2 \lambda_{A_{ii}}^2. \end{cases} \tag{33}$$

Then, the SENMVLI (19) admits a unique solution $(p_1^*, p_2^*, \ldots, p_m^*)$.

Proof. By Lemma 2, it is sufficient to prove that there exist $(p_1^*, p_2^*, \ldots, p_m^*)$ which satisfying (27). For each $i \in \Lambda$, we define $\phi_i: \prod_{i=1}^{m} \mathscr{H}_i \longrightarrow \mathscr{H}_i$ by

$$\phi_i(p_1, p_2, \ldots, p_m) = p_i - h_i(p_i) + \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - (T_i(h_i(p_i)) + \rho_i A_i(p_1, p_2, \ldots, p_m))], \tag{34}$$

for all $(p_1, p_2, \ldots, p_m) \in \prod_{i=1}^{m} \mathscr{H}_i$. Define $\|.\|_*$ on $\prod_{i=1}^{m} \mathscr{H}_i$ by

$$\left\|\left(p_1, p_2, p_3, \ldots, p_m\right)\right\|_* = \sum_{i=1}^{m} \left\|p_i\right\|_i, \forall \left(p_1, p_2, \ldots, p_m\right) \in \prod_{i=1}^{m} \mathscr{H}_i. \tag{35}$$

It is easy to see that $\left(\prod_{i=1}^{m}\mathscr{H}_i, \|.\|_*\right)$ is a Hilbert space. Also, define $G\colon \prod_{i=1}^{m}\mathscr{H}_i \longrightarrow \prod_{i=1}^{m}\mathscr{H}_i$ as follows:

$$G\left(p_1, p_2, \ldots, p_m\right) = \left(\phi_1\left(p_1, p_2, \ldots, p_m\right), \phi_2\left(p_1, p_2, \ldots, p_m\right), \ldots, \phi_m\left(p_1, p_2, \ldots, p_m\right)\right), \tag{36}$$

for all $\left(p_1, p_2, \ldots, p_m\right) \in \prod_{i=1}^{m}\mathscr{H}_i$. First of all, we prove that G is a contraction mapping.

Let $\left(p_1, p_2, \ldots, p_m\right), \left(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m\right) \in \prod_{i=1}^{m}\mathscr{H}_i$ be given. By using (32) and (34) and Theorem 1, for each $i \in \Lambda$, we have

$$\begin{aligned}
&\left\|\phi_i\left(p_1, p_2, \ldots, p_m\right) - \phi_i\left(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m\right)\right\|_i \\
&= \left\|\left[p_i - h_i\left(p_i\right) + \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i\left(.,p_i\right)}\left[g_i\left(p_i\right) - \left(T_i\left(h_i\left(p_i\right)\right) + \rho_i A_i\left(p_1, p_2, \ldots, p_m\right)\right)\right]\right] - \left[\widehat{p}_i - h_i\left(\widehat{p}_i\right) + \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i\left(.,\widehat{p}_i\right)}\left[g_i\left(\widehat{p}_i\right) - \left(T_i\left(h_i\left(\widehat{p}_i\right)\right) + \rho_i A_i\left(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m\right)\right)\right]\right]\right\|_i \\
&\le \left\|\left(p_i - h_i\left(p_i\right)\right) - \left(\widehat{p}_i - h_i\left(\widehat{p}_i\right)\right)\right\|_i + \left\|\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i\left(.,p_i\right)}\left[g_i\left(p_i\right) - \left(T_i\left(h_i\left(p_i\right)\right) + \rho_i A_i\left(\left(p_1, p_2, \ldots, p_m\right)\right)\right)\right] - \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i\left(.,\widehat{p}_i\right)}\left[g_i\left(p_i\right) - \left(T_i\left(h_i\left(p_i\right)\right) + \rho_i A_i\left(\left(p_1, p_2, \ldots, p_m\right)\right)\right)\right]\right\|_i \\
&\quad + \left\|\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i\left(.,\widehat{p}_i\right)}\left[g_i\left(p_i\right) - \left(T_i\left(h_i\left(p_i\right)\right) + \rho_i A_i\left(\left(p_1, p_2, \ldots, p_m\right)\right)\right)\right] - \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i\left(.,\widehat{p}_i\right)}\left[g_i\left(\widehat{p}_i\right) - \left(T_i\left(h_i\left(\widehat{p}_i\right)\right) + \rho_i A_i\left(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m\right)\right)\right]\right\|_i \le \left\|\left(p_i - h_i\left(p_i\right)\right) - \left(\widehat{p}_i - h_i\left(\widehat{p}_i\right)\right)\right\|_i + \xi_i\left\|p_i - \widehat{p}_i\right\|_i \\
&\quad + \left\|\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i\left(.,\widehat{p}_i\right)}\left[g_i\left(p_i\right) - \left(T_i\left(h_i\left(p_i\right)\right) + \rho_i A_i\left(p_1, p_2, \ldots, p_m\right)\right)\right] - \mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i\left(.,\widehat{p}_i\right)}\left[g_i\left(\widehat{p}_i\right) - \left(T_i\left(h_i\left(\widehat{p}_i\right)\right) + \rho_i A_i\left(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m\right)\right)\right]\right\|_i \\
&\quad + \frac{\tau_i}{\alpha_i + \delta_i}\left\|\left[g_i\left(p_i\right) - \left(T_i\left(h_i\left(p_i\right)\right) + \rho_i A_i\left(p_1, p_2, \ldots, p_m\right)\right)\right] - \left[g_i\left(\widehat{p}_i\right) - \left(T_i\left(h_i\left(\widehat{p}_i\right)\right) + \rho_i A_i\left(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m\right)\right)\right]\right\|_i \le \left\|\left(p_i - h_i\left(p_i\right)\right) - \left(\widehat{p}_i - h_i\left(\widehat{p}_i\right)\right)\right\|_i + \xi_i\left\|p_i - \widehat{p}_i\right\|_i \\
&\quad + \frac{\tau_i}{\alpha_i + \delta_i}\left\|\left[\left(g_i\left(p_i\right) - g_i\left(\widehat{p}_i\right)\right) - \left(T_i\left(h_i\left(p_i\right)\right) - T_i\left(h_i\left(\widehat{p}_i\right)\right)\right)\right] - \rho_i\left[A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m\right) - A_i\left(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m\right)\right]\right\|_i \\
&\quad + \frac{\rho_i\tau_i}{\alpha_i + \delta_i}\left[\begin{array}{l} \mathbf{1}\left\|A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m\right) - A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, \widehat{p}_m\right)\right\|_m + \left\|A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, \widehat{p}_m\right) - A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, \widehat{p}_m\right)\right\|_{m-1} \\ + \left\|A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, \widehat{p}_m\right) - A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, \widehat{p}_m\right)\right\|_{m-2} \cdots + \left\|A_i\left(p_1, \widehat{p}_2, \ldots, \widehat{p}_{i-1}, \widehat{p}_i, \widehat{p}_{i+1}, \ldots, \widehat{p}_m\right) - A_i\left(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_{i-1}, \widehat{p}_i, \widehat{p}_{i+1}, \ldots, \widehat{p}_m\right)\right\|_1 \end{array}\right] \\
&\le \left\|\left(p_i - h_i\left(p_i\right)\right) - \left(\widehat{p}_i - h_i\left(\widehat{p}_i\right)\right)\right\|_i + \xi_i\left\|p_i - \widehat{p}_i\right\|_i \\
&\quad + \frac{\tau_i}{\alpha_i + \delta_i}\left\|\left[\left(g_i\left(p_i\right) - g_i\left(\widehat{p}_i\right)\right) - \left(T_i\left(h_i\left(p_i\right)\right) - T_i\left(h_i\left(\widehat{p}_i\right)\right)\right)\right] - \rho_i\left[A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m\right) - A_i\left(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m\right)\right]\right\|_i \\
&\quad + \frac{\tau_i\rho_i}{\alpha_i + \delta_i}\sum_{j\in\Lambda, i\neq j}\left(\left\|A_i\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_m\right) - A_i\left(p_1, p_2, \ldots, p_{j-1}, \widehat{p}_j, p_{j+1}, \ldots, \widehat{p}_m\right)\right\|_i\right) \le \left\|\left(p_i - \widehat{p}_i\right) - \left(h_i\left(p_i\right) - h_i\left(\widehat{p}_i\right)\right)\right\|_i + \xi_i\left\|p_i - \widehat{p}_i\right\|_i \\
&\quad + \frac{\tau_i}{\alpha_i + \delta_i}\left[\left\|T_i\left(h_i\left(p_i\right)\right) - T_i\left(h_i\left(\widehat{p}_i\right)\right)\right\|_i + \left\|\left(g_i\left(p_i\right) - g_i\left(\widehat{p}_i\right)\right) - \rho_i\left[A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m\right) - A_i\left(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m\right)\right]\right\|_i\right] \\
&\quad + \frac{\tau_i\rho_i}{\alpha_i + \delta_i}\sum_{j\in\Lambda, i\neq j}\left(\left\|A_i\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_m\right) - A_i\left(p_1, p_2, \ldots, p_{j-1}, \widehat{p}_j, p_{j+1}, \ldots, \widehat{p}_m\right)\right\|_i\right) \le \left\|\left(p_i - \widehat{p}_i\right) - \left(h_i\left(p_i\right) - h_i\left(\widehat{p}_i\right)\right)\right\|_i + \xi_i\left\|p_i - \widehat{p}_i\right\|_i \\
&\quad + \frac{\tau_i}{\alpha_i + \delta_i}\left[\lambda_{T_i}\lambda_{h_i}\left\|p_i - \widehat{p}_i\right\|_i + \left\|\left(g_i\left(p_i\right) - g_i\left(\widehat{p}_i\right)\right) - \rho_i\left[A_i\left(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m\right) - A_i\left(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m\right)\right]\right\|_i\right] \\
&\quad + \frac{\tau_i\rho_i}{\alpha_i + \delta_i}\sum_{j\in\Lambda, i\neq j}\left(\left\|A_i\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_m\right) - A_i\left(p_1, p_2, \ldots, p_{j-1}, \widehat{p}_j, p_{j+1}, \ldots, \widehat{p}_m\right)\right\|_i\right).
\end{aligned} \tag{37}$$

It follows from μ_{h_i}-strongly monotonicity and λ_{h_i}-Lipschitz continuity of h_i that

$$\begin{aligned}
\left\|p_i - \widehat{p}_i - \left(h_i\left(p_i\right) - h_i\left(\widehat{p}_i\right)\right)\right\|_i^2 &= \left\|p_i - \widehat{p}_i\right\|_i^2 - 2\left\langle h_i\left(p_i\right) - h_i\left(\widehat{p}_i\right), p_i - \widehat{p}_i\right\rangle + \left\|h_i\left(p_i\right) - h_i\left(\widehat{p}_i\right)\right\|_i^2 \\
&\le \left\|p_i - \widehat{p}_i\right\|_i^2 - 2\mu_{h_i}\left\|p_i - \widehat{p}_i\right\|_i^2 + \lambda_{h_i}^2\left\|p_i - \widehat{p}_i\right\|_i^2 \\
&= \left(1 - 2\mu_{h_i} + \lambda_{h_i}^2\right)\left\|p_i - \widehat{p}_i\right\|_i^2,
\end{aligned} \tag{38}$$

i.e.,

$$\left\| p_i - \widehat{p}_i - (h_i(p_i) - h_i(\widehat{p}_i)) \right\|_i \leq \sqrt{\left(1 - 2\mu_{h_i} + \lambda_{h_i}^2\right)} \left\| p_i - \widehat{p}_i \right\|_i. \tag{39}$$

By using the $\lambda_{A_{ii}}$-Lipschitz continuity of A_i and $\mu_{A_{ii}}$-strongly monotonicity of A_i with respect to g_i and λ_{g_i}-Lipschitz continuity of g_i, respectively, we evaluate

$$\begin{aligned}
&\left\| (g_i(p_i) - g_i(\widehat{p}_i)) - \rho_i [A_i(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m) - A_i(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m)] \right\|_i^2 \\
&\leq \left\| (g_i(p_i) - g_i(\widehat{p}_i)) \right\|^2 - 2\rho_i \langle A_i(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m) - A_i(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m), (g_i(p_i) - g_i(\widehat{p}_i)) \rangle \\
&\quad + \rho_i^2 \left\| A_i(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m) - A_i(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m) \rangle \right\|_i^2 \\
&\leq \lambda_{g_i}^2 \left\| p_i - \widehat{p}_i \right\|_i^2 - 2\rho_i \langle A_i(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m) - A_i(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m), g_i(p_i) - g_i(\widehat{p}_i) \rangle \\
&\quad + \rho_i^2 \lambda_{A_{ii}}^2 \left\| p_i - \widehat{p}_i \right\|_i^2 \\
&\leq \lambda_{g_i}^2 \left\| p_i - \widehat{p}_i \right\|_i^2 - 2\rho_i \mu_{A_{ii}} \left\| p_i - \widehat{p}_i \right\|_i^2 + \rho_i^2 \lambda_{A_{ii}}^2 \left\| p_i - \widehat{p}_i \right\|_i^2 \\
&= \left(\lambda_{g_i}^2 - 2\rho_i \mu_{A_{ii}} + \rho_i^2 \lambda_{A_{ii}}^2 \right) \left\| p_i - \widehat{p}_i \right\|_i^2,
\end{aligned} \tag{40}$$

which implies that

$$\begin{aligned}
&\left\| (g_i(p_i) - g_i(\widehat{p}_i)) - \rho_i [A_i(p_1, p_2, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m) - A_i(p_1, p_2, \ldots, p_{i-1}, \widehat{p}_i, p_{i+1}, \ldots, p_m)] \right\|_i \\
&\quad \leq \sqrt{\left(\lambda_{g_i}^2 - 2\rho_i \mu_{A_{ii}} + \rho_i^2 \lambda_{A_{ii}}^2 \right)} \left\| p_i - \widehat{p}_i \right\|_i.
\end{aligned} \tag{41}$$

