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Research Article

Analysis of Climatic Model Using Fractional Optimal Control

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CO_2 emissions are one of the most critical and challenging problems impacting the atmosphere. In order to reduce CO_2 emissions, significant efforts must be put forward in advanced research. In this research, we propose a fractional optimal control approach to effectively deal with the accessible resources and minimize the restriction of variables. The mathematical model is formulated in terms of the Caputo fractional derivative. The formulated problem is numerically solved using a forward-backward sweep approach with the generalized Euler method. We present a numerical problem to obtain the optimal solution and show that the suggested process is very efficient in science and engineering problems.

1. Introduction

CO_2 emissions have been emerged as one of the most critical and complicated issues impacting the atmosphere. It has an indirect influence on the more obvious form of air pollution and smog. By increasing temperature and humidity, it produces smog, which has a detrimental effect on respiratory health and numerous skin illnesses. Fossil fuel combustion is the major source of new CO_2 emissions because it discharges carbon dioxide into the atmosphere, which acts as a blanket over the Earth, absorbing solar heat and raising temperatures. CO_2 emissions have had far-reaching implications, hurting both the ecosystem and the humans who live within it. Numerous authors [1, 2] have investigated fractional differential equations with integer and noninteger derivatives for use in mathematical modeling of CO_2 emissions. Recently, there has been a lot of interest in the modeling of CO_2 emissions utilizing time-dependent controls and optimal control theory [3, 4]. Scientists used mathematical models to describe the changes of CO_2 emissions and deforestation and provided the mathematical modelling to explain the influence of greenhouse gas emissions on the ecosystem [5, 6]. Using the same mathematical model, Nordhaus [7] analyzed the optimal taxing systems to stabilize climate and carbon dioxide emissions. Fractional calculus has been extensively utilized to represent dynamical processes in many various disciplines, including science, engineering, and many more [8, 9]. This is because fractional order derivatives include the memory effect, which is a significant attribute. Hertel and Rosch [10] created general optimum control issues that are motivated by the fractional derivative of the Riemann–Liouville equation. The same authors developed a reliable numerical framework for the mathematical model and presented related optimal control problems using Caputo derivatives [11].

Numerous scholars have been working on identifying climate changes in a certain geographical area and have created multiple efficient models for doing so. They used a numerical scheme to derive the conditions for the optimality system for a general control problem with Caputo derivatives. Verma et al. [12] suggested various nonlinear dynamical models to identify the optimum solutions for carbon dioxide emission reduction. The state model for an optimal control problem contained both first-order and noninteger derivatives, and necessary optimality conditions have been derived for this problem [13, 14].

According to research in the literature, environmental systems and other processes with nonlinear behaviour are good candidates for optimal control problems. Therefore, utilising fractional derivatives and optimal control theory to describe climatic changes has a lot of benefits [15, 16]. This research presents the formulation of a mathematical model with the help of fractional differential equations where the optimal solutions are determined by the application of Pontryagin's principle. The paper is summarized as follows: we provide the formulation of fractional optimal control problem in Section 2. In Section 3, the formulation of controlled CO_2 emissions model is considered and necessary condition for the optimality of model problem is derived. Sections 4 and 5 describe the numerical scheme for the solution of the problem and the experimentation of the results in the form of simulation, and finally, Section 6 deals with the concluding remarks.

2. Fundamental Properties of Fractional Calculus

This section provides a quick overview of the fractional optimal control model's mathematical formulation. Then, in Section 3, we will put the modelling approach to build our CO_2 emissions model. There are a lot of other kinds of fractional derivatives, but Riemann–Liouville (R-L) derivative and Caputo derivative are the two that are most frequently used in engineering and mathematical modelling. We construct the optimization model for this task using Caputo fractional derivatives.

Definition 1. The left R-L fractional derivative is given as [17]

$$ {}_{a}^{L}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x (x-t)^{(m-\alpha-1)} f(t)\mathrm{d}t, \quad (1) $$

where $m-1<\alpha<m$.

Definition 2. The right R-L fractional derivative is given as

$$ {}_{a}^{R}D_{x}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b (t-x)^{(m-\alpha-1)} f(t)\mathrm{d}t. \quad (2) $$

Here order α fulfills $m-1\leq\alpha<m$, where Γ denotes Euler's Gamma function.

