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Research Article

Normal Structure and Some Inequalities of Geometric Parameters in Banach Space X and X^*

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Let X and X^* be a Banach space and its dual, respectively. In this paper, we study the relations between modulus of $W_{1X^*}(\varepsilon)$ and modulus $\zeta_{X^*}(\varepsilon)$ in X^* and normal structure in X, respectively. Among other results, we proved either $W_{1X^*}(\varepsilon) < \varepsilon/2$, for any $0 \le \varepsilon \le 2$, or $\zeta_{X^*}(\varepsilon) < 1 + \varepsilon$, for any $0 \le \varepsilon \le 1$, implies both X and its dual X^* have uniform normal structure.

1. Introduction

Suppose X, $B(X)$, $S(X)$, and X^* be a real Banach space, its unit ball of X, its unit sphere of X, and its dual space of X, respectively. Let $x \in S(X)$, we use $\nabla_x \subseteq S(X^*)$ to denote the set of all norm one supporting functionals at x. Let $x_1, x_2 \in B(X)$, we use $[x_1, x_2]$ to denote the line segment connecting x_1 and x_2. For a 2-dimensional subspace X_2 of X and $x_1, x_2 \in S(X_2)$, we use $\widetilde{x_1, x_2}$, which denotes the curve on $S(X_2)$, connecting x_1 to x_2 counter-clockwise, and $l(\widetilde{x_1, x_2})$ denotes the arc length of the curve $\widetilde{x_1, x_2}$.

For H in X, where H is a bounded subset, $d(H) = \sup\{\|x - y\|: x, y \in H\}$ is used to denote the diameter of H.

The following geometric concepts were introduced in 1948 [1]:

Definition 1. A bounded and convex subset K of X is said to have a normal structure if for every convex subset $H \subseteq K$, there is a point $x_0 \in H$, such that $\sup\{\|x_0 - y\|: y \in H\} < d(H)$.

X is said to have a normal structure, or weak normal structure if each bounded and convex subset $H \subseteq X$, or each weakly compact convex set $K \subseteq X$ has a normal structure.

If there exists a number c with $0 < c < 1$, such that for any bounded closed convex subset $K \subseteq X$, $\sup\{\|x_0 - y\|: y \in K\} \le c \cdot d(K)$ for a $x_0 \in K$, then X is said to have a uniform normal structure.

The mapping $T: C \longrightarrow C$, where C is a subset of X, is called nonexpansive if for all $x, y \in C$, we have $\|Tx - Ty\| \le \|x - y\|$. For fixed point property of a nonexpansive mapping $T: C \longrightarrow C$, please refer [2–4].

> Kirk [3] proved that every nonexpansive mapping T in a convex and weakly compact subset C of X has a fixed point in C.
>
> Gao [5] introduced the following concept: the modulus of W_{1X}-convexity:

Definition 2. Let ∇_x be the set of norm 1 support functionals of $S(X)$ at x and $r_1(x, y) = \inf\{\langle x - y/2, f_x\rangle, \text{for } f_x \in \nabla_x\}$. Then, $W_{1X}(\varepsilon) = \sup\{r_1(x, y), \|x - y\| \le \varepsilon\}$, for $0 \le \varepsilon \le 2$ is called the modulus of W_{1X}-convexity.

In this paper, we use an equivalent definition for W_{1X}-convexity:

Definition 3. $W_{1X}(\varepsilon) = \sup\{\langle x - y/2, f_x\rangle: x, y \in S(X), \|x - y\| \le \varepsilon \text{ for some } f_x \in \nabla_x\}$, where $0 \le \varepsilon \le 2$.

In general, $W_{1X}(\varepsilon)$ and $W_{1X^*}(\varepsilon)$ are not equal, for $0 \le \varepsilon \le 2$.

Gao [6] introduced the modulus of $\zeta_X(\varepsilon)$.

Definition 4. $\zeta_X(\varepsilon) = \sup\{l(\widetilde{x_1, x_2}): x_1, x_2 \in S(X) \text{ satisfy } \langle x_1 - x_2, f_{x_1}\rangle \le \varepsilon \text{ for a } f_{x_1} \in \nabla_{x_1}\}$, where $0 \le \varepsilon \le 2$.

We had $W_{1X}(\varepsilon) \le \varepsilon/2$ and $W_{1X}(\varepsilon) \le \zeta_X(\varepsilon)$ for $0 \le \varepsilon \le 2$. Both $W_{1X}(\varepsilon)$ and $\zeta_X(\varepsilon)$ are nondecreasing functions in $[0, 2]$ [5, 6].

In this article, we show the relation between modulus of $W_{1X^*}(\varepsilon)$ in X^* and normal structure in X, and modulus $\zeta_{X^*}(\varepsilon)$ in X^* and normal structure in X, respectively. More results for fixed points of nonexpansive mapping are obtained. Among other conclusions, we proved X with $W_{1X^*}(\varepsilon) < \varepsilon/2$, where $0 \le \varepsilon \le 2$, or $\zeta_{X^*}(\varepsilon) < 1 + \varepsilon$, where $0 \le \varepsilon \le 1$, implies X and its dual X^* both have the uniform normal structure.