Since for $i \in \Lambda$, A_i is $\nu_{i,j}$-Lipschitz continuous in the j^{th} argument $(j \in \Lambda, j \neq i)$, we have

$$\left\| A_i\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_m\right) - A_i\left(p_1, p_2, \ldots, p_{j-1}, \widehat{p}_j, p_{j+1}, \ldots, p_m\right)_i \leq \nu_{i,j} \right\| p_j - \widehat{p}_{j\,j}. \tag{42}$$

Substituting (39)–(42) in (37), for $i \in \Lambda$, we deduce that

$$\begin{aligned}
\left\| \phi_i(p_1, p_2, \ldots, p_m) - \phi_i(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m) \right\|_i \leq &\left[\xi_i + \sqrt{\left(1 - 2\mu_{h_i} + \lambda_{h_i}^2\right)} + \frac{\tau_i \left(\lambda_{T_i} \lambda_{h_i} + \sqrt{\left(\lambda_{g_i}^2 - 2\rho_i \mu_{A_{ii}} + \rho_i^2 \lambda_{A_{ii}}^2 \right)} \right)}{\alpha_i + \delta_i} \right] \left\| p_i - \widehat{p}_i \right\|_i \\
&+ \frac{\tau_i \rho_i}{\alpha_i + \delta_i} \sum_{j \in \Lambda, i \neq j} \nu_{i,j} \left\| p_j - \widehat{p}_j \right\|_j,
\end{aligned} \tag{43}$$

i.e.,

$$\left\| \phi_i(p_1, p_2, \ldots, p_m) - \phi_i(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m) \right\|_i \leq \Theta_i \left\| p_i - \widehat{p}_{ii} + \frac{\tau_i \rho_i}{\alpha_i + \delta_i} \sum_{j \in \Lambda, i \neq j} \nu_{i,j} \right\| p_j - \widehat{p}_j \Bigg\|_j, \tag{44}$$

where $\Theta_i = \xi_i + \sqrt{(1 - 2\mu_{h_i} + \lambda_{h_i}^2)} + \tau_i (\lambda_{T_i}\lambda_{h_i} + \sqrt{(\lambda_{g_i}^2 - 2\rho_i\mu_{A_{ii}} + \rho_i^2\lambda_{A_{ii}}^2)})/\alpha_i + \delta_i$.

From (36) and (42), we get

$$\begin{aligned}\|G(p_1, p_2, \ldots, p_m) - G(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\|_* &= \sum_{i=1}^{m} \|\phi_i(p_1, p_2, \ldots, p_m) - \phi_i(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\|_i \\ &\leq \sum_{i=1}^{m} \left(\Theta_i \left\| p_i - \widehat{p}_i \right\|_i + \frac{\tau_i \rho_i}{\alpha_i + \delta_i} \sum_{j \in \Lambda, i \neq j} \nu_{i,j} \left\| p_j - \widehat{p}_j \right\|_j \right) \\ &= \left(\Theta_1 + \sum_{k=2}^{m} \frac{\tau_k \rho_k}{\alpha_k + \delta_k} \nu_{k,1} \right) \|p_1 - \widehat{p}_1\|_1 \\ &\quad + \left(\Theta_2 + \sum_{k \in \Lambda, k \neq 2} \frac{\tau_k \rho_k}{\alpha_k + \delta_k} \nu_{k,2} \right) \|p_2 - \widehat{p}_2\|_2 + \cdots + \left(\Theta_m + \sum_{k=1}^{m-1} \frac{\tau_k \rho_k}{\alpha_k + \delta_k} \nu_{k,m} \right) \|p_m - \widehat{p}_m\|_m \\ &\leq \max \left\{ \Theta_i + \sum_{k \in \Lambda, k \neq i} \frac{\tau_k \rho_k}{\alpha_k + \delta_k} \nu_{k,i} \colon i \in \Lambda \right\} \sum_{i=1}^{m} \|p_i - \widehat{p}_i\|_i,\end{aligned} \tag{45}$$

i.e.,

$$\|G(p_1, p_2, \ldots, p_m) - G(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\|_* \leq \Omega \|(p_1, p_2, \ldots, p_m) - (\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\|_*, \tag{46}$$

where $\Omega = \max\left\{\Theta_i + \sum_{k \in \Lambda, k \neq i} (\tau_k \rho_k)/(\alpha_k + \delta_k)\nu_{k,i} \colon i \in \Lambda\right\}$. The condition (33) guarantees that $0 \leq \Omega < 1$. By the inequality (44), we note that G is a contraction mapping. Therefore, there exists a unique point $(p_1^*, p_2^*, \ldots, p_m^*) \in \prod_{i=1}^{m} \mathscr{H}_i$ such that $G(p_1^*, p_2^*, \ldots, p_m^*) = (p_1^*, p_2^*, \ldots, p_m^*)$. From (34) and (36), it follows that $(p_1^*, p_2^*, \ldots, p_m^*)$ satisfies in equation (27), i.e., for each $i \in \Lambda$,

$$h_i(p_i^*) = \mathscr{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(\cdot, p_i^*)} [g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]. \tag{47}$$

By Lemma 2, we conclude that $(p_1^*, p_2^*, \ldots, p_m^*) \in \prod_{i=1}^{m} \mathscr{H}_i$ is a unique solution of SENMVLI (19). This completes the proof. □

4. Proximal Iterative Schemes and Stability Analysis

In this section, we first recall some definitions related to nearly uniformly Lipschitzian mapping. Furthermore, we use a nearly uniformly Lipschitzian mapping $Q_i, i \in \Lambda$, to define a self-mapping $S = (Q_1, Q_2, \ldots, Q_m)$ on $\prod_{i=1}^{m} \mathscr{H}_i$, by using the equivalent alternative formulation (27) to suggest and analyze some proximal iterative algorithms for finding an element of the set of the fixed points of S which is the unique solution of the problem SENMVLI (19).

Definition 7 (see [9]). A nonlinear mapping $Q \colon \mathscr{H} \longrightarrow \mathscr{H}$ is said to be

(i) generalized Lipschitzian if, there exists a constant $k' > 0$ *such that*

$$\|Q(p) - Q(q)\| \leq k' (\|p - q\| + 1), \quad \forall p, q \in \mathscr{H}, \tag{48}$$

(ii) uniformly k-Lipschitzian if, there exists a constant $k > 0$ such that for each $n \in \mathbb{N}$,

$$\left\|Q^n(p) - Q^n(q)\right\| \leq k\|p - q\|, \quad \forall p, q \in \mathscr{H}, \tag{49}$$

(iii) nearly Lipschitzian with respect to the sequence $\{\alpha_n\}$ if, for each $n \in \mathbb{N}$, there exists a constant $\kappa_n > 0$ such that

$$\left\|Q^n(x) - Q^n(y)\right\| \leq \kappa_n(\|p-q\| + \alpha_n), \quad \forall p, q \in \mathscr{H}, \tag{50}$$

where $\{\alpha_n\}$ is a fixed sequence in $[0, \infty]$ with $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$.

For an arbitrary, but fixed $n \in \mathbb{N}$, the infimum of constants κ_n in (50) is called nearly Lipschitz constant and it is denoted by $\beta(Q^n)$. Notice that

$$\beta(Q^n) = \sup\left\{\frac{\left\|Q^n(p) - Q^n(q)\right\|}{\|p-q\| + \alpha_n}: p, q \in \mathscr{H}, p \neq q\right\}. \tag{51}$$

Definition 8 (see [9]). A nearly Lipschitzian mapping $Q\colon \mathscr{H} \longrightarrow \mathscr{H}$ with the sequence $\{(\alpha_n, \beta(Q^n))\}$ is said to be nearly uniformly K-Lipschitzian mapping if, $\beta(Q^n) < K$, for all $n \in \mathbb{N}$. In other words, $\alpha_n = K$, for all $n \in \mathbb{N}$.

For each $i \in \Lambda$, let $Q_i\colon \mathscr{H}_i \longrightarrow \mathscr{H}_i$ be a nearly uniformly K_i-Lipschitzian mapping with the sequence $\{l_{i,n}\}$. We define the self-mapping S of $\prod_{i=1}^m \mathscr{H}_i$ by

$$S(p_1, p_2, \ldots, p_m) = (Q_1 p_1, Q_2 p_2, \ldots, Q_m p_m), \forall (p_1, p_2, \ldots, p_m) \in \prod_{i=1}^m \mathscr{H}_i. \tag{52}$$

Then, $S = (Q_1, Q_2, \ldots, Q_m)\colon \prod_{i=1}^m \mathscr{H}_i \longrightarrow \prod_{i=1}^m \mathscr{H}_i$ is a nearly uniformly $\max\{K_i\colon i \in \Lambda\}$-Lipschitzian mapping with the sequence $\left\{\sum_{i=1}^m l_{i,n}\right\}_{n=1}^{\infty}$ with respect to the norm $\|.\|_*$ in $\prod_{i=1}^m \mathscr{H}_i$. To see this fact, let $(p_1, p_2, \ldots, p_m), (\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m) \in \prod_{i=1}^m \mathscr{H}_i$ be given. Then for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\left\|S^n(p_1, p_2, \ldots, p_m) - S^n(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\right\|_* &= \left\|(Q_1^n p_1, Q_2^n p_2, \ldots, Q_m^n p_m) - (Q_1^n \widehat{p}_1, Q_2^n \widehat{p}_2, \ldots, Q_m^n \widehat{p}_m)\right\|_* \\
&= \left\|(Q_1^n p_1 - Q_1^n \widehat{p}_1, Q_2^n p_2 - Q_2^n \widehat{p}_2, \ldots, Q_m^n p_m - Q_m^n \widehat{p}_m)\right\|_* \\
&= \sum_{i=1}^m \left\|Q_i^n p_i - Q_i^n \widehat{p}_i\right\|_i \leq \sum_{i=1}^m K_i\left(\left\|p_i - \widehat{p}_i\right\|_i + l_{i,n}\right) \\
&\leq \max\{K_i\colon i \in \Lambda\} \sum_{i=1}^m \left(\|p_i - \widehat{p}_i\|_i + l_{i,n}\right) \\
&= \max\{K_i\colon i \in \Lambda\}\left(\left\|(p_1, p_2, \ldots, p_m) - (\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\right\|_* + \sum_{i=1}^m l_{i,n}\right).
\end{aligned} \tag{53}$$

We denote the sets of all the fixed points of $Q_i, i \in \Lambda$ and S by $\mathrm{Fix}(Q_i)$ and $\mathrm{Fix}(S)$, respectively, and the set of all the solutions of SENMVLI (19) by SENMVLI$(\mathscr{H}_i, g_i, h_i, T_i, \psi_i, A_i, i \in \Lambda)$. In view of (52), for any $(p_1, p_2, \ldots, p_m) \in \prod_{i=1}^m \mathscr{H}_i$, $(p_1, p_2, \ldots, p_m) \in \mathrm{Fix}(S)$ if and only if $p_i \in \mathrm{Fix}(Q_i), i \in \Lambda$, i.e., $\mathrm{Fix}(S) = \mathrm{Fix}(Q_1, Q_2, \ldots, Q_m) = \prod_{i=1}^m \mathrm{Fix}(Q_i)$.

If $(p_1^*, p_2^*, \ldots, p_m^*) \in \mathrm{Fix}(S) \cap$ SENMVLI$(\mathscr{H}_i, g_i, h_i, T_i, \psi_i, A_i, i \in \Lambda)$, then by using Lemma 2, one can easily to see that for each $i \in \Lambda$ and for all $n \in \mathbb{N}$,

$$\begin{cases}
p_i^* = Q_i^n p_i^* = p_i^* - h_i(p_i^*) + \mathscr{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(\cdot, p_i^*)}\left[g_i(p_i^*) - \left(T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*)\right)\right] \\
= Q_i^n\left[p_i^* - h_i(p_i^*) + \mathscr{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(\cdot, p_i^*)}\left[g_i(p_i^*) - \left(T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*)\right)\right]\right].
\end{cases} \tag{54}$$

The fixed point formulation (54) enables us to suggest the following proximal iterative algorithm with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $S = (Q_1, Q_2, \ldots, Q_m)$ which is also a unique solution of SENMVLI (19).