Definition 3. The left Caputo fractional derivative is termed as

$$ {}_{a}^{C}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{(m-\alpha-1)} f^{(m)}(t)\mathrm{d}t. \quad (3) $$

Definition 4. The right Caputo fractional derivative is termed as

$$ {}_{x}^{C}D_{b}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (t-x)^{(m-\alpha-1)} f^{(m)}(t)\mathrm{d}t, \quad (4) $$

where α is the order of the Caputo derivative.

3. Fractional Optimal Control Formulation

Agrawal [18] presented the formulation of fractional optimal control for a class of distributed systems. The main goal is to find an optimal control u^* which minimizes objective functional,

$$ J(u) = \int_0^1 F(x,u,t)\mathrm{d}t, \quad (5) $$

subject to fractional dynamics constraints

$$ {}_{0}^{C}D_{t}^{\alpha} x(t) = W(x,u,t), \quad (6) $$

with initial condition

$$ x(0) = x_0, \quad (7) $$

where $x(t)$ is the state variable, $F(x,u,t)$ and $W(x,u,t)$ are the two arbitrary functions, and x_0 is the state variable at time $t=0$. We can obtain the necessary condition only if we manipulate equations (5) and (6) using Lagrange multiplier approach, the variations of calculus, and integration by parts which shows that this equation is now independent from variation of a derivative such that,

$$ \begin{aligned} {}_{0}^{C}D_{t}^{\alpha} Cx(t) &= W(x,u,t), \\ {}_{t}^{C}D_{1}^{\alpha}\lambda &= \frac{\partial F}{\partial x} + \lambda\frac{\partial W}{\partial x}, \\ \frac{\partial F}{\partial x} + \lambda\frac{\partial W}{\partial x} &= 0, \end{aligned} \quad (8) $$

with

$$ \begin{aligned} x(0) &= x_0, \\ \lambda_1 &= 0, \end{aligned} \quad (9) $$

where λ represents the Lagrange multiplier or co-state variable.

4. Controlled CO_2 Emissions Model with Fractional Derivatives

In order to obtain the essential conditions or equations for the fractional optimal control model's optimality, we will use the data from the formulation of the fractional optimal control problem. Our aim is to find an optimal control that reduces the emission of CO_2 and increase forested area which means to find an optimal control u^* that reduces the objective functional,

$$ J(U) = \int_0^1 F(X,U,t)\mathrm{d}t, \quad (10) $$

subject to fractional dynamics constraints

$$ {}_{0}^{C}D_{t}^{\alpha} X(t) = W(X,U,t), \quad (11) $$

with initial condition

$$X(0) = X_0, \tag{12}$$

where

$$\begin{aligned} X(t) &= (x, z, y)^T, \\ X(0) &= (x(0), z(0), y(0))^T, \\ U(t) &= (u_1, u_2)^T, \\ F(X, U, t) &= e^{-\delta t}\left(ax^2 + bu_1^2 + cu_2^2\right), \end{aligned} \tag{13}$$

so, we get

$${}_{C}^{0}D_t^{\alpha} x(t) = -z\alpha_1 + xr\left(1 - \frac{x}{s}\right) + y(-u_2 + \alpha_2), \tag{14}$$

$${}_{C}^{0}D_t^{\alpha} z(t) = -zh + u_1 y, \tag{15}$$

$${}_{C}^{0}D_t^{\alpha} y(t) = y\gamma, \tag{16}$$

where $X(t)$ represents a state vector and $U(t)$ represents a control vector. Initial conditions are $x(0) = x_0, z(0) = z_0$ and $y(0) = y_0$. Manabe and Stouffer [19] formulated the classical optimal control problem at $\alpha = 1$ and therefore the dynamic constraint equations (14)–(16) with above initial condition become as $u_1 = u_2 = 0$.

It is important to point out that there are thorough justifications in the literature for the formulation of necessary conditions for optimality of various fractional dynamical systems [15, 16]. Therefore,

$$\begin{aligned} {}_{C}^{0}D_t^{\alpha} X(t) &= W(X, U, t), \\ {}_{t}^{C}D_b^{\alpha} \lambda(t) &= \frac{\partial F}{\partial X} + \lambda^T \frac{\partial W}{\partial X}, \\ \frac{\partial F}{\partial U} + \lambda^T \frac{\partial W}{\partial U} &= 0, \\ \mathrm{X}(0) &= X_0, \\ \lambda(1) &= 0, \end{aligned} \tag{17}$$

where $\lambda(t) = (\lambda_1, \lambda_2, \lambda_3)^T$ is the co-state vector.

