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Research Article

Subobjects and Compactness in Point-Free Convergence

Emilio Angulo and Frédéric Mynard

NJCU, Department of Mathematics, 2039 Kennedy Blvd, Jersey City, NJ 07305, USA

Correspondence should be addressed to Frédéric Mynard; fmynard@njcu.edu

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We consider subobjects in the context of point-free convergence (in the sense of Goubault-Larrecq and Mynard), characterizing extremal monomorphisms in the opposite category of that of convergence lattices. It turns out that special ones are needed to capture the notion of subspace. We call them standard and they essentially depend on one element of the convergence lattice. We introduce notions of compactness and closedness for general filters on a convergence lattice, obtaining adequate notions for standard extremal monos by restricting ourselves to principal filters. The classical facts that a closed subset of a compact space is compact and that a compact subspace of a Hausdorff space is closed find generalizations in the point-free setting under the form of general statements about filters. We also give a point-free analog of the classical fact that a continuous bijection from a compact pseudotopology to a Hausdorff pseudotopology is a homeomorphism.

1. Introduction

Recall that a convergence ξ on a set X is a relation between the set $\mathbb{FP}X$ of (set-theoretic) filters on X and the set X, denoted by

$$x \in \lim_{\xi}\mathcal{F}, \tag{1}$$

if $\mathcal{F}$ and x are ξ-related, subject to the following two axioms:

$$\begin{aligned}&(\text{monotone})\ \mathcal{F} \subset \mathcal{G} \Rightarrow \lim_{\xi}\mathcal{F} \subset \lim_{\xi}\mathcal{G},\\ &(\text{point axiom})\ x \in \lim_{\xi}\{x\}^{\uparrow},\end{aligned} \tag{2}$$

for every $x \in X$ and every $\mathcal{F}, \mathcal{G} \in \mathbb{FP}X$. Continuity of a map $f\colon (X,\xi) \longrightarrow (Y,\tau)$ is simply preservation of limits, that is,

$$(\text{continuity})\ f\left(\lim_{\xi}\mathcal{F}\right) \subset \lim_{\tau} f[\mathcal{F}], \tag{3}$$

where $f[\mathcal{F}] = \{B \subset Y\colon f^{-1}(B) \in \mathcal{F}\} \in \mathbb{FP}Y$ is the image filter of $\mathcal{F}$ under f.

Let **Conv** denote the category of convergence spaces and continuous maps. This is a topological category in which subspaces are defined as usual as the initial structure for the inclusion map; namely, if $A \subset X$ and (X,τ) is a convergence space, then the *induced convergence* $\tau_{|A}$ on A is defined by $\lim_{\tau_{|A}}\mathcal{F} = \lim_{\tau}\mathcal{F}^{\uparrow X} \cap A$, where $\mathcal{F}^{\uparrow X}$ is the filter generated on X by the filter $\mathcal{F}$ on A.

The category **Top** of topological spaces and continuous maps is a concretely reflective subcategory of **Conv**. See e.g., [1], for a systematic treatment of **Conv** and of classical topology from that viewpoint.

In [2], Goubault-Larrecq and Mynard introduce a point-free generalization of **Conv** in which the function

$$\lim_{\xi}\colon \mathbb{FP}X \to \mathbb{P}X, \tag{4}$$

is abstracted away to a monotone function

$$\lim\colon \mathbb{F}L \longrightarrow L, \tag{5}$$

from (order-theoretic) filters on a lattice L to L. Note that (point axiom) is not part of the axiomatic in this point-free version of convergence spaces, though the notion can also be recovered (as so-called *centered* convergence lattices) in an abstract order-theoretic form.

In the context of this paper, a lattice is an ordered set with all finite joins and meets, including the empty ones, so that our lattices have a greatest element, usually denoted by $\top$ and a least element usually denoted by $\bot$. Lattice morphisms preserve all finite joins and meets including the empty ones, so that our lattice morphisms send the greatest element to greatest element and the least to least. Let **Lat** denote the corresponding category. By a *category of lattices*, we mean a subcategory of **Lat**.

Definition 1. Given a category **C** of lattices, a *convergence* **C**-*object* $(L, \lim)$ is a **C**-object L together with a monotone map $\lim\colon \mathbb{F}L \longrightarrow L$. The objects of the category $\mathbf{C}^{\mathbf{Conv}}$ are the convergence **C**-objects, and the morphisms $\varphi\colon L \longrightarrow L'$ are the **C**-morphisms that are *continuous* in the sense that for every $\mathscr{F} \in \mathbb{F}L'$,

$$\text{(ptfree continuity)} \lim_{L'}\mathscr{F} \leq \varphi\left(\lim_{L}\varphi^{-1}(\mathscr{F})\right), \tag{6}$$

where $\varphi^{-1}(\mathscr{F}) = \{\ell \in L\colon \varphi(\ell) \in \mathscr{F}\}$. The morphism φ is *final* (in the usual categorical sense) if we have equality in (ptfree continuity).

The category **Conv** embeds coreflectively into $(\mathbf{C}^{\mathbf{Conv}})^{\mathrm{op}}$ when **C** is the category **Frm** of frames or **CFrm** of coframes: the powerset-functor $\mathbb{P}\colon \mathbf{Conv} \longrightarrow (\mathbf{C}^{\mathbf{Conv}})^{\mathrm{op}}$ sending (X, ξ) to $(\mathbb{P}X, \lim_{\xi})$ and $f\colon (X, \xi) \longrightarrow (Y, \tau)$ to $\mathbb{P}f = f^{-1}\colon \mathbb{P}Y \longrightarrow \mathbb{P}X$ (in $\mathbf{C}^{\mathbf{Conv}}$) is then right-adjoint to the point-functor $\mathrm{pt}\colon (\mathbf{C}^{\mathbf{Conv}})^{\mathrm{op}} \longrightarrow \mathbf{Conv}$ (the coreflector), where the underlying set of $\mathrm{pt}L$, the set of "points" of L, is the set of $\mathbf{C}^{\mathbf{Conv}}$-morphisms from L to $\mathbb{P}(1)$, hence depending on the choice of **C**. The convergence structure on $\mathrm{pt}L$ is given by

$$\text{(Conv on pt)} \lim_{\mathrm{pt}L}\mathscr{F} = \left(\lim_{L}\mathscr{F}^{\circ}\right)^{\bullet}, \tag{7}$$

where $\ell^{\bullet} = \{\varphi \in \mathrm{pt}L\colon \varphi(\ell) = \{1\}\}$ and $\mathscr{F}^{\circ} = \{\ell \in L\colon \ell^{\bullet} \in \mathscr{F}\}$. Finally, if $\varphi \in \mathbf{C}^{\mathbf{Conv}}(L, L')$, then $\mathrm{pt}\varphi\colon \mathrm{pt}L' \longrightarrow \mathrm{pt}L$ is defined by $\mathrm{pt}(\varphi)(f) = f \circ \varphi$.