Iterative Algorithm 1. For each $i \in \Lambda$, let $\mathscr{H}_i, g_i, h_i, T_i, \psi_i, \zeta_i$ and $\rho_i > 0$ be the same as in SENMVLI (19). For any given $(p_{1,1}, p_{2,1}, \ldots, p_{m,1}) \in \prod_{i=1}^m \mathscr{H}_i$, compute the iterative sequences $\{p_{i,n}\}_{n=1}^{\infty} = \{(p_{1,n}, p_{2,n}, \ldots, p_{m,n})\}_{n=1}^{\infty}$ <i>, </i> $\{u_{i,n}\}_{n=1}^{\infty} = \{(u_{1,n}, u_{2,n}, \ldots, u_{m,n})\}_{n=1}^{\infty}$ <i>, </i> $\{v_{i,n}\}_{n=1}^{\infty} =$

$\{(v_{1,n}, v_{2,n}, \ldots, v_{m,n})\}_{n=1}^{\infty}$ </i> , <//i> $\{s_{i,n}\}_{n=1}^{\infty} = \{(s_{1,n}, s_{2,n}, \ldots, s_{m,n})\}_{n=1}^{\infty}$ by the following iterative process:

$$\begin{cases} p_{i,n+1} = (1-\sigma_n-\varepsilon_n)p_{i,n} + \sigma_n\Big[Q_i^n\Big(u_{i,n} - h_i(u_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(\cdot,u_{i,n})}\big[g_i(u_{i,n}) - \big(T_i(h_i(u_{i,n})) + \rho_i A_i(u_{1,n}, u_{2,n}, \ldots, u_{m,n})\big)\big]\Big) + \eta_{i,n}\Big] + \varepsilon_n\vartheta_{i,n} + r_{i,n}, \\ u_{i,n} = (1-\sigma_n'-\varepsilon_n')p_{i,n} + \sigma_n'\Big[Q_i^n\Big(v_{i,n} - h_i(v_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(\cdot,v_{i,n})}\big[g_i(v_{i,n}) - \big(T_i(h_i(v_{i,n})) + \rho_i A_i(v_{1,n}, v_{2,n}, \ldots, v_{m,n})\big)\big]\Big) + \eta_{i,n}'\Big] + \varepsilon_n'\vartheta_{i,n}' + r_{i,n}', \\ v_{i,n} = (1-\sigma_n''-\varepsilon_n'')p_{i,n} + \sigma_n''\Big[Q_i^n\Big(p_{i,n} - h_i(p_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(\cdot,p_{i,n})}\big[g_i(p_{i,n}) - \big(T_i(h_i(p_{i,n})) + \rho_i A_i(p_{1,n}, p_{2,n}, \ldots, p_{m,n})\big)\big]\Big) + \eta_{i,n}''\Big] + \varepsilon_n''\vartheta_{i,n}'' + r_{i,n}''. \end{cases} \tag{55}$$

Let $\{z_{i,n}\}_{n=1}^{\infty} = \{(z_{1,n}, z_{2,n}, \ldots, z_{m,n})\}_{n=1}^{\infty}$, $\{s_{i,n}\}_{n=1}^{\infty} = \{(s_{1,n}, s_{2,n}, \ldots, s_{m,n})\}_{n=1}^{\infty}$, and $\{t_{i,n}\}_{n=1}^{\infty} = \{(t_{1,n}, t_{2,n}, \ldots, t_{m,n})\}_{n=1}^{\infty}$ be the sequences in $\prod_{i=1}^{m}\mathcal{H}_i$, and define $\{\varphi_{i,n}\}_{n=1}^{\infty} = \{(\varphi_{1,n}, \varphi_{2,n}, \ldots, \varphi_{m,n})\}_{n=1}^{\infty}$ by

$$\begin{cases} \varphi_{i,n} = \Big\| z_{i,n+1} - \Big[(1-\sigma_n-\varepsilon_n)z_{i,n} + \sigma_n\Big(Q_i^n\Big(s_{i,n} - h_i(s_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(\cdot,s_{i,n})}\big[g_i(s_{i,n}) - \big(T_i(h_i(s_{i,n})) + \rho_i A_i(s_{1,n}, s_{2,n}, \ldots, s_{m,n})\big)\big] + \eta_{i,n}\Big)\Big) + \varepsilon_n\vartheta_{i,n} + r_{i,n}\Big]\Big\|_i, \\ s_{i,n} = (1-\sigma_n'-\varepsilon_n')z_{i,n} + \sigma_n'\Big[Q_i^n\Big(t_{i,n} - h_i(t_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(\cdot,t_{i,n})}\big[g_i(t_{i,n}) - \big(T_i(h_i(t_{i,n})) + \rho_i A_i(t_{1,n}, t_{2,n}, \ldots, t_{m,n})\big)\big]\Big) + \eta_{i,n}'\Big] + \varepsilon_n'\vartheta_{i,n}' + r_{i,n}', \\ t_{i,n} = (1-\sigma_n''-\varepsilon_n'')z_{i,n} + \sigma_n''\Big[Q_i^n\Big(z_{i,n} - h_i(z_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(\cdot,z_{i,n})}\big[g_i(z_{i,n}) - \big(T_i(h_i(z_{i,n})) + \rho_i A_i(z_{1,n}, z_{2,n}, \ldots, z_{m,n})\big)\big]\Big) + \eta_{i,n}''\Big] + \varepsilon_n''\vartheta_{i,n}'' + r_{i,n}'', \end{cases} \tag{56}$$

where $S_i\colon \mathcal{H}_i \longrightarrow \mathcal{H}_i$ $i \in \Lambda$ is nearly uniformly Lipschitzian mapping, and the sequences $\{\sigma_n\}, \{\varepsilon_n\}, \{\sigma_n'\}, \{\varepsilon_n'\}, \{\sigma_n''\}$, and $\{\varepsilon''\}$ satisfy the conditions $0 < \sigma_n + \varepsilon_n, \sigma_n' + \varepsilon_n', \sigma_n'' + \varepsilon'' < 1$, $\sum_{n=0}^{\infty}\varepsilon_n < \infty$, *and* $\sum_{n=0}^{\infty}\sigma_n = \infty$. For each $i \in \{1, 2, \ldots, m\}$ and for all $n \in \mathbb{N}$, $\{\vartheta_{i,n}\}, \{\vartheta_{i,n}'\}$, and $\{\vartheta_{i,n}''\}$ are bounded sequences in $\mathcal{H}_i$, and $\{\eta_{i,n}\}, \{\eta_{i,n}'\}, \{\eta_{i,n}''\}, \{r_{i,n}\}, \{r_{i,n}'\}$, and $\{r_{i,n}''\}$ are six sequences in $\mathcal{H}_i$ to take into account the possible inexact computation satisfying the following conditions:

$$\begin{cases} \eta_{i,n} = \widehat{\eta}_{i,n} + \overline{\eta}_{i,n}, \eta_{i,n}' = \widehat{\eta}_{i,n}' + \overline{\eta}_{i,n}', \eta_{i,n}'' = \widehat{\eta}_{i,n}'' + \overline{\eta}_{i,n}'', \\ \big\|(\widehat{\eta}_{1,n}, \widehat{\eta}_{2,n}, \ldots, \widehat{\eta}_{m,n})\big\|_* = \lim\limits_{n\longrightarrow\infty}\big\|(\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}')\|_* = \lim\limits_{n\longrightarrow\infty}\big\|(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}'')\big\|_* = 0, \\ \sum\limits_{n=1}^{\infty}\big\|(\overline{\eta}_{1,n}, \overline{\eta}_{2,n}, \ldots, \overline{\eta}_{m,n})\|_* < \infty, \sum\limits_{n=1}^{\infty}\big\|(\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}')\|_* < \infty, \sum\limits_{n=1}^{\infty}\big\|(\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}'')\|_* < \infty, \\ \sum\limits_{n=1}^{\infty}\big\|(r_{1,n}, r_{2,n}, \ldots, r_{m,n})\|_* < \infty, \sum\limits_{n=1}^{\infty}\big\|(r_{1,n}', r_{2,n}', \ldots, r_{m,n}')\|_* < \infty, \sum\limits_{n=1}^{\infty}\big\|(r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}'')\|_* < \infty. \end{cases} \tag{57}$$

Now, we establish the following strong convergence result for the sequences generated by the proximal iterative Algorithm 1 and stability analysis under some suitable conditions.

Theorem 3. *For each $i \in \Lambda$, let $\mathcal{H}_i, g_i, h_i, T_i, \psi_i, \zeta_i$ and $\rho_i > 0$ be the same as in Theorem 2, and let all the conditions of Theorem 2 hold. For each $i \in \Lambda$, suppose that $Q_i\colon \mathcal{H}_i \longrightarrow \mathcal{H}_i$ is a nearly uniformly K_i-Lipschitzian mapping with the sequence $\{l_{i,n}\}$, and $S\colon \prod_{i=1}^{m}\mathcal{H}_i \longrightarrow \prod_{i=1}^{m}\mathcal{H}_i$ is a nearly uniformly $\max\{K_i\colon i \in \Lambda\}$-Lipschitzian mapping with the sequence $\{\sum_{i=1}^{m} l_{m,i}\}_{m=1}^{\infty}$ defined by (52) such that $\mathrm{Fix}(S) \cap \mathrm{SENMVLI}(\mathcal{H}_i, g_i, h_i, T_i, \psi_i, A_i, i \in \Lambda) \neq \varnothing$. Suppose that $\Omega < \min\{1, 1/K_i\}$, for each $i \in \Lambda$, where Ω is same as in (44). Then,*

(i) the iterative sequence $\{(p_{1,n}, p_{2,n}, \ldots, p_{m,n})\}_{n=1}^{\infty}$ generated by Algorithm 1 converges strongly to a unique element of $\mathrm{Fix}(S) \cap \mathrm{SENMVLI}(\mathcal{H}_i, g_i, h_i, T_i, \psi_i, A_i, i \in \Lambda)$

(ii) Furthermore, if $0 < \kappa < \sigma_n$, then $\lim_{n\longrightarrow\infty}(z_{1,n}, z_{2,n}, \ldots, z_{m,n}) = (p_1^*, p_2^*, \ldots, p_m^*)$ if and only if $\lim_{n\longrightarrow\infty}(\sum_{i=1}^{m}\varphi_{i,n}) = 0$, where $\varphi_{i,n}$ is given in (55); i.e., the sequence $\{(p_{1,n}, p_{2,n}, \ldots, p_{m,n})\}_{n=1}^{\infty}$ generated by (55) is $\mathcal{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(\cdot,p_i)}$ – stable, for each $i \in \Lambda$

Proof

(i) Suppose $(p_1^*, p_2^*, \ldots, p_m^*)$ is a unique solution of SENMVLI (19). For each $i \in \Lambda$, we have

$$h_i(p_i^*) = \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}\left[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]\right)\Big]. \tag{58}$$

Since SEMVLI$(\mathcal{H}_i, g_i, h_i, T_i, \psi_i, A_i, i \in \Lambda)$ is a singleton set and $\mathrm{Fix}(S) \cap \mathrm{SEMVLI}(\mathcal{H}_i, g_i, h_i, T_i, \psi_i, A_i, i \in \Lambda) \neq \varnothing$, we conclude that for each $i \in \Lambda$, $p_i^* \in \mathrm{Fix}(Q_i)$. Hence, for each $n \in \mathbb{N}$ and for each $i \in \Lambda$, we can write

$$\begin{aligned}
p_i^* &= (1-\sigma_n-\varepsilon_n)p_i^* + \sigma_n\Big[Q_i^n\Big(p_i^* - h_i(p_i^*) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]\Big)\Big] + \varepsilon_n p_i^*,\\
&= (1-\sigma_n'-\varepsilon_n')p_i^* + \sigma_n'\Big[Q_i^n\Big(p_i^* - h_i(p_i^*) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]\Big)\Big] + \varepsilon_n' p_i^*,\\
&= (1-\sigma_n''-\varepsilon_n'')p_i^* + \sigma_n''\Big[Q_i^n\Big(p_i^* - h_i(p_i^*) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]\Big)\Big] + \varepsilon_n'' p_i^*,
\end{aligned} \tag{59}$$

where the sequences $\{\sigma_n\}$, $\{\varepsilon_n\}$, $\{\sigma_n'\}$, $\{\varepsilon_n'\}$, $\{\sigma_n''\}$, and $\{\varepsilon_n''\}$ are same as in Algorithm 1.