2. Modulus $W_{1X^*}(\varepsilon)$

Theorem 1 (See [7–9]). *The following proved that*

(a) If X is an uniformly nonsquare space, then X must be a supper-reflexive space, and therefore reflexive space

(b) X^* is a super-reflexive space if and only if X is a super-reflexive space.

The following proved that [5]

Theorem 2 (See [5]). *A space X is uniformly nonsquare if for $0 \le \varepsilon \le 2$, we have $W_{1X}(\varepsilon) < \varepsilon/2$.*

Theorem 3 (See [5]). *A space X has a normal structure if for $0 \le \varepsilon \le 2$, we have $W_{1X}(\varepsilon) < \varepsilon/2$.*

Lemma 1 (Bishop–Phelps–Bollobás [10]). *For a Banach space X and $0 < \varepsilon < 1$, if an element $z \in B(X)$ and an element $h \in S(X^*)$ satisfy the condition $1 - \langle z, h\rangle < \varepsilon^2/4$, then there exist an element $y \in S(X)$ and an element $g \in \nabla_y$ such that we have $\|y - z\| < \varepsilon$, and also $\|g - h\| < \varepsilon$.*

Example 1. Let $X = c_0$, $X^* = l_1$, and $X^{**} = l_\infty$. Then,

(a) $W_{1c_0}(\varepsilon) = \varepsilon/2$, for $0 \le \varepsilon \le 2$

(b) $W_{1l_1}(\varepsilon) = \varepsilon/2$, for $0 \le \varepsilon \le 2$

In fact,

(a) Let $x_1 = (1, 1, 0, \ldots, 0, 0, 0, \ldots) \in S(c_0)$,

$$x_2 = (t, 1, 0, \ldots, 0, 0, 0, \ldots) \in S(c_0), -1 \le t \le 1, \qquad f_1 = (1, 0, 0, \ldots, 0, 0, 0, \ldots) \in \nabla_{x_1} \subseteq S(l_1). \tag{1}$$

We have $\langle x_1 - x_2, f_1\rangle = 1 - t$, and

$$\|x_1 - x_2\|_{c_0} = \|(1 - t, 0, 0, \ldots, 0, 0, 0, \ldots)\|_{c_0} = 1 - t. \tag{2}$$

So, $W_{1c_0}(1 - t) = 1 - t/2$, for $-1 \le t \le 1$.

Let $\varepsilon = 1 - t$, we have $W_{1c_0}(\varepsilon) = \varepsilon/2, 0 \le \varepsilon \le 2$.

(b) Let $x_1 = (0, 1, 0, 0, \ldots, 0, 0, 0, \ldots) \in S(l_1)$,

$$x_2 = \left(\frac{t}{2}, 1 - t, \frac{t}{2}, 0, \ldots, 0, 0, 0, \ldots\right) \in S(l_1), 0 \le t \le 1, \qquad f_1 = (-1, 1, -1, 0, \ldots, 0, 0, 0, \ldots) \in \nabla_{x_1} \subseteq S(l_\infty). \tag{3}$$

We have $\langle x_1 - x_2, f_1\rangle = \langle (-t/2, t, -t/2, 0, \ldots, 0, 0, 0, \ldots), (-1, 1, -1, 0, \ldots, 0, 0, 0, \ldots)\rangle = 2t$, and $\|x_1 - x_2\|_{l_1} = \|(-t/2, t, -t/2, 0, \ldots, 0, 0, 0, \ldots)\|_{l_1} = 2t$.

So, $W_{1l_1}(2t) = t, 0 \le t \le 1$.

Let $\varepsilon = 2t$, we have $W_{1l_1}(\varepsilon) = \varepsilon/2, 0 \le \varepsilon \le 2$.

Example 2. (See [11])

(a) If H is a Hilbert space, we have $W_{1H}(\varepsilon) = (\varepsilon/2)^2$, when $0 \le \varepsilon \le 2$

(b) For space X, $W_{1X}(\varepsilon) = \sup\{1/2(1 - n^+(x, y)): x, y \in S(X), \|x - y\| \le \varepsilon\}$, where $n^+(x, y) = \lim_{t \longrightarrow 0^+} \|x + ty\| - 1/t$.

If X is a reflexive Banach space, or separable Banach space, or one of those spaces that admit an equivalent smooth norm, then $U(X^*)$ is weak∗ sequentially compact ([12], Ch. XIII).