In the same work [2], adjunctions between **Conv**, $\mathbf{C}^{\mathbf{Conv}}$, and some of their important subcategories are proved, and it is also shown that, if **C** is the category of coframes, $\mathbf{C}^{\mathbf{Conv}}$ is a topological category. Though the point-free analog of the category of topological spaces in $\mathbf{C}^{\mathbf{Conv}}$ is not related in a straightforward way with the classical approach to point-free topology [2], ([2], section 8.5) shows how the latter can be recovered, realizing the category of locales as a reflective subcategory of the opposite of the category of strong topological coframes in the $\mathbf{C}^{\mathbf{Conv}}$ context. This new framework already proved further versatility, as the category of convergence approach spaces in the sense of [3] can also be faithfully represented in $\mathbf{C}^{\mathbf{Conv}}$ [4].

In the present work, we explore the basic concepts of subspace, compactness, closedness, and Hausdorffness in the setting of $\mathbf{C}^{\mathbf{Conv}}$. More specifically, we characterize the extremal monomorphisms of $(\mathbf{C}^{\mathbf{Conv}})^{\mathrm{op}}$, that is, the extremal epimorphism of $\mathbf{C}^{\mathbf{Conv}}$, as the **C**-extremal epimorphism that are also final in $\mathbf{C}^{\mathbf{Conv}}$. Though this provides an adequate notion of subobject, a stricter notion is necessary to capture exactly those subobjects of the convergence lattice $(\mathbb{P}X, \lim_{\xi})$ coming from a convergence space (X, ξ) that represent a subspace of (X, ξ). We call such extremal epis *standard*. They essentially depend on a single element of the convergence lattice, like picking a subspace of X depends on picking an element of $\mathbb{P}X$.

In the context of convergence spaces, the notions of closedness and compactness have been generalized to families of subsets rather than single subsets, e.g., [1, 5, 6] and references therein, to the effect that a subset has the property if and only if its principal filter does. Similarly, in the point-free context, we consider compactness and closedness for filters, and the case of a principal filter provides notions for standard extremal epis. We also consider a second direction of generalization, defining compactness for morphisms, though few results are obtained in this direction. This avenue may be explored further in future work.

Throughout this work, we are going to use definitions and notations from [2] for convergence lattices, from [7, 8] for categorical notions, and from [9, 10] for lattice theory and locales. Moreover, though we work with categories **C** of lattices, some notions involve infinite suprema or infima. Whenever they appear, the existence of such infima or suprema is implicitly assumed, so that the notion is most naturally considered within convergence lattices that are complete lattices, though the morphisms remain those of $\mathbf{C}^{\mathbf{Conv}}$.

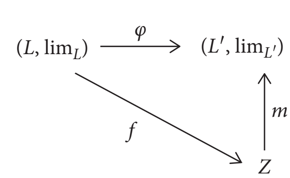
2. Subobjects in Categories of Convergence Lattices

As pointed out in ([10], III. 1) and ([8], 1.2.2.6 2), the extremal monomorphisms are the categorical candidates to represent the *subobjects* of a topological category. Because we are working in $(\mathbf{C}^{\mathbf{Conv}})^{\mathrm{op}}$, the extremal monomorphisms there are the extremal epimorphisms in $\mathbf{C}^{\mathbf{Conv}}$. Recall that an epimorphism e is *extremal* if whenever $e = m \circ f$ for a morphism f and a monomorphism m, m is an isomorphism.

It turns out that these extremal epimorphisms have the expected topological behavior of being final.

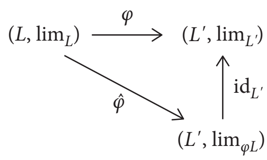
Theorem 1. *Let* **C** *be a category of lattices. The extremal epimorphisms of* $\mathbf{C}^{\mathbf{Conv}}$ *are exactly extremal epimorphisms of* **C** *that are final in* $\mathbf{C}^{\mathbf{Conv}}$.

Proof. Assume that $\varphi\colon (L, \lim_{L}) \longrightarrow (L', \lim_{L'})$ is a $\mathbf{C}^{\mathbf{Conv}}$-extremal epimorphism and let $m \circ f$ be a **C**-factorization of φ, where m is a **C**-monomorphism.

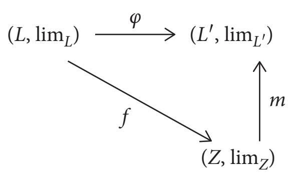


Note that, even if **C** is not a category of coframes, we can always construct the final structures for sources with only one morphism ([2], Corollary 3.3). Here we can give Z the final structure $\lim_{Z}\mathscr{F} = f(\lim_{L} f^{-1}(\mathscr{F}))$. As f is final and $\varphi = m \circ f$ is a morphism, we conclude that m is a $\mathbf{C}^{\mathbf{Conv}}$-morphism. Therefore, m is a $\mathbf{C}^{\mathbf{Conv}}$-isomorphism because φ is an extremal epimorphism in $\mathbf{C}^{\mathbf{Conv}}$. In particular, m is an isomorphism of $\mathbf{C}$. Hence, φ is an extremal epimorphism of $\mathbf{C}$.

Moreover, $\varphi\colon (L, \lim_L) \longrightarrow (L', \lim_{L'})$ is final. To see this, consider the next commutative triangle



Assume conversely that $\varphi\colon (L, \lim_L) \longrightarrow (L', \lim_{L'})$ is an extremal epimorphism of $\mathbf{C}$ that is final in $\mathbf{C}^{\mathbf{Conv}}$. Let $m \circ f$ be a $\mathbf{C}^{\mathbf{Conv}}$ factorization of φ with m a $\mathbf{C}^{\mathbf{Conv}}$-monomorphism:



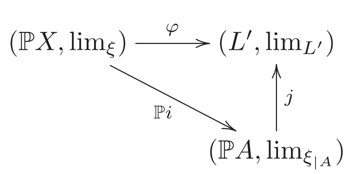
Then m is in particular a $\mathbf{C}$-monomorphism, hence a $\mathbf{C}$-isomorphism, which means that $f = m^{-1}\varphi$ as maps, and therefore, because φ is final in $\mathbf{C}^{\mathbf{Conv}}$, we conclude that m^{-1} is a $\mathbf{C}^{\mathbf{Conv}}$ morphism. □

Remark 1. Note that when $\mathbf{C}$ is the category of frames, the extremal epimorphisms of $\mathbf{C}$ are the onto $\mathbf{C}$-morphisms ([10], Proposition 1.1.3). This is also true for the category of coframes.