Now, let $L = \max\left\{\sup_{n\geq 1}\|\vartheta_{i,n} - p_i^*\|_i, i \in \Lambda\right\}$, $L' = \max\left\{\sup_{n\geq 1}\|\vartheta_{i,n}' - p_i^*\|_i, i \in \Lambda\right\}$ and $L'' = \max\left\{\sup_{n\geq 1}\|\vartheta_{i,n}'' - p_i^*\|_i, i \in \Lambda\right\}$. Using Algorithm 1 and (44), it follows that

$$\begin{aligned}
\|p_{i,n+1} - p_i^*\|_i &= \left\| \begin{array}{l} \Big[(1-\sigma_n-\varepsilon_n)p_{i,n} + \sigma_n\Big(Q_i^n\Big(u_{i,n} - h_i(u_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,u_{i,n})}[g_i(u_{i,n}) - (T_i(h_i(u_{i,n})) + \rho_i A_i(u_{1,n}, u_{2,n}, \ldots, u_{m,n}))]\Big) + \eta_{i,n}\Big) + \varepsilon_n\vartheta_{i,n} + r_{i,n}\Big] \\ -\Big[(1-\sigma_n-\varepsilon_n)p_i^* + \sigma_n\Big[Q_i^n\Big(p_i^* - h_i(p_i^*) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]\Big)\Big] + \varepsilon_n p_i^*\Big] \end{array} \right\|_i \\
&\leq (1-\sigma_n-\varepsilon_n)\|p_{i,n} - p_i^*\|_i + \sigma_n \left\| \begin{array}{l} Q_i^n\Big(u_{i,n} - h_i(u_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_{i,n})}[g_i(u_{i,n}) - (T_i(h_i(u_{i,n})) + \rho_i A_i(u_{1,n}, u_{2,n}, \ldots, u_{m,n}))]\Big) \\ -Q_i^n\Big(p_i^* - h_i(p_i^*) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]\Big) \end{array} \right\|_i \\
&\quad \cdot \|\vartheta_{i,n} - p_i^*\|_i + \sigma_n\|\eta_{i,n}\|_i + \|r_{i,n}\|_i \\
&\leq (1-\sigma_n-\varepsilon_n)\|p_{i,n} - p_i^*\|_i + \sigma_n K_i \left\{ \left\| \begin{array}{l} \Big(u_{i,n} - h_i(u_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,u_{i,n})}[g_i(u_{i,n}) - (T_i(h_i(u_{i,n})) + \rho_i A_i(u_{1,n}, u_{2,n}, \ldots, u_{m,n}))]\Big) \\ -\Big(p_i^* - h_i(p_i^*) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]\Big) \end{array} \right\|_i + l_{i,n} \right\} \\
&\quad \cdot \|\vartheta_{i,n} - p_i^*\|_i + \sigma_n\|\eta_{i,n}\|_i + \|r_{i,n}\|_i \\
&\leq (1-\sigma_n-\varepsilon_n)\|p_{i,n} - p_i^*\|_i + \sigma_n K_i \left\| \begin{array}{l} \Big(u_{i,n} - h_i(u_{i,n}) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,u_{i,n})}[g_i(u_{i,n}) - (T_i(h_i(u_{i,n})) + \rho_i A_i(u_{1,n}, u_{2,n}, \ldots, u_{m,n}))]\Big) \\ -\Big(p_i^* - h_i(p_i^*) + \mathcal{J}_{\rho_i,T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]\Big) \end{array} \right\|_i \\
&\quad + \sigma_n K_i l_{i,n} + \varepsilon_n L + \sigma_n\big(\|\widehat{\eta}_{i,n}\|_i + \|\overline{\eta}_{i,n}\|_i\big) + \|r_{i,n}\|_i
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\sigma_n-\varepsilon_n)\|p_{i,n}-p_i^*\|_i+\sigma_n K_i\left(\Theta_i\|u_{i,n}-\widehat{p}_i\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|u_{j,n}-\widehat{p}_j\|_j\right)\\
&\quad+\sigma_n K_i l_{i,n}+\varepsilon_n L+\sigma_n\left(\|\widehat{\eta}_{i,n}\|_i+\|\overline{\eta}_{i,n}\|_i\right)+\|r_{i,n}\|_i\\
&\leq (1-\sigma_n-\varepsilon_n)\|p_{i,n}-p_i^*\|_i+\sigma_n K_i\left(\Theta_i\|u_{i,n}-\widehat{p}_i\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|u_{j,n}-\widehat{p}_j\|_j\right)\\
&\quad+\sigma_n K_i l_{i,n}+\sigma_n\left(\|\widehat{\eta}_{i,n}\|_i+\|\overline{\eta}_{i,n}\|_i\right)+\|r_{i,n}\|_i+\varepsilon_n L.
\end{aligned}
\tag{60}
$$

Using the similar arguments of (60), we can establish that for each $i\in\Lambda$,

$$
\|u_{i,n+1}-p_i^*\|_i\leq (1-\sigma_n'-\varepsilon_n')\|p_{i,n}-p_i^*\|_i+\sigma_n' K_i\left(\Theta_i\|v_{i,n}-p_i^*\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|v_{j,n}-\widehat{p}_j\|_j\right)\|\widehat{\eta}_{i,n}'\|_i+\|\overline{\eta}_{i,n}'\|_i+\|r_{i,n}'\|_i+\varepsilon_n' L',
\tag{61}
$$

$$
\begin{aligned}
\|v_{i,n+1}-p_i^*\|_i\leq{}& (1-\sigma_n''-\varepsilon_n'')\|p_{i,n}-p_i^*\|_i+\sigma_n'' K_i\left(\Theta_i\|p_{i,n}-p_i^*\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|p_{j,n}-\widehat{p}_j\|_j\right)\\
&+\sigma_n'' K_i l_{i,n}+\sigma_n''\|\widehat{\eta}_{i,n}''\|_i+\|\overline{\eta}_{i,n}''\|_i+\|r_{i,n}''\|_i+\varepsilon_n'' L''.
\end{aligned}
\tag{62}
$$

Let $K=\max\{K_i\colon i\in\Lambda\}$. Combining (60)–(62), we obtain

$$
\begin{aligned}
\left\|(p_{1,n+1},p_{2,n+1},\ldots,p_{m,n+1})-(p_1^*,p_2^*,\ldots,p_m^*)\right\|_*\leq{}& (1-\sigma_n-\varepsilon_n)\left\|(p_{1,n},p_{2,n},\ldots,p_{m,n})-(p_1^*,p_2^*,\ldots,p_m^*)\right\|_*\\
&+\sigma_n K\sum_{i=1}^m\left(\Theta_i\|u_{i,n}-\widehat{p}_i\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|u_{j,n}-\widehat{p}_j\|_j\right)\\
&+\sigma_n K\sum_{i=1}^m l_{i,n}+\sigma_n\left\|(\widehat{\eta}_{1,n},\widehat{\eta}_{2,n},\ldots,\widehat{\eta}_{m,n})\right\|_*\\
&+\left\|(\overline{\eta}_{1,n},\overline{\eta}_{2,n},\ldots,\overline{\eta}_{m,n})\right\|_*+\|r_{1,n},r_{2,n},\ldots,r_{m,n}\|_*\\
&+m\varepsilon_n L=(1-\sigma_n-\varepsilon_n)\left\|(p_{1,n},p_{2,n},\ldots,p_{m,n})-(p_1^*,p_2^*,\ldots,p_m^*)\right\|_*
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[\left(\Theta_1+\sum_{k=2}^{m}\frac{\tau_k\rho_k}{\alpha_k+\delta_k}\nu_{k,1}\right)\|u_{1,n}-p_1^*\|_1\\
&+left\left(\Theta_2+\sum_{k\in\Lambda,k\neq 2}^{m}\frac{\tau_k\rho_k}{\alpha_k+\delta_k}\nu_{k,2}\right)\|u_{2,n}-p_2^*\|_2+\cdots+\left(\Theta_m+\sum_{k=1}^{m-1}\frac{\tau_k\rho_k}{\alpha_k+\delta_k}\nu_{k,m}\right)\|u_{m,n}-p_m^*\|_m]\\
&+\sigma_nK\sum_{i=1}^{m}l_{i,n}+\sigma_n\left\|\left(\widehat{\eta}_{1,n},\widehat{\eta}_{2,n},\ldots,\widehat{\eta}_{m,n}\right)\right\|_*+\left\|\overline{\eta}_{1,n},\overline{\eta}_{2,n},\ldots,\overline{\eta}_{m,n}\right\|_*+\left\|\left(r_{1,n},r_{2,n},\ldots,r_{m,n}\right)\right\|_*+m\varepsilon_nL\\
\leq{}&(1-\sigma_n-\varepsilon_n)\left\|\left(p_{1,n},p_{2,n},\ldots,p_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*+\sigma_nK\Omega\sum_{i=1}^{m}\|u_{i,n}-p_i^*\|_i+\sigma_nK\sum_{i=1}^{m}l_{i,n}\\
&+\sigma_n\left\|\left(\widehat{\eta}_{1,n},\widehat{\eta}_{2,n},\ldots,\widehat{\eta}_{m,n}\right)\right\|_*+\left\|\overline{\eta}_{1,n},\overline{\eta}_{2,n},\ldots,\overline{\eta}_{m,n}\right\|_*+\left\|\left(r_{1,n},r_{2,n},\ldots,r_{m,n}\right)\right\|_*+m\varepsilon_nL\\
={}&(1-\sigma_n-\varepsilon_n)\left\|\left(p_{1,n},p_{2,n},\ldots,p_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&+\sigma_nK\Omega\left\|\left(u_{1,n},u_{2,n},\ldots,u_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*+\sigma_nK\sum_{i=1}^{m}l_{i,n}\\
&+\sigma_n\left\|\left(\widehat{\eta}_{1,n},\widehat{\eta}_{2,n},\ldots,\widehat{\eta}_{m,n}\right)\right\|_*+\left\|\overline{\eta}_{1,n},\overline{\eta}_{2,n},\ldots,\overline{\eta}_{m,n}\right\|_*+\left\|\left(r_{1,n},r_{2,n},\ldots,r_{m,n}\right)\right\|_*+m\varepsilon_nL.
\end{aligned}
\tag{63}
$$

Applying equivalent logics of (63), we can compute that

$$
\begin{aligned}
\left\|\left(u_{1,n+1},u_{2,n+1},\ldots,u_{m,n+1}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\leq{}&(1-\sigma_n'-\varepsilon_n')\left\|\left(p_{1,n},p_{2,n},\ldots,p_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&+\sigma_n'K\Omega\left\|\left(v_{1,n},v_{2,n},\ldots,v_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&+\sigma_n'K\sum_{i=1}^{m}l_{i,n}+\sigma_n'\left\|\left(\widehat{\eta}_{1,n}',\widehat{\eta}_{2,n}',\ldots,\widehat{\eta}_{m,n}'\right)\right\|_*\\
&+\left\|\left(\overline{\eta}_{1,n}',\overline{\eta}_{2,n}',\cdots,\overline{\eta}_{m,n}'\right)\right\|_*+\left\|r_{1,n}',r_{2,n}',\ldots,r_{m,n}'\right\|_*+m\varepsilon_n'L',
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
\left\|\left(v_{1,n+1},v_{2,n+1},\ldots,v_{m,n+1}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\leq{}&(1-\sigma_n''-\varepsilon_n'')\left\|\left(p_{1,n},p_{2,n},\ldots,p_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&+\sigma_n''K\Omega\left\|\left(p_{1,n},p_{2,n},\ldots,p_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&+\sigma_n''K\sum_{i=1}^{m}l_{i,n}+\sigma_n''\left\|\left(\widehat{\eta}_{1,n}'',\widehat{\eta}_{2,n}'',\ldots,\widehat{\eta}_{m,n}''\right)\right\|_*\\
&+\left\|\left(\overline{\eta}_{1,n}'',\overline{\eta}_{2,n}'',\ldots,\overline{\eta}_{m,n}''\right)\right\|_*+\left\|r_{1,n}'',r_{2,n}'',\ldots,r_{m,n}''\right\|_*+m\varepsilon_n''L''.
\end{aligned}
\tag{65}
$$

Since $(1-\sigma_n''(1-K\Omega))\leq 1$, therefore (65) becomes

$$\begin{aligned}
\left\|\left(v_{1,n+1}, v_{2,n+1}, \ldots, v_{m,n+1}\right)-\left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \leq{} & \left(1-\sigma_n''(1-K\Omega)\right)\left\|\left(p_{1,n}, p_{2,n}, \ldots, p_{m,n}\right)-\left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
& +\sigma_n'' K \sum_{i=1}^m l_{i,n}+\sigma_n''\left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_*+\left\|\left(\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right)\right\|_* \\
& +\left\|r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right\|_*+m\varepsilon_n'' L'' \\
\leq{} & \left\|\left(p_{1,n}, p_{2,n}, \ldots, p_{m,n}\right)-\left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_*+\sigma_n'' K \sum_{i=1}^m l_{i,n} \\
& +\sigma_n''\left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_*+\left\|\left(\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right)\right\|_* \\
& +\left\|r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right\|_*+m\varepsilon_n'' L''.
\end{aligned} \tag{66}$$

Since $(1-\sigma_n''(1-K\Omega)) \leq 1$, therefore using (64) and (66) becomes

$$\begin{aligned}
\left\|\left(u_{1,n+1}, u_{2,n+1}, \ldots, u_{m,n+1}\right)-\left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \leq{} & \left(1-\sigma_n'-\varepsilon_n'\right)\left\|\left(p_{1,n}, p_{2,n}, \ldots, p_{m,n}\right)-\left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
& +\sigma_n' K\Omega\Big[\left\|\left(p_{1,n}, p_{2,n}, \ldots, p_{m,n}\right)-\left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
& +\sigma_n''\left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_*+\left\|\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right\|_* \\
& +\sigma_n'' K \sum_{i=1}^m l_{i,n}+\left\|\left(r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right)\right\|_*+m\varepsilon_n'' L''\Big] \\
& +\sigma_n' K \sum_{i=1}^m l_{i,n}+\sigma_n'\left\|\left(\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}'\right)\right\|_*+\left\|\left(\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}'\right)\right\|_* \\
& +\left\|r_{1,n}', r_{2,n}', \ldots, r_{m,n}'\right\|_*+m\varepsilon_n' L' \\
\leq{} & \left(1-\sigma_n'(1-K\Omega)\right)\left\|\left(p_{1,n}, p_{2,n}, \ldots, p_{m,n}\right)-\left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
& +\sigma_n'\sigma_n'' K\Omega\left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_*+\sigma_n'\sigma_n'' K^2\Omega \sum_{i=1}^m l_{i,n} \\
& +\sigma_n' K\Omega\left\|\left(r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right)\right\|_*+\sigma_n' K\Omega\left\|\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right\|_* \\
& +\sigma_n' K \sum_{i=1}^m l_{i,n}+\sigma_n'\left\|\left(\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}'\right)\right\|_*+\left\|\left(\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}'\right)\right\|_* \\
& +\left\|r_{1,n}', r_{2,n}', \ldots, r_{m,n}'\right\|_*+m\sigma_n'\varepsilon_n'' K\Omega L''+m\varepsilon_n' L' \\
\leq{} & \left\|\left(p_{1,n}, p_{2,n}, \ldots, p_{m,n}\right)-\left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
& +\sigma_n'\sigma_n'' K\Omega\left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_*+\sigma_n'\sigma_n'' K^2\Omega \sum_{i=1}^m l_{i,n} \\
& +\sigma_n' K\Omega\left\|\left(r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right)\right\|_*+\sigma_n' K\Omega\left\|\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right\|_* \\
& +\sigma_n' K \sum_{i=1}^m l_{i,n}+\sigma_n'\left\|\left(\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}'\right)\right\|_* \\
& +\left\|\left(\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}'\right)\right\|_*+\left\|r_{1,n}', r_{2,n}', \ldots, r_{m,n}'\right\|_*+m\sigma_n'\varepsilon_n'' K\Omega L''+m\varepsilon_n' L'.
\end{aligned} \tag{67}$$