We can obtain CO_2 emissions system in an enlarged version by applying the compact form of the prerequisites listed previously.

$$\begin{aligned} {}_{C}^{0}D_t^{\alpha} x(t) &= -z\alpha_1 + xr\left(1 - \frac{x}{s}\right) + y(-u_2 + \alpha_2), \\ {}_{C}^{0}D_t^{\alpha} z(t) &= -zh + u_1 y, \\ {}_{C}^{0}D_t^{\alpha} y(t) &= y\gamma. \end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned} {}_{t}^{C}D_b^{\alpha} \lambda_1(t) &= -\frac{\partial H}{\partial x} = -\lambda_1 r + \frac{2r\lambda_1 x}{s} - 2\mathrm{axe}^{-\delta t}, \\ {}_{t}^{C}D_b^{\alpha} \lambda_2(t) &= -\frac{\partial H}{\partial z} = -\lambda_1 \alpha_1 + \lambda_2 h, \\ {}_{t}^{C}D_b^{\alpha} \lambda_3(t) &= -\frac{\partial H}{\partial y} = -\lambda_1(\alpha_2 - u_2) - \lambda_2 u_1 - \lambda_3 \gamma. \end{aligned} \tag{19}$$

Moreover, the optimal controls are given by

Table 1: Description of variables and control functions.

Variables	Description
$x(t)$	Represents atmospheric carbon dioxide (CO_2)
$y(t)$	Represents forest area
$x(t)$	Represents GDP (gross domestics product)
$u_1(t)$	Represents optimal reforestation effort
$u_2(t)$	Represents optimal technological effort

$$\begin{aligned} u_1^* &= \frac{-\lambda_2 y}{2be^{-\delta t}}, \\ u_2^* &= \frac{-\lambda_1 y}{2ce^{-\delta t}}, \end{aligned} \tag{20}$$

where $(*)$ denotes the optimal values of u_1 and u_2. $x(0) = x_0$, $z(0) = z_0$ and $y(0) = y_0$, $\lambda_1(t_f) = 0$, $\lambda_2(t_f) = 0$, $\lambda_3(t_f) = 0$.

5. Numerical Results and Discussion

In this segment, we provide an explanation of a few numerical simulations and their results. We develop a forward-backward sweep algorithm using the RK approach to analyze the biological model in optimality system. In this work, a forward-backward approach using the generalized Euler scheme is employed to compute the numerical solution of the optimality systems. The generalised Euler method has recently been used in a TB model along with the forward-backward algorithm. We use Mathematica software 11 to illustrate the graphical results of various fractional order. Table 1 shows the description of variables and control functions, whereas Table 2 demonstrate the parameters of the model.

The numerical simulations are performed using the initial conditions, and model parameter values are listed as follows: $x(0) = 398$ million tons of carbon dioxide, $y(0) = 2787$ billion international dollars, $z(0) = 43$ million·m^3, $a = 0.1, h = 0.0001, s = 700, u_2 = 0.0008, u_1 = 0.00012$, $\gamma = 0.035$, $r = 0.15$, $\alpha_1 = 0.0006$, $c = 1E + 9, b = 3.5E + 9$, and $\alpha_2 = 0.00005$. The graphical representations demonstrate that we provide a significant reduction in the rate of CO_2 and reforestation increases with decrease in GDP when $\alpha = 1$ and noninteger orders ($\alpha = 0.98, 0.96, 0.94, 0.92, 0.9, 0.88, 0.86$) with time dependent control. The model's formulation for fractional optimal control demonstrates that by maximizing investments in clean technology research and reforestation activities, significant reductions in CO_2 emissions can be attained. Figure 1 represents the graphical result of million tons of CO_2, Figure 2 shows the graphical of million·m^3/year, Figure 3 shows the graphical representation of billion US dollars, Figure 4 represents emission of CO_2 with and without control, Figure 5 represents optimal solutions for forest with and without control, Figure 6 shows the comparison of GDP with and without control, Figure 7 shows the comparison of the amount of CO_2 without control with different values of α, Figure 8 represents

Table 2: Description of parameters.