Note that with this notion of subobject, there may be more subobjects of a convergence space (X, ξ) seen as a convergence lattice $(\mathbb{P}X, \lim_\xi)$ than subspaces of (X, ξ).

Proposition 1. *Let (X, ξ) be a convergence space. Given an extremal epimorphism $\varphi\colon (\mathbb{P}X, lim_\xi) \longrightarrow L'$ of $\mathbf{C}^{\mathbf{Conv}}$, the following are equivalent:*

(1) There is $A \subset X$ with inclusion map $i\colon A \longrightarrow X$ such that



commutes, where $j\colon (\mathbb{P}A, lim_{\xi_{|A}}) \longrightarrow L'$ is an isomorphism (equivalently a monomorphism) of $\mathbf{C}^{\mathbf{Conv}}$.

(2) $\ell_\varphi \coloneqq \wedge\varphi^{-1}(\top_{L'})$ satisfies $\varphi(\ell_\varphi) = \top_{L'}$ and φ is injective on $\downarrow\ell_\varphi$.

Proof. (1)⟹(2): note that in (1) j is an isomorphism whenever it is a monomorphism because φ is an extremal epimorphism. Assuming (1),

$$\varphi^{-1}(\top_{L'}) = (\mathbb{P}i)^{-1} \circ j^{-1}(\top_{L'})$$
$$\&9; = (\mathbb{P}i)^{-1}(A) = \{B \subset X\colon B \subset A\}, \tag{8}$$

because $j^{-1}(\top_{L'}) = \{\top_{\mathbb{P}A}\}$ as j is an isomorphism. Hence, $\wedge\varphi^{-1}(\top_{L'}) = A$ and $\varphi(A) = j(\mathbb{P}i(A)) = j(A) = \top_{L'}$ and moreover, if $B \neq C \in \downarrow\ell_\varphi$, that is, $B \neq C \in \mathbb{P}A$, then $\mathbb{P}i(B) = B \cap A = B$ and $\mathbb{P}i(C) = C \cap A = C$ are different elements of $\mathbb{P}A$ and thus their images under the isomorphism j are different elements of L'. Hence, $\varphi(B) \neq \varphi(C)$.

(2)⟹(1): let $A = \wedge\varphi^{-1}(\top_{L'})$. Consider the map $j\colon (\mathbb{P}A, \lim_{\xi_{|A}}) \longrightarrow L'$ defined by $j(B) = \varphi(B)$ for every $B \in \mathbb{P}A \subset \mathbb{P}X$. As $\varphi(A) = \top_{L'}$ and φ is a morphism of $\mathbf{C}$, so is j. Moreover, φ is injective on $\downarrow A$ so that j is a monomorphism of $\mathbf{C}$ and for every $D \in \mathbb{P}X$,

$$\begin{aligned} \varphi(D) &= \varphi(D \cap X) \\ &= \varphi(D)\wedge\varphi(X) = \varphi(D) \wedge \varphi(A) \\ &= \varphi(D \cap A) = (j \circ \mathbb{P}i)(D). \end{aligned} \tag{9}$$

Moreover, j is continuous because $\mathbb{P}i$ is final and $\varphi = j \circ \mathbb{P}i$ is continuous.

Recall that an epimorphism $e\colon L \longrightarrow L'$ is *split* if it has a right-inverse, that is, if there is a morphism $s\colon L' \longrightarrow L$ such that $e \circ s = \mathrm{id}_{L'}$. Split epimorphisms are extremal. Note that when A is a subspace of (X, ξ) as in Proposition 1, then $\mathbb{P}i\colon (\mathbb{P}X, \lim_\xi) \longrightarrow (\mathbb{P}A, \lim_{\xi_{|A}})$ is also a split epimorphism, taking $s = e \circ j^{-1}$, where $e\colon \mathbb{P}A \longrightarrow \mathbb{P}X$ is $e(B) = B$. However, there are split epimorphisms that do not correspond to subspaces (e.g., Example 1 below). □

Definition 2. An extremal epimorphism $\varphi\colon L \longrightarrow L'$ of $\mathbf{C}^{\mathbf{Conv}}$is *standard* if $\ell_\varphi \coloneqq \wedge\varphi^{-1}(\top_{L'})$ satisfies $\varphi(\ell_\varphi) = \top_{L'}$ and φ is injective on $\downarrow\ell_\varphi$.

Proposition 2. *An extremal epimorphism $\varphi\colon L \longrightarrow L'$ of $\mathbf{C}^{\mathbf{Conv}}$ is standard if and only if L' is isomorphic to a sublattice of L of the form $\downarrow\ell$ for some $\ell \in L$ with the limit given by*

$$\lim_{\downarrow\ell}\mathcal{F} = \ell \wedge \lim_L\uparrow\mathcal{F}, \tag{10}$$

for every $\mathcal{F} \in \mathbb{F}(\downarrow\ell)$.

Proof. For every $\ell \in L$, the map $\varphi\colon L \longrightarrow \downarrow\ell$ defined by $\varphi(m) = m\wedge\ell$ is a standard extremal epimorphism if $\lim_{\downarrow\ell}$ is given by (10). Indeed, φ is onto and is the identity on $\downarrow\ell$, so that $\ell = \ell_\varphi$, $\varphi(\ell) = \ell = \top_{\downarrow\ell}$ and φ is injective on $\downarrow\ell$.

Conversely, given a standard extremal epimorphism $\varphi\colon L \longrightarrow L'$ of $\mathbf{C}^{\mathbf{Conv}}$, we can show that L' is isomorphic to $\downarrow\ell_\varphi$ with the corresponding limit given by (10).

However, not all extremal epimorphisms are standard as shown in the following. □

Proposition 3. *Given a convergence lattice (L, lim_L) with at least two elements and $\varphi \in ptL$, then $\varphi\colon L \longrightarrow \mathbb{P}(1)$ is a split (hence extremal) epimorphism of $\mathbf{C}^{\mathbf{Conv}}$.*

Proof. Let $s\colon \mathbb{P}(1)=\{\top,\bot\}\longrightarrow L$ be defined by $s(\bot)=\bot_L$ and $s(\top)=\top_L$. This is continuous because $\lim_{\mathbb{P}(1)}\{\top\}=\top$ so that for every proper filter $\mathcal{F}\in\mathbb{F}L$, $\lim_L\mathcal{F}\le s(\lim_{\mathbb{P}(1)}s^{-1}(\mathcal{F}))=\top_L$ because $s^{-1}(\mathcal{F})=\{\top\}$. Hence, $\varphi\circ s=\mathrm{id}_{\mathbb{P}(1)}$. □