Using (63) and (67) becomes

$$
\begin{aligned}
&\left\|\left(p_{1,n+1}, p_{2,n+1}, \ldots, p_{m,n+1}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \leq \left(1 - \sigma_n - \varepsilon_n\right)\left\|\left(p_{1,n}, p_{2,n}, \ldots, p_{m,n}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
&+ \sigma_n K\Omega\left[\begin{array}{c} \left\|\left(p_{1,n}, p_{2,n}, \ldots, p_{m,n}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* + \sigma_n'\sigma_n'' K\Omega\left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_* + \sigma_n'\sigma_n'' K^2\Omega\sum_{i=1}^m l_{i,n} + \sigma_n' K\Omega\left\|\left(r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right)\right\|_* + \sigma_n' K\Omega\left\|\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right\|_* \\
+ \sigma_n' K\sum_{i=1}^m l_{i,n} + \sigma_n'\left\|\left(\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}'\right)\right\|_* + \left\|\left(\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}'\right)\right\|_* + \left\|r_{1,n}', r_{2,n}', \ldots, r_{m,n}'\right\|_* + m\sigma_n'\varepsilon_n'' K\Omega L'' + m\varepsilon_n' L'
\end{array}\right] \\
&+ \sigma_n K\sum_{i=1}^m l_{i,n} + \left\|\left(\overline{\eta}_{1,n}, \overline{\eta}_{2,n}, \ldots, \overline{\eta}_{m,n}\right)\right\|_* + \left\|r_{1,n}, r_{2,n}, \ldots, r_{m,n}\right\|_* + m\varepsilon_n L \\
&\leq \left(1 - \sigma_n\left(1 - K\Omega\right)\right)\left\|\left(p_{1,n}, p_{2,n}, \ldots, p_{m,n}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
&+ \sigma_n\left(1 - K\Omega\right)\left[\frac{1}{(1 - K\Omega)}\left(\begin{array}{c} \sigma_n'\sigma_n'' K^2\Omega^2\left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_* + \sigma_n' K^2\Omega^2\left\|\left(\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right)\right\|_* + \sigma_n' K^2\Omega^2\left\|r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right\|_* + \sigma_n' K\Omega\left\|\left(\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}'\right)\right\|_* + K\Omega\left\|\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}'\right\|_* \\
+ K\Omega\left\|\left(r_{1,n}', r_{2,n}', \ldots, r_{m,n}'\right)\right\|_* + \left\|\widehat{\eta}_{1,n}, \widehat{\eta}_{2,n}, \ldots, \widehat{\eta}_{m,n}\right\|_* + \left(\sigma_n'\sigma_n'' K^3\Omega^2 + \sigma_n' K^2\Omega + K\right)\sum_{i=1}^m l_{i,n} + m\sigma_n'\varepsilon_n'' K^2\Omega^2 L'' + m\varepsilon_n' K\Omega L'
\end{array}\right)\right] \\
&+ \left\|\left(\overline{\eta}_{1,n}, \overline{\eta}_{2,n}, \ldots, \overline{\eta}_{m,n}\right)\right\|_* + \left\|r_{1,n}, r_{2,n}, \ldots, r_{m,n}\right\|_* + m\varepsilon_n L.
\end{aligned} \tag{68}
$$

On setting,

$$
\begin{aligned}
\wp_{n+1} &= \left\|\left(p_{1,n+1}, p_{2,n+1}, \ldots, p_{m,n+1}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_*; \varpi_n = \sigma_n\left(1 - K\Omega\right); \\
\varrho_n &= \left\|\left(\overline{\eta}_{1,n}, \overline{\eta}_{2,n}, \ldots, \overline{\eta}_{m,n}\right)\right\|_* + \left\|r_{1,n}, r_{2,n}, \ldots, r_{m,n}\right\|_* + m\varepsilon_n L,
\end{aligned} \tag{69}
$$

$$
\nu_n = \left[\frac{1}{(1 - K\Omega)}\left(\begin{array}{c} \sigma_n'\sigma_n'' K^2\Omega^2\left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_* + \sigma_n' K^2\Omega^2\left\|\left(\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right)\right\|_* + \sigma_n' K^2\Omega^2\left\|\left(r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right)\right\|_* \\
+ \sigma_n' K\Omega\left\|\left(\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}'\right)\right\|_* + K\Omega\left\|\left(\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}'\right)\right\|_* + K\Omega\left\|\left(r_{1,n}', r_{2,n}', \ldots, r_{m,n}'\right)\right\|_* + \left\|\left(\widehat{\eta}_{1,n}, \widehat{\eta}_{2,n}, \ldots, \widehat{\eta}_{m,n}\right)\right\|_* \\
+ \left(\sigma_n'\sigma_n'' K^3\Omega^2 + \sigma_n' K^2\Omega K\right)\sum_{i=1}^m l_{i,n} + m\sigma_n'\varepsilon_n'' K^2\Omega^2 L'' + m\varepsilon_n' K\Omega L'
\end{array}\right)\right]. \tag{70}
$$

Equation (68) can be written as

$$
\wp_{n+1} \leq \left(1 - \varpi_n\right)\wp_n + \nu_n\varpi_n + \varrho_n, \tag{71}
$$

where $\Omega = \max\left\{\Theta_i + \sum_{k\in\Lambda, k\neq i}\left(\tau_k\rho_k\right)/\left(\alpha_k + \delta_k\right)\nu_{k,i}\colon i \in \Lambda\right\}$. Since $\lim_{n\longrightarrow\infty} l_{i,n} = 0$, for each $i \in \Lambda$, and $\sum_{i=1}^m \varepsilon_n < \infty$, in view of (57), it follows that the conditions of Lemma 1 are satisfied. Therefore, using Lemma 1 and (68), we deduce that $(p_{1,n+1}, p_{2,n+1}, \ldots, p_{m,n+1}) \longrightarrow (p_1^*, p_2^*, \ldots, p_m^*)$, as $n \longrightarrow \infty$, and so the sequence $\{(p_{1,n+1}, p_{2,n+1}, \ldots, p_{m,n+1})\}_{n=1}^{\infty}$ generated by Algorithm 1 converges strongly to the only element $(p_1^*, p_2^*, \ldots, p_m^*)$ of the singleton set $Fix(S) \cap \mathrm{SEMVI}(\mathscr{H}_i, g_i, h_i, T_i, \psi_i, A_i, i \in \Lambda)$. This completes the part (I).

(ii) Next, we prove the second conclusion. Let $H(p_i^*) = S_i^n(p_i^* - h_i\ (p_i^*) + \mathscr{J}_{\rho_i, T_{i,n}}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))])$. Using (59) and Algorithm 1, we have

$$
\begin{aligned}
\left\|z_{i,n+1} - p_i^*\right\|_i &\leq \left\|z_{i,n+1} - \left[\left(1 - \sigma_n - \varepsilon_n\right)z_{i,n} + \sigma_n\left(H\left(s_{i,n}\right) + \eta_{i,n}\right) + \varepsilon_n\vartheta_{i,n} + r_{i,n}\right]\right\| \\
&\quad + \left\|\left[\left(1 - \sigma_n - \varepsilon_n\right)z_{i,n} + \sigma_n\left(H\left(s_{i,n}\right) + \eta_{i,n}\right) + \varepsilon_n\vartheta_{i,n} + r_{i,n}\right] - \left[\left(1 - \sigma_n - \varepsilon_n\right)p_i^* + \sigma_n H\left(p_i^*\right) + \varepsilon_n p_n^*\right]\right\|_i \\
&\leq \varphi_{i,n} + \left(1 - \sigma_n - \varepsilon_n\right)\left\|z_{i,n} - p_i^*\right\|_i + \sigma_n K_i\left(\Theta_i\left\|s_{i,n} - p_i^*\right\|_i + \frac{\tau_i\rho_i}{\alpha_i + \delta_i}\sum_{j\in\Lambda, i\neq j}\nu_{i,j}\left\|s_{j,n} - p_j^*\right\|_j\right) \\
&\quad + \sigma_n K_i l_{i,n} + \sigma_n\left\|\widehat{\eta}_{i,n}\right\|_i + \left\|\overline{\eta}_{i,n}\right\|_i + \left\|r_{i,n}\right\|_i + \varepsilon_n L.
\end{aligned} \tag{72}
$$

On similar manner of (73), we can deduce that

$$\begin{aligned}\|s_{i,n+1}-p_i^*\|_i &\le (1-\sigma_n'-\varepsilon_n')\|z_{i,n}-p_i^*\|_i+\sigma_n'K_i\left(\Theta_i\|t_{i,n}-p_i^*\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|t_{j,n}-p_j^*\|_j\right)\\&\quad+\sigma_n'K_il_{i,n}+\sigma_n'\|\hat{\eta}_{i,n}'\|_i+\|\overline{\eta}_{i,n}'\|_i+\|r_{i,n}'\|_i+\varepsilon_n'L',\end{aligned}\tag{73}$$

$$\begin{aligned}\|t_{i,n+1}-p_i^*\|_i &\le (1-\sigma_n''-\varepsilon_n'')\|z_{i,n}-p_i^*\|_i+\sigma_n''K_i\left(\Theta_i\|p_{i,n}-p_i^*\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|z_{j,n}-p_j^*\|_j\right)\\&\quad+\sigma_n''K_il_{i,n}+\sigma_n''\|\hat{\eta}_{i,n}''\|_i+\|\overline{\eta}_{i,n}''\|_i+\|r_{i,n}''\|_i+\varepsilon_n''L''.\end{aligned}\tag{74}$$

Using the similar arguments of (63) and combining (72)–(74), we obtain that

$$\begin{aligned}&\left\|(z_{1,n+1},z_{2,n+1},\dots,z_{m,n+1})-(p_1^*,p_2^*,\dots,p_m^*)\right\|_*\\&\le\sum_{i=1}^m\Bigg[\varphi_{i,n}+(1-\sigma_n-\varepsilon_n)\|z_{i,n}-p_i^*\|_i+\sigma_nK_i\left(\Theta_i\|s_{i,n}-p_i^*\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|s_{j,n}-p_j^*\|_j\right)\\&\quad+\sigma_nK_il_{i,n}+\sigma_n\|\hat{\eta}_{i,n}\|_i+\|\overline{\eta}_{i,n}\|_i+\|r_{i,n}\|_i+\varepsilon_nL\Bigg]\\&\le\sum_{i=1}^m\varphi_{i,n}+(1-\sigma_n-\varepsilon_n)\left\|(z_{1,n},z_{2,n},\dots,z_{m,n})-(p_1^*,p_2^*,\dots,p_m^*)\right\|_*\\&\quad+\sigma_nK\Omega\left\|(s_{1,n},s_{2,n},\dots,s_{m,n})-(p_1^*,p_2^*,\dots,p_m^*)\right\|_*+\sigma_nK\sum_{i=1}^m l_{i,n}\\&\quad+\sigma_n\|(\hat{\eta}_{1,n},\hat{\eta}_{2,n},\dots,\hat{\eta}_{m,n})\|_*+\|(\overline{\eta}_{1,n},\overline{\eta}_{2,n},\dots,\overline{\eta}_{m,n})\|_*\\&\quad+\|(r_{1,n},r_{2,n},\dots,r_{m,n})\|_*+m\varepsilon_nL.\end{aligned}\tag{75}$$

Similarly, we have

$$\begin{aligned}\left\|(s_{1,n+1},s_{2,n+1},\dots,s_{m,n+1})-(p_1^*,p_2^*,\dots,p_m^*)\right\|_* &\le (1-\sigma_n'-\varepsilon_n')\left\|(z_{1,n},z_{2,n},\dots,z_{m,n})-(p_1^*,p_2^*,\dots,p_m^*)\right\|_*\\&\quad+\sigma_n'K\Omega\left\|(t_{1,n},t_{2,n},\dots,t_{m,n})-(p_1^*,p_2^*,\dots,p_m^*)\right\|_*\\&\quad+\sigma_n'K\sum_{i=1}^m l_{i,n}+\sigma_n'\left\|(\hat{\eta}_{1,n}',\hat{\eta}_{2,n}',\dots,\hat{\eta}_{m,n}')\right\|_*+\|\overline{\eta}_{1,n}',\overline{\eta}_{2,n}',\dots,\overline{\eta}_{m,n}'\|_*\\&\quad+\left\|(r_{1,n}',r_{2,n}',\dots,r_{m,n}')\right\|_*+m\varepsilon_n'L'.\end{aligned}\tag{76}$$