Parameter	Definition
r	Represents the emission rate of atmospheric carbon dioxide (CO_2)
s	Represents the carrying capacity of the CO_2 in the atmosphere
h	Represents the forest depletion rate
γ	Represents exponential growth rate of CO_2

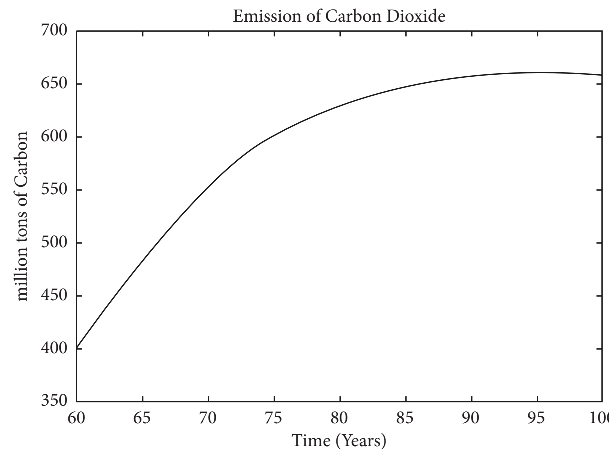

Figure 1: Graphical representation of million tons of CO_2.

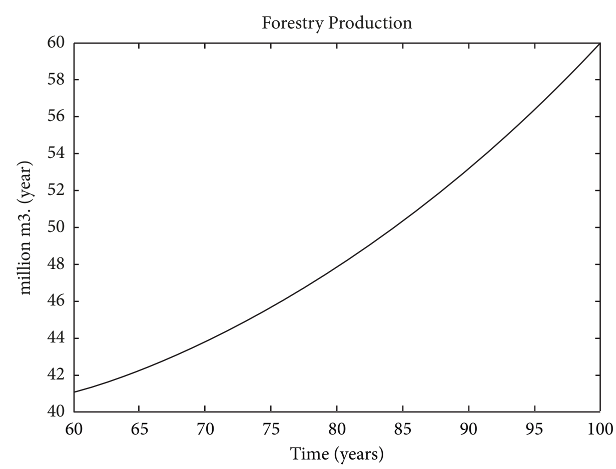

Figure 2: Graphical representation of million·m^3/year.

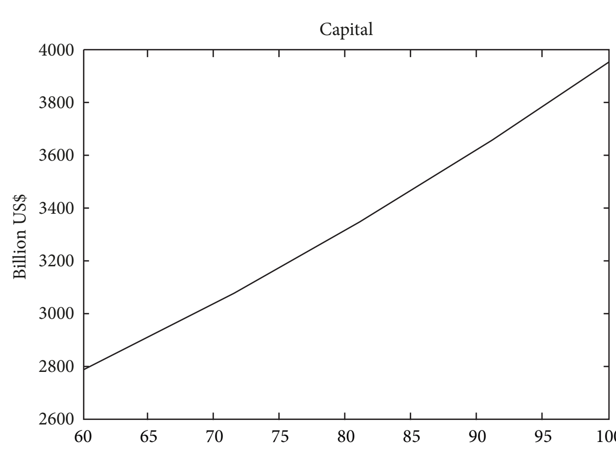

Figure 3: Graphical representation of billion US dollars.

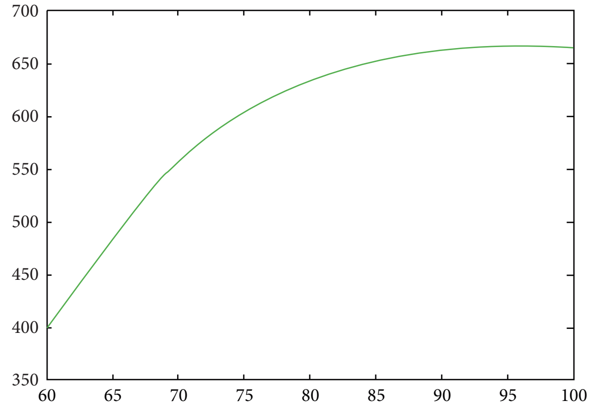

Figure 4: Emission of CO_2 with and without control.

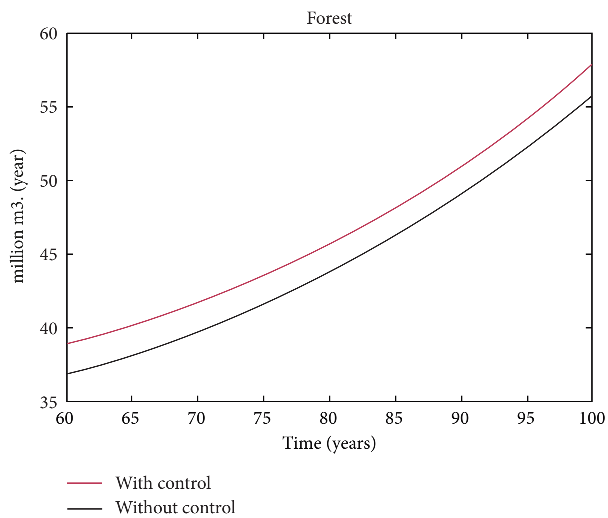

Figure 5: Optimal solutions for forest with and without control.