Example 1. (a split epimorphism of $\mathbf{Lat}^{\mathrm{conv}}$ that is not standard). Note that if $\mathbf{C}$ is the category $\mathbf{Lat}$ of lattices and (X,ξ) is a convergence space, then a point $\varphi\colon\mathbb{P}X\longrightarrow\mathbb{P}(1)$ of $(\mathbb{P}X,\lim_\xi)$ can be identified with the filter $\mathcal{F}=\varphi^{-1}(\top)$, which is prime because φ is a lattice morphism and satisfies $\lim_\xi\mathcal{F}\in\mathcal{F}$ by the continuity condition. As prime filters of $\mathbb{P}X$ are ultrafilters, points of $(\mathbb{P}X,\lim_\xi)$ are ultrafilters $\mathcal{U}$ on X satisfying $\lim_\xi\mathcal{U}\in\mathcal{U}$. There may be such nonprincipal ultrafilters on a convergence space. For instance, on an infinite Noetherian topological space (X,ξ), all free ultrafilters are points of $(\mathbb{P}X,\lim_\xi)$ (see [11] for details), which are nonstandard extremal epimorphisms, for $\wedge\varphi^{-1}(\top)=\cap\mathcal{U}=\varnothing$ and $\varphi(\varnothing)=\bot$.

Let us now examine the action of the functor pt on extremal epimorphism of $\mathbf{C}^{\mathbf{Conv}}$.

Theorem 2. *If $\varphi\colon L\longrightarrow L'$ is a final epimorphism of $\mathbf{C}^{\mathbf{Conv}}$ (in particular an extremal epimorphism), then $pt\varphi\colon ptL'\longrightarrow ptL$ is one-to-one and initial; that is, ptL' is (homeomorphic to) a subspace of ptL.*

Proof. ptφ is one-to-one. Indeed, if $f,g\colon L\longrightarrow\mathbb{P}(1)$ are two points of L' and $\mathrm{pt}\varphi(f)=\mathrm{pt}\varphi(g)$, that is, $f\circ\varphi=g\circ\varphi$, then $f=g$ because φ is an epimorphism. If φ is final, then ptφ is initial. Indeed, if $f\colon(X,\xi)\longrightarrow\mathrm{pt}L'$ with $\mathrm{pt}\varphi\circ f\colon(X,\xi)\longrightarrow\mathrm{pt}L$ continuous, then $\mathrm{pt}\varphi(f(x))\in\lim_{\mathrm{pt}L}\mathrm{pt}\varphi\circ f[\mathcal{F}]$ whenever $x\in\lim_\xi\mathcal{F}$, that is, $\mathrm{pt}\varphi(f(x))(\lim_L(\mathrm{pt}\varphi(f[\mathcal{F}]))^\circ)=\top$ in $\mathbb{P}(1)$, equivalently, $f(x)\circ\varphi(\lim_L(\mathrm{pt}\varphi(f[\mathcal{F}]))^\circ)=\top$. We want to show that $f(x)\in\lim_{\mathrm{pt}L'}f[\mathcal{F}]$, that is, $f(x)(\lim_{L'}(f[\mathcal{F}])^\circ)=\top$ in $\mathbb{P}(1)$. Since φ is final, $\lim_{L'}(f[\mathcal{F}])^\circ=\varphi(\lim_L\varphi^{-1}((f[\mathcal{F}])^\circ))$, so that $f(x)(\lim_{L'}(f[\mathcal{F}])^\circ)=f(x)\circ\varphi(\lim_L\varphi^{-1}((f[\mathcal{F}])^\circ))$. Hence, it is enough to show that $(\mathrm{pt}\varphi(f[\mathcal{F}]))^\circ=\varphi^{-1}((f[\mathcal{F}])^\circ)$, which follows from

$$\begin{aligned}
&\ell\in\varphi^{-1}(f[\mathcal{F}]^\circ)\Longleftrightarrow\varphi(\ell)\in(f[\mathcal{F}])^\circ\\
&\Longleftrightarrow(\varphi(\ell))^\bullet\in f[\mathcal{F}]\\
&\Longleftrightarrow\exists F\in\mathcal{F}\,(f(F)\subset(\varphi(\ell)))^\bullet)\\
&\Longleftrightarrow\exists F\in\mathcal{F},\quad\forall x\in F\,(f(x)(\varphi(\ell))=\top)\\
&\Longleftrightarrow\exists F\in\mathcal{F},\quad\forall x\in F\,(\mathrm{pt}\varphi(f(x))(\ell)=\top)\\
&\Longleftrightarrow\ell^\bullet\in\mathrm{pt}\varphi(f[\mathcal{F}])\\
&\Longleftrightarrow\ell\in(\mathrm{pt}\varphi(f[\mathcal{F}]))^\circ.
\end{aligned}\tag{11}$$

□

3. Variants of Compactness in Convergence Lattices

We shall define notions of *adherence* (the adherence defined here is the natural generalization to subsets of L of what is called raw adherence and denoted by adh^0 in [2]), *compactness* and *closedness,* in the point-free convergence setting. To this end, we say that two subsets A and B of a lattice L *mesh*, in symbols $A\#B$, if $a\wedge b>\bot$ for every $a\in A$ and every $b\in B$. We also write $A^\#=\{\ell\in L\colon\{\ell\}\#A\}$. Note that if two filters $\mathcal{F}$ and $\mathcal{G}$ on L mesh, then there is the smallest filter that contains them both.

Definition 3. Let $(L,\lim_L)$ be a convergence lattice and $\mathcal{F}\in\mathbb{F}L$, we define the adherence of $\mathcal{F}$ as

$$\mathrm{adh}_L\mathcal{F}=\bigvee_{\substack{\mathcal{G}\in\mathbb{F}L\\ \mathcal{G}\#\mathcal{F}}}\lim\mathcal{G}.\tag{12}$$

Note that $\vee_{\substack{\mathcal{G}\in\mathbb{F}L\\ \mathcal{G}\supset\mathcal{F}}}\lim\mathcal{G}\le\mathrm{adh}\mathcal{F}$ because $\{\mathcal{G}\in\mathbb{F}L\colon\mathcal{G}\supset\mathcal{F}\}\subset\{\mathcal{G}\in\mathbb{F}L\colon\mathcal{G}\#\mathcal{F}\}$, and if $\mathcal{G}\#\mathcal{F}$ there is $\mathcal{H}=\uparrow\{g\wedge f\colon g\in\mathcal{G},f\in\mathcal{F}\}\supset\mathcal{F}$ with $\lim\mathcal{H}\ge\lim\mathcal{G}$, so that

$$\mathrm{adh}\mathcal{F}=\bigvee_{\substack{\mathcal{G}\in\mathbb{F}L\\ \mathcal{G}\supset\mathcal{F}}}\lim\mathcal{G}.\tag{13}$$

Remark 2. As is well known, under the Axiom of Choice, denoting $\mathbb{U}L$ the set of maximal filters on L, the set $\mathbb{U}(\mathcal{F})=\{\mathcal{U}\in\mathbb{U}L\colon\mathcal{U}\supset\mathcal{F}\}$ is always nonempty and thus the adherence only depends on maximal filters via:

$$\mathrm{adh}\mathcal{F}=\bigvee_{\mathcal{U}\in\mathbb{U}\mathcal{F}}\lim\mathcal{U}.\tag{14}$$

Lemma 1. *If $f\colon M\longrightarrow L$ is a $\mathbf{C}^{\mathbf{Conv}}$-morphism, then*

$$\mathrm{adh}_L\mathcal{F}\le f\left(\mathrm{adh}_M f^{-1}(\mathcal{F})\right),\tag{15}$$

for every $\mathcal{F}\in\mathbb{F}L$.