Since $(1-\sigma_n''(1-K\Omega))\leq 1$, therefore we obtain

$$\begin{aligned}
\left\|\left(t_{1,n+1},t_{2,n+1},\ldots,t_{m,n+1}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_* &\leq (1-\sigma_n''-\varepsilon_n'')\left\|\left(z_{1,n},z_{2,n},\ldots,z_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&\quad+\sigma_n''K\Omega\left\|\left(z_{1,n},z_{2,n},\ldots,z_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&\quad+\sigma_n''K\sum_{i=1}^{m}l_{i,n}+\sigma_n''\left\|\left(\widehat{\eta}_{1,n}'',\widehat{\eta}_{2,n}'',\ldots,\widehat{\eta}_{m,n}''\right)\right\|_*+\left\|\overline{\eta}_{1,n}'',\overline{\eta}_{2,n}'',\ldots,\overline{\eta}_{m,n}''\right\|_*\\
&\quad+\left\|\left(r_{1,n}'',r_{2,n}'',\ldots,r_{m,n}''\right)\right\|_*+m\varepsilon_n''L''\\
&\leq (1-\sigma_n''(1-K\Omega))\left\|\left(z_{1,n},z_{2,n},\ldots,z_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&\quad+\sigma_n''\left\|\left(\widehat{\eta}_{1,n}'',\widehat{\eta}_{2,n}'',\ldots,\widehat{\eta}_{m,n}''\right)\right\|_*+\left\|\overline{\eta}_{1,n}'',\overline{\eta}_{2,n}'',\ldots,\overline{\eta}_{m,n}''\right\|_*\\
&\quad+\left\|\left(r_{1,n}'',r_{2,n}'',\ldots,r_{m,n}''\right)\right\|_*+\sigma_n''K\sum_{i=1}^{m}l_{i,n}+m\varepsilon_n''L''\\
&\leq\left\|\left(z_{1,n},z_{2,n},\ldots,z_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&\quad+\sigma_n''K\sum_{i=1}^{m}l_{i,n}+\sigma_n''\left\|\left(\widehat{\eta}_{1,n}'',\widehat{\eta}_{2,n}'',\ldots,\widehat{\eta}_{m,n}''\right)\right\|_*+\left\|\overline{\eta}_{1,n}'',\overline{\eta}_{2,n}'',\ldots,\overline{\eta}_{m,n}''\right\|_*\\
&\quad+\left\|\left(r_{1,n}'',r_{2,n}'',\ldots,r_{m,n}''\right)\right\|_*+m\varepsilon_n''L''.
\end{aligned}\tag{77}$$

Since $(1-\sigma_n'(1-K\Omega))\leq 1$, therefore using (76) and (77) becomes

$$\begin{aligned}
\left\|\left(s_{1,n+1},s_{2,n+1},\ldots,s_{m,n+1}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_* &\leq\left\|\left(z_{1,n},z_{2,n},\ldots,z_{m,n}\right)-\left(p_1^*,p_2^*,\ldots,p_m^*\right)\right\|_*\\
&\quad+\sigma_n'\sigma_n''K\Omega\left\|\left(\widehat{\eta}_{1,n}'',\widehat{\eta}_{2,n}'',\ldots,\widehat{\eta}_{m,n}''\right)\right\|_*+\sigma_n'\sigma_n''K^2\Omega\sum_{i=1}^{m}l_{i,n}\\
&\quad+\sigma_n'K\Omega\left\|\left(r_{1,n}'',r_{2,n}'',\ldots,r_{m,n}''\right)\right\|_*+\sigma_n'K\Omega\left\|\overline{\eta}_{1,n}'',\overline{\eta}_{2,n}'',\ldots,\overline{\eta}_{m,n}''\right\|_*\\
&\quad+\sigma_n'K\sum_{i=1}^{m}l_{i,n}+\sigma_n'\left\|\left(\widehat{\eta}_{1,n}',\widehat{\eta}_{2,n}',\ldots,\widehat{\eta}_{m,n}'\right)\right\|_*+\left\|\left(\overline{\eta}_{1,n}',\overline{\eta}_{2,n}',\ldots,\overline{\eta}_{m,n}'\right)\right\|_*\\
&\quad+\left\|r_{1,n}',r_{2,n}',\ldots,r_{m,n}'\right\|_*+m\sigma_n'\varepsilon_n''K\Omega L''+m\varepsilon_n'L'.
\end{aligned}\tag{78}$$

Using (77) and (78), (75) implies that

$$\begin{aligned}
&\left\|\left(z_{1,n+1}, z_{2,n+1}, \ldots, z_{m,n+1}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \leq \sum_{i=1}^{m} \varphi_{i,n} + \left(1 - \sigma_n - \varepsilon_n\right)\left\|\left(z_{1,n}, z_{2,n}, \ldots, z_{m,n}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
&+ \sigma_n K\Omega\left[\begin{array}{c} \left\|\left(z_{1,n}, z_{2,n}, \ldots, z_{m,n}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* + \sigma_n'\sigma_n'' K\Omega \left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_* + \sigma_n'\sigma_n'' K^2\Omega \sum_{i=1}^{m} l_{i,n} \\ + \sigma_n' K\Omega \left\|\left(r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right)\right\|_* + \sigma_n' K\Omega \left\|\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right\|_* + \sigma_n' K \sum_{i=1}^{m} l_{i,n} + \sigma_n' \left\|\left(\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}'\right)\right\|_* + \left\|\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}'\right\|_* + \left\|\left(r_{1,n}', r_{2,n}', \ldots, r_{m,n}'\right)\right\|_* + m\sigma_n'\varepsilon_n'' K\Omega L'' + m\varepsilon_n' L' \end{array}\right] \\
&+ \sigma_n K \sum_{i=1}^{m} l_{i,n} + \sigma_n \left\|\left(\eta_{1,n}, \eta_{2,n}, \ldots, \eta_{m,n}\right)\right\|_* + \left\|\left(r_{1,n}, r_{2,n}, \ldots, r_{m,n}\right)\right\|_* + m\varepsilon_n L \\
&\leq \left(1 - \sigma_n (1 - K\Omega)\right)\left\|\left(z_{1,n}, z_{2,n}, \ldots, z_{m,n}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* + \sigma_n (1 - K\Omega) \\
&\left[\frac{\sum_{i=1}^{m}\varphi_{i,n}}{\kappa (1 - K\Omega)} + \frac{1}{(1 - K\Omega)}\left(\begin{array}{c} \sigma_n'\sigma_n'' K^2\Omega^2 \left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_* + \sigma_n' K^2\Omega^2 \left\|\left(\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}''\right)\right\|_* + \sigma_n' K^2\Omega^2 \left\|r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right\|_* + \sigma_n' K\Omega \left\|\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}'\right\|_* + K\Omega \left\|\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}'\right\|_* \\ + K\Omega \left\|\left(r_{1,n}', r_{2,n}', \ldots, r_{m,n}'\right)\right\|_* + \left\|\widehat{\eta}_{1,n}, \widehat{\eta}_{2,n}, \ldots, \widehat{\eta}_{m,n}\right\|_* + \left(\sigma_n'\sigma_n'' K^3\Omega^2 + \sigma_n' K^2\Omega + K\right) \sum_{i=1}^{m} l_{i,n} + m\sigma_n'\varepsilon_n'' K^2\Omega^2 L'' + m\varepsilon_n' K\Omega L' \end{array}\right)\right] \\
&+ \left\|\left(\overline{\eta}_{1,n}, \overline{\eta}_{2,n}, \ldots, \overline{\eta}_{m,n}\right)\right\|_* + \left\|r_{1,n}, r_{2,n}, \ldots, r_{m,n}\right\|_* + m\varepsilon_n L,
\end{aligned} \tag{79}$$

Since it is given that $\lim_{n\longrightarrow\infty}\sum_{i=1}^{m}\varphi_{i,n} = 0$, therefore, $\lim_{n\longrightarrow\infty}(z_{1,n+1}, z_{2,n+1}, \ldots, z_{m,n+1}) = (p_1^*, p_2^*, \ldots, p_m^*)$, and the result follows.

Conversely, assume that $\lim_{n\longrightarrow\infty}(z_{1,n+1}, z_{2,n+1}, \ldots, z_{m,n+1}) = (p_1^*, p_2^*, \ldots, p_m^*)$. From (59) and $\lim_{n\longrightarrow\infty} l_{i,n} = 0$, we have

$$\begin{aligned}
\varphi_{i,n} &= \left\|z_{i,n+1} - \left[(1 - \sigma_n - \varepsilon_n)z_{i,n} + \sigma_n\left(H\left(t_{i,n}\right) + \eta_n\right) + \varepsilon_n\vartheta_{i,n} + r_n\right]\right\|_i \\
&\leq \left\|z_{i,n+1} - p_i^*\right\|_i + \left\|\left[(1 - \sigma_n - \varepsilon_n)z_{i,n} + \sigma_n\left(H\left(t_{i,n}\right) + \eta_n\right) + \varepsilon_n\vartheta_{i,n} + r_n\right] - p_i^*\right\|_i \\
&\leq \left\|z_{i,n+1} - p_i^*\right\|_i + \left\|\left[(1 - \sigma_n - \varepsilon_n)z_{i,n} + \sigma_n\left(H\left(t_{i,n}\right) + \eta_n\right) + \varepsilon_n\vartheta_{i,n} + r_n\right] - \left[(1 - \sigma_n - \varepsilon_n)p_i^* + \sigma_n H\left(p_i^*\right) + \varepsilon_n p_i^*\right]\right\|_i \\
&\quad \cdot \left\|z_{i,n+1} - p_i^*\right\|_i + (1 - \sigma_n - \varepsilon_n)\left\|z_{i,n} - p_i^*\right\|_i + \sigma_n\left\|H\left(t_{i,n}\right) - H\left(p_i^*\right)\right\|_i \\
&\quad + \sigma_n\left\|\widehat{\eta}_{i,n}\right\|_i + \left\|\overline{\eta}_{i,n}\right\|_i + \varepsilon_n\left\|\vartheta_{i,n} - p_i^*\right\|_i + \left\|r_{i,n}\right\|_i.
\end{aligned} \tag{80}$$

By using (80), we have

$$\begin{aligned}
\sum_{i=1}^{m}\varphi_{i,n} &\leq \sum_{i=1}^{m} \left\|z_{i,n+1} - p_i^*\right\|_i + (1 - \sigma_n - \varepsilon_n)\left\|z_{i,n} - p_i^*\right\|_i + \sigma_n H\left(t_{i,n}\right) - H\left(p_i^*\right)\right\|_i \\
&\quad + \left(\sigma_n\left\|\widehat{\eta}_{i,n}\right\|_i + \left\|\overline{\eta}_{i,n}\right\|_i + \varepsilon_n\left\|\vartheta_{i,n} - p_i^*\right\|_i + \left\|r_{i,n}\right\|_i\right) \\
&\leq \left\|\left(z_{1,n+1}, z_{2,n+1}, \ldots, z_{m,n+1}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
&\quad + \left(1 - \sigma_n (1 - K\Omega)\right)\left\|\left(z_{1,n}, z_{2,n}, \ldots, z_{m,n}\right) - \left(p_1^*, p_2^*, \ldots, p_m^*\right)\right\|_* \\
&\quad + \sigma_n\Big(\sigma_n'\sigma_n'' K^2\Omega^2\left\|\left(\widehat{\eta}_{1,n}'', \widehat{\eta}_{2,n}'', \ldots, \widehat{\eta}_{m,n}''\right)\right\|_* \\
&\quad + \sigma_n' K^2\Omega^2\left\|\left(\overline{\eta}_{1,n}'', \overline{\eta}_{2,n}'', \ldots, \overline{\eta}_{m,n}\right)\right\|_* + \sigma_n' K^2\Omega^2\left\|\left(r_{1,n}'', r_{2,n}'', \ldots, r_{m,n}''\right)\right\|_* \\
&\quad + \sigma_n' K\Omega\left\|\left(\widehat{\eta}_{1,n}', \widehat{\eta}_{2,n}', \ldots, \widehat{\eta}_{m,n}'\right)\right\|_* + K\Omega\left\|\left(\overline{\eta}_{1,n}', \overline{\eta}_{2,n}', \ldots, \overline{\eta}_{m,n}'\right)\right\|_* \\
&\quad + K\Omega\left\|\left(r_{1,n}', r_{2,n}', \ldots, r_{m,n}'\right)\right\|_* + \left\|\left(\widehat{\eta}_{1,n}, \widehat{\eta}_{2,n}, \ldots, \widehat{\eta}_{m,n}\right)\right\|_* \\
&\quad + \left(\sigma_n'\sigma_n'' K^3\Omega^2 + \sigma_n' K^2\Omega + K\right)\sum_{i=1}^{m} l_{i,n} + m\sigma_n'\varepsilon_n'' K^2\Omega^2 L'' + m\varepsilon_n' K\Omega L'\Big) \\
&\quad + \left\|\left(\overline{\eta}_{1,n}, \overline{\eta}_{2,n}, \ldots, \overline{\eta}_{m,n}\right)\right\|_* + \left\|\left(r_{1,n}, r_{2,n}, \ldots, r_{m,n}\right)\right\|_* + m\varepsilon_n L,
\end{aligned} \tag{81}$$

which implies that $\lim_{n\longrightarrow\infty}(\sum_{i=1}^{m}\varphi_{i,n})=0$.