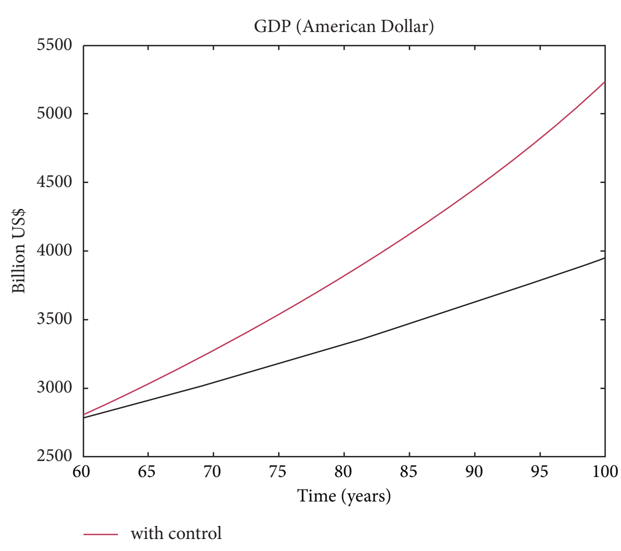

Figure 6: Comparison of GDP with and without control.

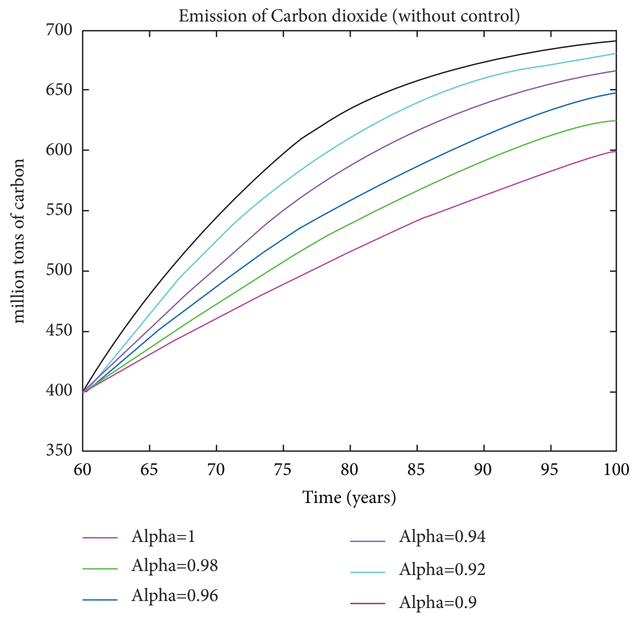

FIGURE 7: Comparison of the amount of CO_2 without control with different values of alpha.