Proof. Since $\lim_L\mathcal{G}\le f(\lim_M f^{-1}(\mathcal{G}))$ for every $\mathcal{G}\in\mathbb{F}L$ with $\mathcal{G}\supset\mathcal{F}$,

$$\begin{aligned}
\mathrm{adh}_L\mathcal{F}&=\bigvee_{\mathcal{G}\supset\mathcal{F}}\lim_L\mathcal{G}\le\bigvee_{\mathcal{G}\supset\mathcal{F}}f\left(\lim_M f^{-1}(\mathcal{G})\right)\\
&\le f\left(\bigvee_{\mathcal{G}\supset\mathcal{F}}\lim_M f^{-1}(\mathcal{G})\right)\\
&\le f\left(\bigvee_{\mathcal{H}\supset f^{-1}(\mathcal{F})}\lim_M\mathcal{H}\right)=f\left(\mathrm{adh}_M f^{-1}(\mathcal{F})\right).
\end{aligned}\tag{16}$$

Abstracting from the case of $L=(\mathbb{P}X,\lim_\xi)$, we say that $\ell\in L$ is *compact* if every $\mathcal{F}\in\mathbb{F}L$ with $\ell\in\mathcal{F}$ satisfies $\mathrm{adh}_L\mathcal{F}\wedge\ell>\bot_L$. We shall generalize the notion of compactness in several directions. On one hand, compactness of an element ℓ should coincide with compactness of the corresponding standard extremal epimorphism $-\wedge\ell\colon L\longrightarrow\downarrow\ell$ for an appropriate notion of compactness of $\varphi\colon L\longrightarrow L'$. On the other hand, compactness has been extended to families of subsets in a very useful fashion in the context of **Conv** (See, e.g., [1, 5, 6, 12] and references therein) and this can be extended to the point-free setting. We start with the latter, as the corresponding notions will be useful in analyzing the former concept. □

3.1. Compactness for Filters

Definition 4. Let $(L, \lim_L)$ be a convergence lattice and $\mathbb{D} \subset \mathbb{F}L$. A filter $\mathcal{F} \in \mathbb{F}L$ is $\mathbb{D}$-*compact at* $A \subset L$ if

$$\forall \mathcal{D} \in \mathbb{D}, \quad \mathcal{D} \# \mathcal{F} \Longrightarrow \mathrm{adh}\mathcal{D} \in A^{\#}. \tag{17}$$

When $\mathbb{D} = \mathbb{F}L$, we omit the prefix $\mathbb{D}$. When $A = \mathcal{F}$ we omit "at $\mathcal{F}$." Hence, $\mathcal{F}$ is *compact* if

$$\forall \mathcal{H} \in \mathbb{F}L, \quad \mathcal{H} \# \mathcal{F} \Longrightarrow \mathrm{adh}\mathcal{H} \in \mathcal{F}^{\#}. \tag{18}$$

In the case where $A = \{\top\}$, we omit "at A" and add the suffix "oid." So $\mathcal{F}$ is $\mathbb{D}$-*compactoid* means that

$$\forall \mathcal{H} \in \mathbb{D}, \mathcal{H} \# \mathcal{F} \Longrightarrow \mathrm{adh}\mathcal{H} > \bot. \tag{19}$$

A related notion is that of near $\mathbb{D}$-compactness as introduced (in the case $L = \mathbb{P}X$) in [5]. A filter $\mathcal{F}$ on L is *nearly* $\mathbb{D}$-*compact at* $A \subset L$ if

$$\forall \mathcal{D} \in \mathbb{D}, \quad \mathcal{D} \supset \mathcal{F} \Longrightarrow \mathrm{adh}\mathcal{D} \in A^{\#}. \tag{20}$$

Note that with these definitions, an element $\ell \in L$ is compact if and only if its principal filter $\uparrow \ell$ is a compact filter. Hence, relativizations of compactness with respect to a class of filters $\mathbb{D}$ (yielding notions of countable compactness, Lindelöfness, etc.) or with respect to a subset A of L can be applied to a single element, identifying it with its principal filter. As a result, we say that a convergence lattice L is $\mathbb{D}$-compact if $\{\top\}$ is $\mathbb{D}$-*compactoid*, that is, $\mathrm{adh}\mathcal{D} > \bot$ for every proper filter $\mathcal{D} \in \mathbb{D}$.

Remark 3. Note that when $\mathbb{D} = \mathbb{F}$ is the class of all filters, then $\mathbb{D}$-compactness and near $\mathbb{D}$-compactness are equivalent. Indeed, if $\mathcal{F}$ is nearly compact (at A) and $\mathcal{H} \# \mathcal{F}$ there is a filter $\mathcal{G}$ finer than $\mathcal{F}$ and $\mathcal{H}$. By near compactness, $\mathrm{adh}\mathcal{G} \in A^{\#}$. Since $\mathcal{G} \supset \mathcal{H}$, $\mathrm{adh}\mathcal{G} \leq \mathrm{adh}\mathcal{H}$ and thus $\mathrm{adh}\mathcal{H} \in A^{\#}$.

It is clear that the image under the $\mathbb{P}$ functor of a compact convergence space is $\mathbf{C}^{\mathbf{Conv}}$-compact (that is, $\mathcal{F} = \{\top\}$ is compact).

Definition 5. Let $(L, \lim_L)$ be a convergence lattice. A filter $\mathcal{F} \in \mathbb{F}L$ is *closed* if

$$\mathrm{adh}\mathcal{F} \leq \wedge \mathcal{F}. \tag{21}$$

Remark 4. Note that this is a generalization to filters of the notion of *closed element* of a convergence lattice $(L, \lim)$ as defined in [4, 13] (which is different from the notion of closed element introduced in [2], where the present notion is called quasi-closed), where $\ell \in L$ is *closed* if

$$\ell \in \mathcal{F} \Longrightarrow \lim \mathcal{F} \leq \ell, \tag{22}$$

for every $\mathcal{F} \in \mathbb{F}L$. Indeed, $\ell \in L$ is a closed element if and only if its principal filter $\uparrow \ell$ is closed in the sense of Definition 5.