Hence, the sequence $\{(p_{1,n}, p_{2,n}, \ldots, p_{m,n})\}_{n=1}^{\infty}$ generated by (56) is $\mathcal{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}-$ stable, for each $i\in\Lambda$. This completes the proof. □

5. Proximal Dynamical System

In this section, we consider the proximal dynamical system technique to study the existence and uniqueness of the solution of SENMVLI (19). In Section 3, we have shown that the SENMVLI (19) is equivalent to a fixed point problem. By using this equivalent result, we suggest and analyze the following proximal dynamical system associated with the SENMVLI (19). For each $i\in\Lambda$, we define the residue vector as follows:

$$\mathcal{G}_i(p_i)=h_i(p_i)-\mathcal{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}\left[g_i(p_i)-(T_i(h_i(p_i))+\rho_iA_i(p_1,p_2,\ldots,p_m))\right]. \tag{82}$$

It is evident form Lemma 2 that SENMVLI (19) has a solution $(p_1, p_2, \ldots, p_m)\in\prod_{i=1}^{m}\mathcal{H}_i$ if and only if $(p_1, p_2, \ldots, p_m)$ is a zero of the equation

$$\mathcal{G}_i(p_i)=0, \text{for each } i\in\Lambda. \tag{83}$$

Using the residue vector equations (82) and (83) and fixed point formulation (27), we suggest the following proximal dynamical system:

$$\begin{aligned}\frac{\mathrm{d}p_i}{\mathrm{d}t}&=-\omega_i\mathcal{G}_i(p_i)\\&=\omega_i\left\{\mathcal{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}\left[g_i(p_i)-(T_i(h_i(p_i))+\rho_iA_i(p_1,p_2,\ldots,p_m))\right]-h_i(p_i)\right\}, p_i(t_0)=c_i\in\mathcal{H}_i,\end{aligned} \tag{84}$$

associated with SENMVLI (19), where $\omega_i>0$ is parameter. We call the proximal dynamical system (84) as extended nonlinear mixed variational-like proximal dynamical system. Here, the right hand side is related to the proximal operator and is discontinuous on the boundary of $\mathcal{H}_i$. From the definition, it is definite that the solution of the extended nonlinear mixed variational-like proximal dynamical system (84) belongs to the constraint set $\mathcal{H}_i$. This points out that the approximate results such as the existence, uniqueness, and continuous dependence of the solution of (84) can be investigated.

To state our results, we need the following well-known concepts.

Definition 9 (see [23]). The dynamical system is said to be converge to the solution set Γ^* of the problem (19), if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\longrightarrow\infty}\operatorname{dist}(p_i(t),\Gamma^*)=0, \tag{85}$$

where $\operatorname{dist}(p_i(t),\Gamma^*)=\inf_{q_i\in\Gamma^*}\|p_i-q_i\|$.

If the solution set Γ^* has a unique solution $(p_1^*, p_2^*, \ldots, p_m^*)\in\prod_{i=1}^{m}\mathcal{H}_i$, then (84) implies that $\lim_{t\longrightarrow\infty}(p_1(t), p_2(t), \ldots, p_m(t))=(p_1^*, p_2^*, \ldots, p_m^*)$.

Definition 10 (see [23]). The dynamical system is said to be globally exponentially stable with degree θ *at* p^*, if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\|p(t)-p^*\|\le c_0\|p(t)-p^*\|\exp(-\theta(t-t_0)), \forall t\ge t_0, \tag{86}$$

where c_0 and θ are positive constants, independent of the initial point.

Lemma 3 (see [23]). *Let $\hat{p}$ and $\hat{q}$ be real-valued nonnegative continuous functions with domain $\{t: t\ge t_0\}$ and let $\alpha(t)=\alpha_0(|t-t_0|)$, where α_0 is a monotone increasing function. If, for all $t\ge t_0$,*

$$\hat{p}(t)\le\alpha(t)+\int_{t_0}^{t}\hat{p}(s)\hat{q}(s)\mathrm{d}s, \tag{87}$$

holds, then

$$\hat{p}(t)\le\alpha(t)\exp\left\{\int_{t_0}^{t}\hat{q}(s)\mathrm{d}s\right\}. \tag{88}$$

The existence and uniqueness of the solution of SENMVLI (19) is shown in Theorem 2. Now, by combining Lemma 3 and Theorem 2, we obtain the unique solution of extended nonlinear mixed variational-like proximal dynamical system (84).

Theorem 4. *For each $i\in\Lambda$, let $\mathcal{H}_i, g_i, h_i, T_i, \psi_i, \zeta_i$, and $\rho_i>0$ be the same as in Theorem 2 and let all the conditions of Theorem 2 hold. For each $i\in\Lambda$, suppose that Q_i: $\mathcal{H}_i\longrightarrow\mathcal{H}_iK_i$-Lipschitzian mapping with the sequence $\{l_{i,n}\}$, and S: $\prod_{i=1}^{m}\mathcal{H}_i$ is a nearly uniformly $\longrightarrow\prod_{i=1}^{m}\mathcal{H}_i$ is* *a nearly uniformly* $\max\{K_i: i \in \Lambda\}$*-Lipschitzian mapping with the sequence* $\{\sum_{i=1}^m l_{i,n}\}_{m=1}^{\infty}$ *defined by (52) such that* $\mathrm{Fix}(S) \cap \mathrm{SEMVLI}(\mathscr{H}_i, g_i, h_i, T_i, \psi_i, A_i, i \in \Lambda) \neq \emptyset$. *Suppose that* $\Omega < \min\{1, 1/K_i\}$, *for each* $i \in \Lambda$, *where* Ω *is the same as in (42). Then, there exists a unique continuous solution* $(p_1(t), p_2(t), \ldots, p_m(t))$ *of extended nonlinear mixed variational-like proximal dynamical system (84) with* $p_i(t_0) = c_i$ *over* $[t_0, \infty]$.

Proof. By Lemma 2, we have $h_i(p_i) = \mathscr{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - (T_i(h_i(p_i)) + \rho_i A_i(p_1, p_2, \ldots, p_m))]$ is a solution of SENMVLI (19). For each $i \in \Lambda$, we define $\mathscr{F}_i: \prod_{i=1}^m \mathscr{H}_i \longrightarrow \mathscr{H}_i$ by

$$\mathscr{F}_i(p_1, p_2, p_3, \ldots, p_m) = \omega_i\left\{\mathscr{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - (T_i(h_i(p_i)) + \rho_i A_i(p_1, p_2, \ldots, p_m))] - h_i(p_i)\right\}, \tag{89}$$

for all $(p_1, p_2, \ldots, p_m) \in \prod_{i=1}^m \mathscr{H}_i$. Define $\|.\|_*$ on $\prod_{i=1}^m \mathscr{H}_i$ by

$$\|(p_1, p_2, p_3, \ldots, p_m)\|_* = \sum_{i=1}^m \|p_i\|_i, \forall (p_1, p_2, \ldots, p_m) \in \prod_{i=1}^m \mathscr{H}_i. \tag{90}$$

It is easy to see that $(\prod_{i=1}^m \mathscr{H}_i, \|.\|_*)$ is a Hilbert space. Also, define $G: \prod_{i=1}^m \mathscr{H}_i \longrightarrow \prod_{i=1}^m \mathscr{H}_i$ as follows:

$$\mathscr{G}(p_1, p_2, \ldots, p_m) = (\mathscr{F}_1(p_1, p_2, \ldots, p_m), \mathscr{F}_2(p_1, p_2, \ldots, p_m), \ldots, \mathscr{F}_m(p_1, p_2, \ldots, p_m)), \tag{91}$$

for all $(p_1, p_2, \ldots, p_m) \in \prod_{i=1}^m \mathscr{H}_i$.

To prove that $\mathscr{G}(p_1, p_2, \ldots, p_m)$ is a locally Lipschitz continuous mapping, suppose that $(p_1, p_2, \cdots, p_m) \neq (\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m) \in \prod_{i=1}^m \mathscr{H}_i$ are given. By using (32) and (34) and Theorem 1, for each $i \in \Lambda$, we have

$$\begin{aligned}
&\|\mathscr{F}_i(p_1, p_2, \ldots, p_m) - \mathscr{F}_i(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\|_i \\
&= \omega_i\Bigg[\Bigg\|\Bigg(\mathscr{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - (T_i(h_i(p_i)) + \rho_i A_i(p_1, p_2, \ldots, p_m))] - h_i(p_i)\Bigg) \\
&\quad - \Bigg(\mathscr{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,\widehat{p}_i)}[g_i(\widehat{p}_i) - (T_i(h_i(\widehat{p}_i)) + \rho_i A_i(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m))] - h_i(\widehat{p}_i)\Bigg)\Bigg\|_i\Bigg] \\
&\leq \omega_i\|p_i - \widehat{p}_i\|_i + \omega_i\Bigg\|\Bigg(p_i - h_i(p_i) + \mathscr{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - T_i(h_i(p_i)) \\
&\quad + \rho_i A_i(p_1, p_2, \ldots, p_m))] - \Bigg(\widehat{p}_i - h_i(\widehat{p}_i) + \mathscr{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - (T_i(h_i(p_i)) \\
&\quad + \rho_i A_i(p_1, p_2, \ldots, p_m))]\|_i.
\end{aligned} \tag{92}$$

Using the same technique of (35)–(39) and (42) and (92) becomes

$$\begin{aligned}
&\|\mathscr{F}_i(p_1, p_2, \ldots, p_m) - \mathscr{F}_i(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\|_i \\
&\leq \omega_i\|p_i - \widehat{p}_i\|_i + \omega_i\left(\Theta_i\|p_i - \widehat{p}_i\|_i + \frac{\tau_i \rho_i}{\alpha_i + \delta_i}\sum_{j\in\Lambda, i\neq j} \nu_{i,j}\|p_j - \widehat{p}_j\|_j\right),
\end{aligned} \tag{93}$$

where $\Theta_i = \xi_i + \sqrt{(1 - 2\mu_{h_i} + \lambda_{h_i}^2)} + (\tau_i(\lambda_{T_i}\lambda_{h_i} + \sqrt{(\lambda_{g_i}^2 - 2\rho_i\mu_{A_{ii}} + \rho_i^2\lambda_{A_{ii}}^2)}))/(\alpha_i + \delta_i)$. Applying the similar arguments of (36) and (44), we have

$$\begin{aligned}
&\|\mathscr{G}(p_1, p_2, \ldots, p_m) - \mathscr{G}(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\|_* \\
&= \sum_{i=1}^m \|\mathscr{F}_i(p_1, p_2, \ldots, p_m) - \mathscr{F}_i(\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m)\|_i \\
&\leq \max\{\omega_i: i \in \Lambda\}\left(\sum_{i=1}^m \|p_1 - \widehat{p}_1\|_i + \max\{\Theta_i\right. \\
&\quad \left. + \sum_{k\in\Lambda, k\neq i} \frac{\tau_k \rho_k}{\alpha_k + \delta_k}\nu_{k,i}: i \in \Lambda\right\}\sum_{i=1}^m \|p_1 - \widehat{p}_1\|_i \\
&\leq \omega\left(\sum_{i=1}^m \|p_1 - \widehat{p}_1\|_i + \Omega\sum_{i=1}^m \|p_1 - \widehat{p}_1\|_i\right),
\end{aligned} \tag{94}$$

i.e.,

$$\|\mathscr{G}(p_1,p_2,\ldots,p_m)-\mathscr{G}(\widehat{p}_1,\widehat{p}_2,\ldots,\widehat{p}_m)\|_* \le \omega(1+\Omega)\|(p_1,p_2,\ldots,p_m)-(\widehat{p}_1,\widehat{p}_2,\ldots,\widehat{p}_m)\|_*, \tag{95}$$

where $\Omega=\max\{\Theta_i+\sum_{k\in\Lambda,k\neq i}(\tau_k\rho_k)/(\alpha_k+\delta_k)\nu_{k,i}\colon i\in\Lambda\}$ and $\omega=\max\{\omega_i\colon i\in\Lambda\}$. Therefore, $\mathscr{G}$ is a locally Lipschitz continuous mapping. Hence, for each $(c_1,c_2,\ldots,c_m)\in\prod_{i=1}^m\mathscr{H}_i$, there exists a unique and continuous solution $(p_1(t),p_2(t),\ldots,p_m(t))$ of the extended nonlinear mixed variational-like proximal dynamical system (84), defined in an interval $t_0\le t\le\mathscr{B}$ with the initial condition $p_i(t_0)=c_i$, for each $i\in\Lambda$. Suppose that $[t_0,\mathscr{B}]$ is the maximal interval of the existence of the solutions of (84). Now, we have to show that $\mathscr{B}=\infty$. For any $(p_1,p_2,\ldots,p_m)\in\prod_{i=1}^m\mathscr{H}_i$, we have

$$\begin{aligned}
\left\|\frac{dp_i}{dt}\right\|_i &= \|\mathscr{F}_i(p_1,p_2,\ldots,p_m)\|_i\\
&= \omega_i\left\|\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i)-(T_i(h_i(p_i))+\rho_iA_i(p_1,p_2,\ldots,p_m))]-h_i(p_i)\right\|_i\\
&\le \omega_i\|\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i)-(T_i(h_i(p_i))+\rho_iA_i(p_1,p_2,\ldots,p_m))]\\
&\quad -\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*)-(T_i(h_i(p_i^*))+\rho_iA_i(p_1^*,p_2^*,\ldots,p_m^*))]\|_i\\
&\quad +\omega_i\|p_i-p_i^*\|_i\\
&\le \omega_i\Big[\|\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i)-(T_i(h_i(p_i))+\rho_iA_i(p_1,p_2,\ldots,p_m))]\\
&\quad -\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i)-(T_i(h_i(p_i))+\rho_iA_i(p_1,p_2,\ldots,p_m))]\|_i\\
&\quad +\|\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i)-(T_i(h_i(p_i))+\rho_iA_i(p_1,p_2,\ldots,p_m))]\\
&\quad -\mathscr{J}_{\rho_i,T_i}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*)-(T_i(h_i(p_i^*))+\rho_iA_i(p_1^*,p_2^*,\ldots,p_m^*))]\|_i\Big]\\
&\le \omega_i\|p_i-p_i^*\|_i+\omega_i\left(\Theta_i\|p_i-p_i^*\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|p_j-p_j^*\|_j\right).
\end{aligned} \tag{96}$$