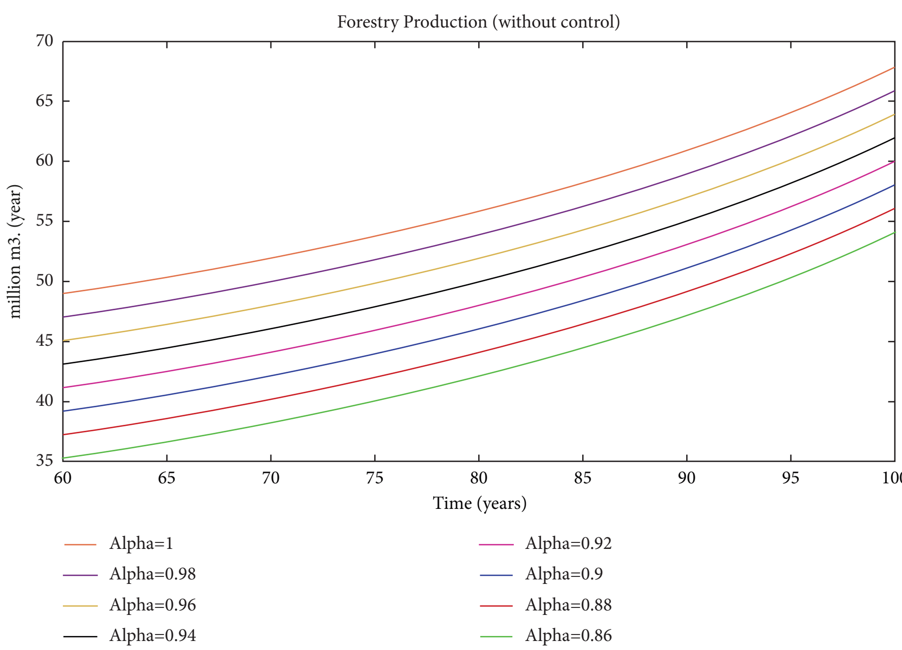

FIGURE 8: Comparison of the forestation rate without control with different values of alpha.

comparison of the forestation rate without control with different values of α, Figure 9 represents comparison of GDP without control with different values of α, Figure 10 represents comparison of the amount of CO_2 with control with different values of α, Figure 11 represents comparison of the forestation rate with control with different values of α, and Figure 12 represents comparison of GDP with control with different values of α.

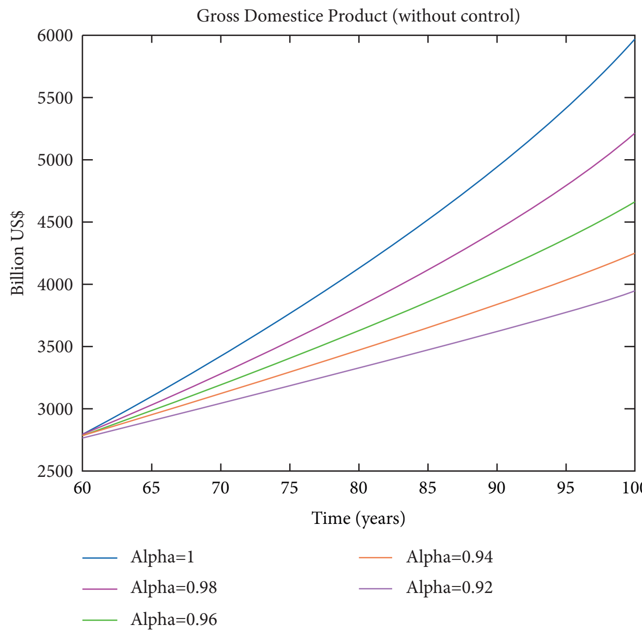

Figure 9: Comparison of GDP without control with different values of alpha.

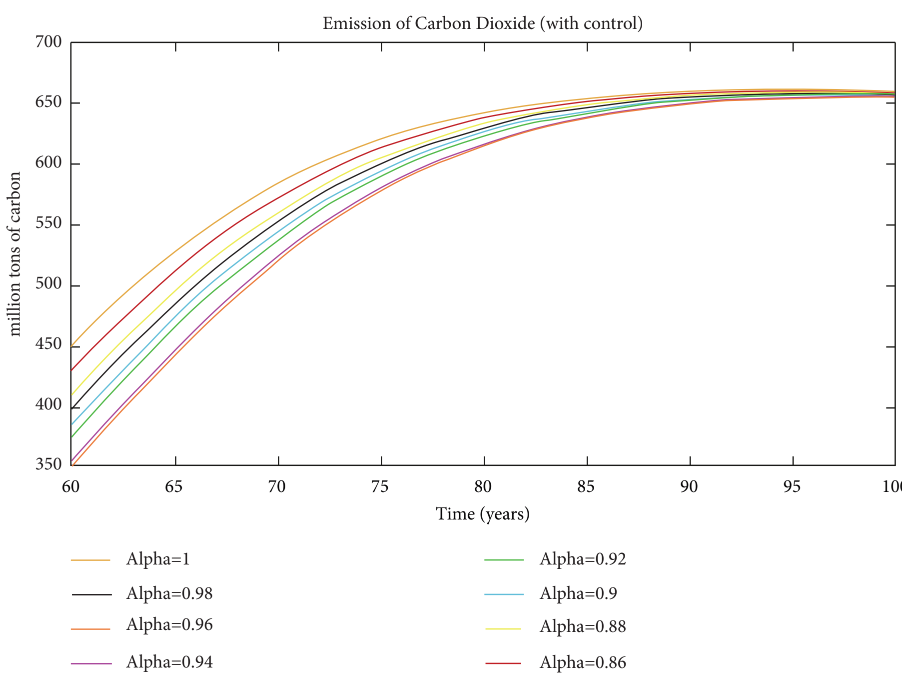

Figure 10: Comparison of the amount of CO_2 with control with different values of alpha.