Hence, a subspace $(A, \xi_{|A})$ of a convergence space (X, ξ) is closed if and only if A is a closed element of $(\mathbb{P}X, \lim_\xi)$ if and only if the principal filter of A in the convergence lattice $(\mathbb{P}X, \lim_\xi)$ is closed in the sense of the previous definition.

Theorem 3. *Let (L, lim_L) be a convergence lattice and $\mathbb{D} \subset \mathbb{F}L$, let $\mathcal{G} \supset \mathcal{F}$ be filters on L, where $\mathcal{F}$ is $\mathbb{D}$-compactoid and $\mathcal{G}$ is closed. Then, $\mathcal{G}$ is nearly $\mathbb{D}$-compact.*

Proof. Let $\mathcal{D} \in \mathbb{D}$ with $\mathcal{D} \supset \mathcal{G}$. Note that $\mathrm{adh}\mathcal{G} \geq \mathrm{adh}\mathcal{D}$. Then, $\mathcal{D} \# \mathcal{F}$ because $\mathcal{G} \supset \mathcal{F}$, and $\mathcal{F}$ is $\mathbb{D}$-compactoid, hence,

$$\mathrm{adh}\mathcal{G} \geq \mathrm{adh}\mathcal{D} > \bot. \tag{23}$$

As $\mathcal{G}$ is closed, $\bot < \mathrm{adh}\mathcal{D} \leq \wedge \mathcal{G}$ so that

$$\mathrm{adh}\mathcal{D} \wedge g = \mathrm{adh}\mathcal{D} > \bot, \tag{24}$$

for every $g \in \mathcal{G}$ and the conclusion follows.

In the case where $\mathbb{D} = \mathbb{F}L$, we obtain, in view of Remark 3: □

Corollary 1. *If $\mathcal{F}$ is compactoid, $\mathcal{G} \supset \mathcal{F}$, and $\mathcal{G}$ is closed then $\mathcal{G}$ is compact.*

In particular, in the case $\mathcal{F} = \{\top\}$ we have the following.

Corollary 2. *If $(L, \lim_L)$ is a compact convergence lattice then every closed filter on L is compact.*

In particular, applying this fact to principal filters, the fact that closed subspaces of a compact convergence space are closed extends to the point-free setting: every closed element of a compact convergence lattice is compact.

The point-free version of the classical fact that a continuous image of a compact set is compact will not extend straightforwardly to all morphisms in an arbitrary category of convergence lattices, but we can give a version in convergence frames.

To this end, note that a class of filters $\mathbb{D}$ consists of a set $\mathbb{D}L \subset \mathbb{F}L$ of filters on each lattice L.

Definition 6. We say that a class $\mathbb{D}$ of filters is *admissible* if given a lattice morphism $f: L \longrightarrow L'$, $f(\mathcal{D}) \in \mathbb{D}L'$ whenever $\mathcal{D} \in \mathbb{D}L$ and $f^{-1}(\mathcal{H}) \in \mathbb{D}L$ whenever $\mathcal{H} \in \mathbb{D}L'$.

Note that, in particular, if $\mathbb{D}$ is admissible and $L' \subset L$, then every $\mathcal{G} \in \mathbb{D}L'$ generates a filter of $\mathbb{D}L$.

Recall that frames are pseudocomplemented; that is, every $\ell \in L$ has a *pseudocomplement* $\ell^* := \vee\{m \in L: \ m \wedge \ell = \bot\}$, which satisfies $\ell \wedge \ell^* = \bot$. Note that if ℓ has a pseudocomplement, then $\ell \in \mathcal{F}^{\#}$ if and only if $\ell^* \notin \mathcal{F}$. In general, a lattice or frame morphism does not need to preserve pseudocomplements, though $\varphi(\ell^*) \leq (\varphi(\ell))^*$ whenever ℓ and $\varphi(\ell)$ have pseudcomplements. On the other hand, a morphism of Heyting algebras preserves pseudocomplements.

Lemma 2. *Let L and L' be lattices and $\varphi: L \longrightarrow L'$ be a lattice morphism. Let $\mathcal{F} \in \mathbb{F}L'$ and $\ell \in L$. Then,*

$$\varphi(\ell) \in \mathcal{F}^{\#} \Longrightarrow \ell \in \left(\varphi^{-1}(\mathcal{F})\right)^{\#}. \tag{25}$$

If ℓ is pseudocomplemented and φ preserves pseudocomplements, then the converse is true.

Proof. If $\varphi(\ell)\wedge m \neq \perp$ then $\ell\wedge a \neq \perp$ for every $a \in \varphi^{-1}(m)$, for $\varphi(\ell\wedge a) = \varphi(\ell)\wedge\varphi(a) = \varphi(\ell)\wedge m \neq \perp$ and $\varphi(\perp_L) = \perp_{L'}$. In particular, if $\varphi(\ell) \in \mathscr{F}^{\#}$ then $\ell \in (\varphi^{-1}(\mathscr{F}))^{\#}$.

Conversely, if $\ell \in \varphi^{-1}(\mathscr{F})^{\#}$ then $\ell^* \notin \varphi^{-1}(\mathscr{F})$, that is, $\varphi(\ell^*) \notin \mathscr{F}$. If φ respects pseudocomplements $\varphi(\ell^*) = \varphi(\ell)^* \notin \mathscr{F}$ equivalently, $\varphi(\ell) \in \mathscr{F}^{\#}$. □

Theorem 4. *Let $g\colon (L,\lim_L) \longrightarrow (H,\lim_H)$ be a* $\mathbf{Frm}^{\mathbf{Conv}}$ *morphism that preserves pseudocomplements. If $\mathscr{F} \in \mathbb{F}(H)$ is compact, then $g^{-1}(\mathscr{F})$ is compact.*

Note that in the classical case where $L = \mathbb{P}Y$, $H = \mathbb{P}X$, and $g = f^{-1}$, where $f\colon X \longrightarrow Y$, then L and H are Boolean algebras, hence complemented, and g respects complements, hence pseudocomplements.

Proof. Let $\ell \in L$ such that $g(\ell) \in \mathscr{F}$. If $\mathscr{H}\# g^{-1}[\mathscr{F}]$, then $g[\mathscr{H}]\#\mathscr{F}$ by Lemma 2. Thus, $\mathrm{adh} g[\mathscr{H}]\#\mathscr{F}$ because $\mathscr{F}$ is compact. In particular, $\mathrm{adh} g[\mathscr{H}]\wedge g(\ell) > \perp$, that is,

$$\left(\bigvee_{\mathscr{W}\# g[\mathscr{H}]} \lim\mathscr{W}\right)\wedge g(\ell) \neq \perp. \tag{26}$$

Since L is a frame, this implies that there is $\mathscr{W}\# g[\mathscr{H}]$ with $\lim\mathscr{W}\wedge g(\ell) \neq \perp$.