Now, we calculate

$$\begin{aligned}\|\mathscr{G}(p_1,p_2,\ldots,p_m)\|_* &= \sum_{i=1}^{m}\|\mathscr{F}_i(p_1,p_2,\ldots,p_m)\|_i = \sum_{i=1}^{m}\left\|\frac{\mathrm{d}p_i}{\mathrm{d}t}\right\|_i\\
&\le \sum_{i=1}^{m}\omega_i\left(\|p_i-p_i^*\|_i+\left(\Theta_i\|p_i-p_i^*\|_i+\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda,i\neq j}\nu_{i,j}\|p_j-p_j^*\|_j\right)\right)\\
&\le \max\{\omega_i\colon i\in\Lambda\}\left(\sum_{i=1}^{m}\|p_i-p_i^*\|_i+\max\{\Theta_i\right.\\
&\quad\left.+\sum_{k\in\Lambda,k\neq i}\frac{\tau_k\rho_k}{\alpha_k+\delta_k}\nu_{k,i}\colon i\in\Lambda\right\}\sum_{i=1}^{m}\|p_j-p_j^*\|_i\\
&\le \omega\left(\sum_{i=1}^{m}\|p_i-p_i^*\|_i+\Omega\sum_{i=1}^{m}\|p_i-p_i^*\|_i\right)\\
&\le \omega(1+\Omega)\sum_{i=1}^{m}\|p_i-p_i^*\|_i\\
&= \omega(1+\Omega)\|(p_1,p_2,\ldots,p_m)-(p_1^*,p_2^*,\ldots,p_m^*)\|_*\\
&\le \omega(1+\Omega)\|(p_1,p_2,\ldots,p_m)\|_*+\omega(1+\Omega)\|(p_1^*,p_2^*,\ldots,p_m^*)\|_*,\end{aligned}\tag{97}$$

and therefore,

$$\begin{aligned}\|(p_1(t),p_2(t),\ldots,p_m(t))\|_* &\le \|(p_1(t_0),p_2(t_0),\ldots,p_m(t_0))\|_*\\
&\quad+\int_{t_0}^{t}\|\mathscr{G}(p_1(s),p_2(s),\ldots,p_m(s))\|_*\mathrm{d}s\\
&\le \left(\|(p_1(t_0),p_2(t_0),\ldots,p_m(t_0))\|_*+k_1(t-t_0)\right)\\
&\quad+k_2\int_{t_0}^{t}\|(p_1(s),p_2(s),\ldots,p_m(s))\|_*\mathrm{d}s,\end{aligned}\tag{98}$$

where $k_1=\omega(1+\Omega)\|(p_1^*,p_2^*,\ldots,p_m^*)\|_*$ and $k_2=\omega(1+\Omega)$.
Using Lemma 3, we have

$$\|(p_1(t),p_2(t),\ldots,p_m(t))\|_*\le\left(\|(p_1(t_0),p_2(t_0),\ldots,p_m(t_0))\|_*+k_1(t-t_0)\right)e^{k_2(t-t_0)},\forall t\in[t_0,\mathscr{B}]\Big).\tag{99}$$

Hence, the solution is bounded for $t\in(t_0,\mathscr{B})$, if $\mathscr{B}$ is finite. Thus, $\mathscr{B}=\infty$. This completes the proof. □

Applying the approach of Xia and Feng [31, 32], we now show that the trajectory of the solution of extended nonlinear mixed variational-like proximal dynamical system (84) converges to a unique solution of SENMVLI (19).

Theorem 5. *For each* $i\in\Lambda$, *let* $\mathscr{H}_i, g_i, h_i, T_i, \psi_i, \zeta_i$, *and* $\rho_i>0$ *be the same as in Theorem 2 and let all the conditions of Theorem 2 hold. If the following conditions are satisfied:*

$$\begin{cases}\chi_i=\mu_{h_i}+\dfrac{\tau_i\rho_i}{\alpha_i+\delta_i}\displaystyle\sum_{j\in\Lambda,i\neq j}\nu_{i,j},\\[2ex]
\tau_i\sqrt{\left(\lambda_{g_i}^2-2\rho_i\mu_{A_{ii}}+\rho_i^2\lambda_{A_{ii}}^2\right)}<\left[(\chi_i-\xi_i)(\alpha_i+\delta_i)-\tau_i\lambda_{T_i}\lambda_{h_i}\right],\end{cases}\tag{100}$$

then, the extended nonlinear mixed variational-like proximal dynamical system (84) converges globally exponentially to a unique solution of SENMVLI (19).

Proof. In Theorem 2, we prove the existence of a unique solution $(p_1^*, p_2^*, \ldots, p_m^*)$ of the problem SENMVLI (19). By Lemma 2, we have $h_i(p_i^*) = \mathcal{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i^*)}[g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))]$. Also, in view of Theorem 4, the extended mixed variational-like resolvent dynamical system (82) has a unique solution $(p_1(t), p_2(t), \ldots, p_m(t))$ over $[t_0, \mathscr{B}]$ for any fixed $c_i \in \mathscr{H}_i$, for each $i \in \Lambda$. Let $(p_1(t), p_2(t), \ldots, p_m(t)) = (p_1(t, t_0; c_1), p_2(t, t_0; c_2), \ldots, p_m(t, t_0; c_m))$ be the solution of the initial value problem (84) with $p_i(t_0) = c_i$, for each $i \in \Lambda$. Now, we consider the Lyapunov function $\mathscr{L}$ defined on $\prod_{i=1}^m \mathscr{H}_i$ by

$$\mathscr{L}(p_1, p_2, \ldots, p_m) = \|(p_1, p_2, \ldots, p_m) - (p_1^*, p_2^*, \ldots, p_m^*)\|\|_*^2, \forall (p_1, p_2, \ldots, p_m) \in \prod_{i=1}^m \mathscr{H}_i. \tag{101}$$

For each $i \in \Lambda$, from (82), (91)–(95), and (101), and using μ_{h_i}-strongly monotonicity of h_i, we have

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial p_i}\frac{dp_i}{dt} &= 2\left\langle p_i(t) - p_i^*, \frac{dp_i}{dt}\right\rangle_i \\
&= 2\omega_i \langle p_i(t) - p_i^*, \mathcal{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - (T_i(h_i(p_i)) + \rho_i A_i(p_1, p_2, \ldots, p_m))] - h_i(p_i)\rangle_i \\
&= -2\omega_i \langle p_i(t) - p_i^*, h_i(p_i) - h_i(p_i^*)\rangle_i \\
&\quad + 2\omega_i \langle p_i(t) - p_i^*, \mathcal{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)}[g_i(p_i) - (T_i(h_i(p_i)) + \rho_i A_i(p_1, p_2, \ldots, p_m))] - h_i(p_i^*)\rangle_i \\
&\leq -2\omega_i \mu_{h_i}\|p_i(t) - p_i^*\|_i^2 + 2\omega_i\|p_i(t) - p_i^*\|_i\Big\|\mathcal{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i)} g_i(p_i) - (T_i(h_i(p_i)) \\
&\quad + \rho_i A_i(p_1, p_2, \ldots, p_m) - \mathcal{J}_{\rho_i, T_i}^{\partial_{\zeta_i}\psi_i(.,p_i^*)} g_i(p_i^*) - (T_i(h_i(p_i^*)) + \rho_i A_i(p_1^*, p_2^*, \ldots, p_m^*))\|_i \\
&\leq -2\omega_i\left(\mu_{h_i} - \frac{\xi_i(\alpha_i + \delta_i) + \tau_i\left(\lambda_{T_i}\lambda_{h_i} + \sqrt{\left(\lambda_{g_i}^2 - 2\rho_i\mu_{A_{ii}} + \rho_i^2\lambda_{A_{ii}}^2\right)}\right)}{\alpha_i + \delta_i}\right)\|p_i(t) - p_i^*\|\|_i^2 \\
&\quad - \omega_i\frac{\tau_i\rho_i}{\alpha_i + \delta_i}\sum_{j\in\Lambda, i\neq j}\nu_{i,j}\|p_j - \widehat{p}_j\|\|_j^2 \\
&\leq -2\omega_i\left(\Phi_i\Big\|p_i(t) - p_i^*\|\|_i^2 + \frac{\tau_i\rho_i}{\alpha_i + \delta_i}\sum_{j\in\Lambda, i\neq j}\nu_{i,j}\Big\|p_j(t) - p_j^*\|\|_j^2\right),
\end{aligned} \tag{102}$$

where $\Phi_i = \mu_{h_i} - \xi_i(\alpha_i + \delta_i) + \tau_i(\lambda_{T_i}\lambda_{h_i} + \sqrt{(\lambda_{g_i}^2 - 2\rho_i\mu_{A_{ii}} + \rho_i^2\lambda_{A_{ii}}^2)})/(\alpha_i + \delta_i)$. By using (102), we have

$$\begin{aligned}\frac{d\mathscr{L}}{dt} &= \sum_{i=1}^{m}\frac{\partial \mathscr{L}_i}{\partial p_i}\frac{dp_i}{dt} = 2\sum_{i=1}^{m}\langle p_i(t) - p_i^*, \frac{dp_i}{dt}\rangle_i \\ &\leq -2\sum_{i=1}^{m}\left[\omega_i\Phi_i\left\|p_i(t) - p_i^*\right\|_i^2 + \omega_i\frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda, i\neq j}\nu_{i,j}\left\|p_j - \widehat{p}_j\right\|_j^2\right], \\ &\leq -2\omega\max\left\{\Phi_i + \frac{\tau_i\rho_i}{\alpha_i+\delta_i}\sum_{j\in\Lambda, i\neq j}\nu_{i,j}\colon i = 1,2,\ldots,m\right\}\sum_{i=1}^{m}\left\|p_i(t) - p_i^*\right\|_i^2,\end{aligned} \tag{103}$$

i.e.,

$$\frac{d}{dt}\left\|(p_1,p_2,\ldots,p_m) - (p_1^*,p_2^*,\ldots,p_m^*))\right\|\|_*^2 \leq -2\omega\Delta\left\|(p_1,p_2,\ldots,p_m) - (p_1^*,p_2^*,\ldots,p_m^*))\right\|\|_*^2, \tag{104}$$

where $\Delta = \max\left\{\Phi_i + (\tau_i\rho_i)/(\alpha_i+\delta_i)\sum_{j\in\Lambda, i\neq j}\nu_{i,j}\colon i\in\Lambda\right\}$ and $\omega = \{\omega_i\colon i\in\Lambda\}$. Therefore, we have

$$\left\|((p_1,p_2,\ldots,p_m) - p_1^*,p_2^*,\ldots,p_m^*)\right\|_* \leq \left\|(p_1,p_2,\ldots,p_m) - (p_1^*,p_2^*,\ldots,p_m^*)\right\|_* e^{-\omega\Delta(t-t_0)}. \tag{105}$$

Using the conditions (32) and (100), we conclude that $\Delta > 0$. Hence, the trajectory of the solution of extended nonlinear mixed variational-like proximal dynamical system (84) converges globally exponentially to a unique solution of SENMVLI (19). This completes the proof. □

6. Conclusion

In this work, we have studied a new system of extended nonlinear mixed variational-like inequalities involving different nonlinear mappings in the setting of real Hilbert spaces. Using the proximal operator technique, we have shown that the system of extended nonlinear mixed variational-like inequalities is equivalent to the corresponding fixed point problem, and applying this equivalent result, we have proved the existence and uniqueness of solution of the system of extended nonlinear mixed variational-like inequalities. Making use of equivalent fixed point formulation and nearly uniformly Lipschizian mapping, we have proposed a new three-step iterative algorithm with mixed errors to examine the convergence and stability analysis of the suggested iterative algorithm under some suitable conditions. Finally, we have analyzed and suggested a proximal dynamical system associated with the system of extended nonlinear mixed variational-like inequalities. We have shown that the trajectory of the solution of the proximal dynamical system converges globally exponentially to a unique solution of the considered problem. We would like to emphasize that the problem considered in this article can be further investigated from different aspects such as sensitivity analysis, well-posedness, approximation, and numerical analysis. The concepts and method of the proposed operator splitting scheme may be extended for solving the system of quasi variational-like inequalities, system of equilibrium problems, and other related generalized systems.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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