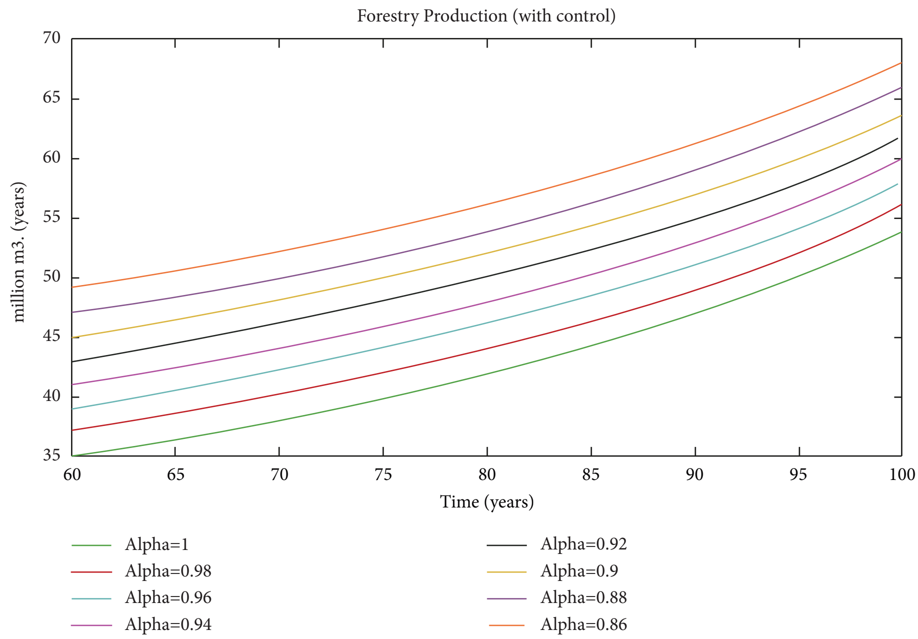

Figure 11: Comparison of the forestation rate with control with different values of alpha.

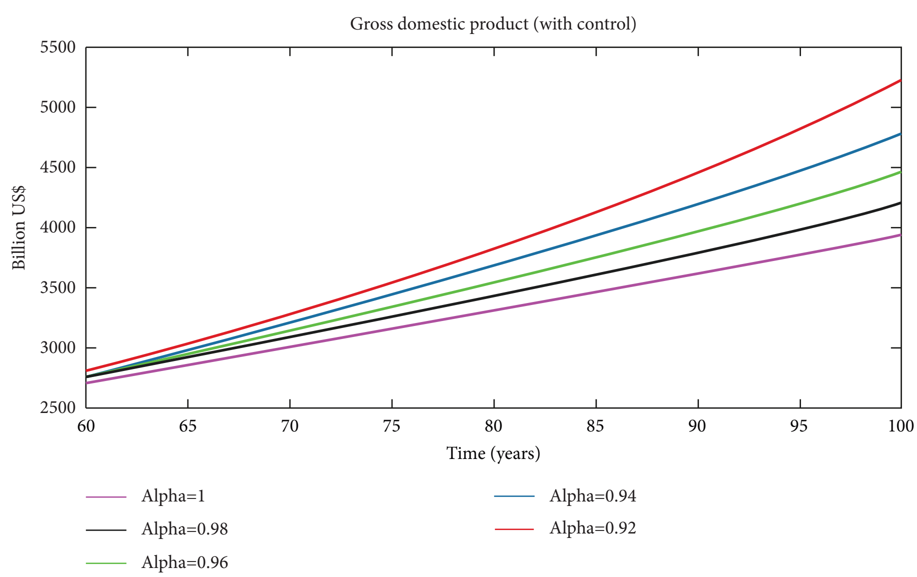

Figure 12: Comparison of GDP with control with different values of alpha.

6. Conclusion

In this study, a mathematical model that displays CO_2 emissions, deforestation rate, and GDP has been taken into consideration. The model is characterized by fractional derivatives. The forward-backward sweep approach combined with the extended Euler method was used to numerically solve the optimality problem. The results of the models demonstrate that when control is implemented, the rate of CO_2 emissions is reduced and the reforestation rate increases. The amount of CO_2 is reduced more than with any other method when all time dependent controls are employed in simulations. The numerical solutions show that the optimum control problem for fractional orders is significantly more precise than involving integer orders.