Now, $\lim\mathscr{W} \leq g(\lim g^{-1}(\mathscr{W}))$ by (ptfree continuity). Thus,

$$\perp \neq \lim\mathscr{W}\wedge g(\ell) \leq g\left(\lim g^{-1}(\mathscr{W})\wedge\ell\right), \tag{27}$$

so that $\lim g^{-1}(\mathscr{W})\wedge\ell \neq \perp$, because g is a lattice morphism, hence sends $\perp$ to $\perp$.

Recall that a convergence space is *Hausdorff* if $\lim\mathscr{F}$ consists of at most one point for every filter $\mathscr{F}$ on the space. The following definition gives an abstraction to the point-free setting. □

Definition 7. We say a convergence lattice $(L,\lim_L)$ is *Hausdorff* if for every $\mathscr{F} \in \mathbb{F}(L)$ either $\lim\mathscr{F} = \perp$ or $\lim\mathscr{F}$ is an atom of L; that is, $\lim\mathscr{F}$ is minimal in $L\backslash\{\perp\}$.

It is straightforward that the image under $\mathbb{P}$ of any Hausdorff convergence space is a Hausdorff convergence lattice.

Theorem 5. *Let (L, lim_L) be a Hausdorff convergence frame and let $\mathscr{F}$ be a compact filter on L. Then, $\mathscr{F}$ is closed.*

Proof. We have to prove that

$$\bigvee_{\mathscr{F}\subset\mathscr{G}} \lim\mathscr{G} = \mathrm{adh}\mathscr{F} \leq \wedge\mathscr{F}. \tag{28}$$

It will be enough to prove that

$$\lim\mathscr{G} \leq f, \tag{29}$$

for any $\mathscr{G}\supset\mathscr{F}$ and any $f \in \mathscr{F}$. For any such $\mathscr{G}$ and f, $\mathrm{adh}\mathscr{G}\#\mathscr{F}$ by compactness of $\mathscr{F}$. Hence,

$$\bigvee_{\mathscr{H}\supset\mathscr{G}} (\lim\mathscr{H})\wedge f > \perp, \tag{30}$$

equivalently,

$$\bigvee_{\mathscr{H}\supset\mathscr{G}} (\lim\mathscr{H}\wedge f) > \perp, \tag{31}$$

because L is a frame. Hence,

$$\forall f \in \mathscr{F}\,\exists\mathscr{H}_f\supset\mathscr{G}\left[\lim\mathscr{H}_f\wedge f > \perp\right]. \tag{32}$$

Because L is Hausdorff, it follows that

$$\lim\mathscr{G} \leq \lim\mathscr{H}_f = \lim\mathscr{H}_f\wedge f \leq f. \tag{33}$$

□

Remark 5. The same result can be obtained without using the assumption that L be a frame, using maximal filters as in Remark 2, hence using the Axiom of Choice.

3.2. Pseudotopological Convergence Lattices and Minimality of Compact Hausdorff Structures. We call a convergence lattice *pseudotopological*, if for every $\mathscr{F} \in \mathbb{F}(L)$,

$$\lim\mathscr{F} \geq \bigwedge_{\mathscr{H}\#\mathscr{F}} \mathrm{adh}\mathscr{H}. \tag{34}$$

Note that the reverse inequality in (34) is always true (of course, (34) implicitly assumes that the infimum involved exists. Hence, pseudotopological convergence lattices are most naturally considered in the context of complete lattices).

Given a convergence lattice $(L,\lim_L)$, we define its pseudotopological modification $\lim_L^S\colon \mathbb{F}(L) \longrightarrow L$ by

$$\lim_L^S\mathscr{F} \coloneqq \bigwedge_{\mathscr{H}\#\mathscr{F}} \mathrm{adh}_L\mathscr{H}. \tag{35}$$

In view of Remark 2, $\lim_L^S\mathscr{F} = \wedge_{\mathscr{U}\in\mathbb{U}(\mathscr{F})}\lim_L\mathscr{U}$, under the ultrafilter principle, in particular under the Axiom of Choice.

Lemma 3. *Let $(L,\lim_L)$ be a convergence lattice. Under the Axiom of Choice, $(L,\lim_L)$ is Hausdorff if and only if $(L,\lim_L^S)$ is Hausdorff.*

Proof. Since $\lim_L\mathscr{F} \leq \lim_L^S\mathscr{F}$, $\lim_L\mathscr{F}$ is either $\perp$ or an atom if this is the case for $\lim_L^S\mathscr{F}$. Conversely, assume $(L,\lim_L^S)$ is not Hausdorff, that is, there is $\mathscr{F} \in \mathbb{F}L$ and $\ell \in L$ with $\perp < \ell < \lim_L^S\mathscr{F}$. Under the Axiom of Choice, this means that $\lim_L\mathscr{U} > \ell > \perp$ for every $\mathscr{U} \in \mathbb{U}(\mathscr{F})$ and thus $(L,\lim_L)$ is not Hausdorff. □

Remark 6. Note that the corresponding **Conv** statement that a convergence ξ is Hausdorff if and only if its pseudotopological modification $S\xi$ is, can be proved without invoking the ultrafilter principle, but we were not able to obtain the general point-free analog in a choice-free manner.

Another classical convergence result ([1], Corollary IX. 2.8), to the effect that a continuous bijection from a compact pseudotopology to a Hausdorff pseudotopology is a homeomorphism, finds a natural point-free generalization.

Theorem 6. *Assume the Axiom of Choice. Let $\varphi\colon (L,\lim_L) \longrightarrow (L',\lim_{L'})$ be a bijective* $\mathbf{C}^{\mathbf{Conv}}$ *morphism,*

$(L, \lim_L)$ *be Hausdorff, and* $(L', \lim_{L'})$ *be compact. Then,* $\varphi\colon (L, \lim_L^S) \longrightarrow (L', \lim_{L'}^S)$ *is an isomorphism.*

Keep in mind that if φ is a bijective morphism of $\mathbf{C}$, then $\mathscr{U}$ is a maximal filter if and only if $\varphi(\mathscr{U})$ is maximal, and φ preserves all infima and suprema.

Proof. We shall prove continuity of φ^{-1}, that is,

$$\lim_L^S \mathscr{F} \leq \varphi^{-1}\left(\lim_{L'}^S \varphi(\mathscr{F})\right), \tag{36}$$

for all $\mathscr{F} \in \mathbb{F}(L)$.

If $\mathscr{U} \in \mathbb{U}(L)$ then $\varphi(\mathscr{U}) \in \mathbb{U}(L')$ and thus $\lim_{L'} \varphi(\mathscr{U}) \neq \bot$ by compactness of L'. Moreover, $\lim_{L'} \varphi(\mathscr{U}) \leq \varphi(\lim_L \varphi^{-1}(\varphi(\mathscr{U})))$ by continuity of φ applied to the filter $\varphi(\mathscr{U})$, which is equivalent to $\varphi^{-1}(\lim_{L'} \varphi(\mathscr{U})) \leq \lim_L \mathscr{U}$ because φ is bijective. Hence, $\bot \neq \varphi^{-1}(\lim_{L'} \varphi(\mathscr{U})) \leq \lim_L \mathscr{U}$. Because $(L, \lim_L)$ is Hausdorff, $\lim_L \mathscr{U}$ is minimal in $L \backslash \{\bot\}$ and therefore, $\varphi^{-1}(\lim_{L'} \varphi(\mathscr{U})) = \lim_L \mathscr{U}$.

Now take $\mathscr{F} \in \mathbb{F}(L)$. Applying the fact that φ is a $\mathbf{C}$-isomorphism and thus φ^{-1} preserves all infima while $\mathbb{U}(\varphi(\mathscr{F})) = \{\varphi(\mathscr{U})\colon \mathscr{U} \in \mathbb{U}(\mathscr{F})\}$, we obtain

$$\begin{aligned}\lim_L^S \mathscr{F} &= \bigwedge_{\mathscr{U} \in \mathbb{U}(\mathscr{F})} \lim_L \mathscr{U} = \bigwedge_{\mathscr{U} \in \mathbb{U}(\mathscr{F})} \varphi^{-1}(\lim_{L'} \varphi(\mathscr{U})) \\ &= \varphi^{-1}\left(\bigwedge_{\mathscr{U} \in \mathbb{U}(\mathscr{F})} \lim_{L'} \varphi(\mathscr{U})\right) \\ &= \varphi^{-1}\left(\bigwedge_{\mathscr{W} \in \mathbb{U}(\varphi(\mathscr{F}))} \lim_{L'} \mathscr{W}\right) = \varphi^{-1}\left(\lim_{L'}^S \varphi(\mathscr{F})\right).\end{aligned} \tag{37}$$

□

3.3. Compact Morphisms. A morphism $\varphi\colon L \longrightarrow L'$ of $\mathbf{C}^{\mathbf{Conv}}$ is *compact* if

$$\varphi\left(\mathrm{adh}_L \varphi^{-1}(\mathscr{F})\right) \neq \bot_{L'}, \tag{38}$$

for every $\mathscr{F} \in \mathbb{F}L'$. This terminology comes from the following.

Proposition 4. *A standard extremal epimorphism* $\varphi\colon L \longrightarrow L'$ *of* $\mathbf{C}^{\mathbf{Conv}}$ *is compact if and only if its defining element* $\ell_\varphi = \wedge \varphi^{-1}(\top_{L'})$ *is a compact element.*

Proof. In view of Proposition 2, we may assume $L' = \downarrow \ell_\varphi$ and $\varphi(m) = m \wedge \ell_\varphi$. Hence, φ is compact if and only if $\ell_\varphi \wedge \mathrm{adh}_L \uparrow \mathscr{F} \neq \bot$ for every $\mathscr{F} \in \mathbb{F}(\downarrow \ell_\varphi)$, where $\uparrow \mathscr{F}$ is the filter $\mathscr{F}$ generates on L. Of course, it is equivalent to ask for $\ell_\varphi \wedge \mathrm{adh}_L \mathscr{G} \neq \bot$ for every $\mathscr{G} \in \mathbb{F}L$ with $\ell_\varphi \in \mathscr{G}$, for $\mathscr{F} = \left\{g \wedge \ell_\varphi\colon g \in \mathscr{G}\right\} \in \mathbb{F}(\downarrow \ell_\varphi)$ with $\uparrow \mathscr{F} = \mathscr{G}$.

The following observation can be seen as an alternative abstract version (compare Theorem 4) of the fact that the continuous image of a compact space is compact (remember that $f\colon M \longrightarrow L$ is an abstraction of a map $\mathbb{P}g\colon \mathbb{P}Y \longrightarrow \mathbb{P}X$ induced by a map $g\colon X \longrightarrow Y$). □

Proposition 5. *If* $f\colon M \longrightarrow L$ *is a* $\mathbf{C}^{\mathbf{Conv}}$*-morphism and* $\varphi\colon L \longrightarrow L'$ *is compact, then* $\varphi \circ f\colon M \longrightarrow L'$ *is compact.*

Proof. Let $\mathscr{F} \in \mathbb{F}L'$. We need to show that

$$\varphi \circ f\left(\mathrm{adh}_M f^{-1}\left(\varphi^{-1}(\mathscr{F})\right)\right) \neq \bot_{L'}, \tag{39}$$

In view of Lemma 1, $\mathrm{adh}_L \varphi^{-1}(\mathscr{F}) \leq f(\mathrm{adh}_M f^{-1}(\varphi^{-1}(\mathscr{F}))))$. Hence, by compactness of φ,

$$\varphi \circ f\left(\mathrm{adh}_M f^{-1}\left(\varphi^{-1}(\mathscr{F})\right) \geq \varphi\left(\mathrm{adh}_L \varphi^{-1}(\mathscr{F})\right)\right) > \bot_{L'}. \tag{40}$$

In particular, in the case of a standard extremal epimorphism φ, we can identify L' with $\downarrow \ell$ for a compact element $\ell \in L$ and φ with $\varphi(m) = m \wedge \ell$. That $\varphi \circ f$ is compact means that

$$f\left(\mathrm{adh}_M f^{-1}(\mathscr{F})\right) \wedge \ell > \bot_L, \tag{41}$$

for every $\mathscr{F} \in \mathbb{F}L$ with $\ell \in \mathscr{F}$. Now if $f^{-1}(\ell) \subset \mathscr{G} \in \mathbb{F}L$, then $\ell \in \mathscr{F} = f(\mathscr{G})$, so that, in view of $\mathscr{G} \subset f^{-1} f(\mathscr{G})$,

$$f(\mathrm{adh}_M \mathscr{G}) \wedge \ell \geq f\left(\mathrm{adh}_M f^{-1} f(\mathscr{G})\right) \wedge \ell > \bot_L, \tag{42}$$

and thus $\mathrm{adh}_M \mathscr{G} \wedge m > \bot_M$ for every $m \in f^{-1}(\ell)$, that is, $\mathrm{adh}_M \mathscr{G} \# f^{-1}(\ell)$. In other words, $f^{-1}(\ell)$ is a compact filter.

There are various remaining problems to consider related to compactness in the context of point-free convergence, most notably an analog of the Tychonoff theorem (which first requires an adequate notion mimicking products) and an analog of the Čech–Stone compactification. We hope to address these issues in a future work. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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