Data Availability

All the data are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] W. Xie, W.-Z. Wu, C. Liu, T. Zhang, and Z. Dong, "Forecasting fuel combustion-related co2 emissions by a novel continuous fractional nonlinear grey Bernoulli model with grey wolf optimizer," *Environmental Science and Pollution Research*, vol. 28, no. 28, pp. 38128–38144, 2021.

[2] M. Gao, H. Yang, Q. Xiao, and M. Goh, "A novel fractional grey riccati model for carbon emission prediction," *Journal of Cleaner Production*, vol. 282, Article ID 124471, 2021.

[3] M. A. L. Caetano, D. F. M. Gherardi, and T. Yoneyama, "An optimized policy for the reduction of co2 emission in the brazilian legal amazon," *Ecological Modelling*, vol. 222, no. 15, pp. 2835–2840, 2011.

[4] H. S. Asad, R. K. K. Yuen, J. Liu, and J. Wang, "Adaptive modeling for reliability in optimal control of complex hvac systems," *Building Simulation*, vol. 12, pp. 1095–1106, 2019.

[5] R. S. Tol, "On the optimal control of carbon dioxide emissions: an application of fund," *Environmental Modeling & Assessment*, vol. 2, no. 3, pp. 151–163, 1997.

[6] A. J. Jarvis, P. C. Young, D. T. Leedal, and A. Chotai, "A robust sequential co2 emissions strategy based on optimal control of atmospheric co2 concentrations," *Climatic Change*, vol. 86, no. 3-4, pp. 357–373, 2008.

[7] W. D. Nordhaus, "Reflections on the economics of climate change," *The Journal of Economic Perspectives*, vol. 7, no. 4, pp. 11–25, 1993.

[8] K. R. Stollery, "Constant utility paths and irreversible global warming," *Canadian Journal of Economics*, vol. 31, no. 3, pp. 730–742, 1998.

[9] R. Gorenflo and F. Mainardi, "Fractional calculus and stable probability distributions," *Archives of Mechanics*, vol. 50, no. 3, pp. 377–388, 1998.

[10] T. W. Hertel and S. D. Rosch, "Climate change, agriculture, and poverty," *Applied Economic Perspectives and Policy*, vol. 32, no. 3, pp. 355–385, 2010.

[11] M. Verma and C. Gautam, "Optimal mitigation of atmospheric carbon dioxide through forest management programs: a modeling study," *Computational and Applied Mathematics*, vol. 41, no. 7, pp. 320–327, 2022.

[12] M. Verma, A. K. Verma, and A. Misra, "Mathematical modeling and optimal control of carbon dioxide emissions from energy sector," *Environment, Development and Sustainability*, vol. 23, no. 9, pp. 13919–13944, 2021.

[13] R. C. Pullar, R. M. Novais, A. P. Caetano, M. A. Barreiros, S. Abanades, and F. A. C. Oliveira, "A review of solar thermochemical co2 splitting using ceria-based ceramics with designed morphologies and microstructures," *Frontiers of Chemistry*, vol. 7, p. 601, 2019.

[14] H. Kheiri and M. Jafari, "Optimal control of a fractional-order model for the hiv/aids epidemic," *International Journal of Biomathematics*, vol. 11, no. 07, Article ID 1850086, 2018.

[15] Y. Sekerci, "Climate change effects on fractional order prey-predator model," *Chaos, Solitons & Fractals*, vol. 134, Article ID 109690, 2020.

[16] M. A. L. Caetano, D. F. M. Gherardi, and T. Yoneyama, "Optimal resource management control for co2 emission and reduction of the greenhouse effect," *Ecological Modelling*, vol. 213, no. 1, pp. 119–126, 2008.

[17] M. C. Caputo and D. F. Torres, "Duality for the left and right fractional derivatives," *Signal Processing*, vol. 107, pp. 265–271, 2015.

[18] O. P. Agrawal, "Fractional optimal control of a distributed system using eigenfunctions," *Journal of Computational and Nonlinear Dynamics*, vol. 3, no. 2, 2008.

[19] S. Manabe and R. J. Stouffer, "Multiple-century response of a coupled ocean-atmosphere model to an increase of atmospheric carbon dioxide," *Journal of Climate*, vol. 7, no. 1, pp. 5–23, 1994.