Lemma 2 (See [13]). *Let $B(X^*)$ be weak* sequentially compact but X fails to have a weak normal structure, then there are sequence $\{x_n\} \subseteq S(X)$ and sequence $\{f_n\} \subseteq S(X^*)$ such that for any $\varepsilon > 0$,*

(a) $\|f_i - f_j\| > 2 - \varepsilon$, if $i \ne j$; if $i \ne j$

(b) $\langle x_i, f_i\rangle = 1$, if $1 \le i \le \infty$

(c) $|\langle x_j, f_i\rangle| < \varepsilon$, if $i \ne j$

(d) $|\|x_i - x_j\| - 1| < \varepsilon$

Theorem 4. *Let X be a Banach space and $B(X^*)$ weak∗ sequentially compact but X fails to have a weak normal structure, then for some $0 \le \varepsilon \le 2$, we have $W_{1X^*}(\varepsilon) \ge \varepsilon/2$.*

Proof. Let $0 \le t \le 1$, and $tf_1 + (1 - t)f_2 \in [f_1, f_2]$ where $[f_1, f_2]$ is a line segment which connect f_1 and f_2 in $B(X^*)$, and let $tf_1 + kf_2 \in \widetilde{f_1, f_2}$, where $\widetilde{f_1, f_2}$ is an arc between f_1 and f_2 on $S(X^*)$. Then, from the convexity of $B(X^*)$, we get $k \ge 1 - t$.

Since $\langle x_1 - x_2/1 + \varepsilon, f_1\rangle \ge 1 - \varepsilon$, from Bishop–Phelps–Bollobás theorem, there exist $y_1 \in X$, and $g_1 \in X^*$ with $\|y_1\| \le \varepsilon$, and $\|g_1\| \le \varepsilon$ such that

$$\left\langle \frac{x_1 - x_2}{1 + \varepsilon} + y_1, f_1 + g_1\right\rangle = 1, \text{Therefore } \frac{x_1 - x_2}{1 + \varepsilon} + y_1 \in \nabla_{f_1 + g_1}. \tag{4}$$

We have $\|(f_1+g_1)-(tf_1+kf_2)\| = \|(1-t)f_1-kf_2+g_1\| \le 1-t+k+\varepsilon$.

But, $\langle x_1-x_2/1+\varepsilon+y_1, (f_1+g_1)-(tf_1+kf_2)\rangle$

$$\begin{aligned} &= \langle \frac{x_1-x_2}{1+\varepsilon}+y_1, (1-t)f_1-kf_2+g_1\rangle, \\ &= \langle \frac{x_1-x_2}{1+\varepsilon}, (1-t)f_1-kf_2\rangle + \langle y_1, (1-t)f_1-kf_2\rangle \\ &\quad + \langle \frac{x_1-x_2}{1+\varepsilon}+y_1, g_1\rangle, \\ &\ge 1-t+k-2\varepsilon. \end{aligned} \tag{5}$$

So, $\sup\{\langle x, f_x-f_y/2\rangle : f_x, f_y \in S(X^*), \|f_x-f_y\| \le 1-t+k+\varepsilon \text{ for some } x \in \nabla_{f_x}\} \ge 1-t+k/2-4\varepsilon$.

Since ε can be arbitrarily small as we need, from continuity of $W_{1X^*}(\varepsilon)$, we get $W_{1X^*}(1-t+k) \ge 1-t+k/2$.

Let $\varepsilon = 1-t+k$, it is clear that $0 \le \varepsilon \le 2$, we have $W_{1X^*}(\varepsilon) > \varepsilon/2$, for any $0 \le \varepsilon \le 2$. □

Theorem 5. *Let $B(X^*)$ be weak* sequentially compact and $W_{1X^*}(\varepsilon) < \varepsilon/2$, then for an arbitrary $0 \le \varepsilon \le 2$, X has a normal structure.*

Proof. From Theorem 2, if $W_{1X^*}(\varepsilon) < \varepsilon/2$ for an arbitrary $0 \le \varepsilon \le 2$ we have X^* as uniformly nonsquare. Therefore, both X and X^* are super reflexive. So, the normal structure and the weak normal structure coincide. □

3. Modulus $\zeta_{X^*}(\varepsilon)$

For a 2-dimensional subspace X_2 of space X, it is clear that $S(X_2)$ is a simple closed curve which is symmetric about the origin and unique up to orientation. For more properties of curves, please see [5, 7, 14] and [15].

Theorem 6 (See [2, 4, 15]). *For a 2-dimensional Banach space X_2. The following statements are true:*

(a) $6 \le l(S(X_2)) \le 8$

(b) *$l(S(X_2)) = 6$ if and only if $S(X_2)$ is an affine regular hexagon*

(c) *$l(S(X_2)) = 8$ if and only if $S(X_2)$ is a parallelogram*

Lemma 3 (See [16]). *If $x_1, x_2 \in B(X)$ with $0 < \epsilon < 1$ such that $\|x_1+x_2\|/2 > 1-\varepsilon$, then for all $0 \le c \le 1$ and $z = cx_1+(1-c)x_2 \in [x_1, x_2]$, it follows that $\|z\| > 1-2\epsilon$.*

For the following, we assume $l(\widetilde{x_1, x_2}) \le 1/2 l(S(X_2))$ where $x_1, x_2 \in S(X)$.

Example 3. (a) $\zeta_{c_0}(\varepsilon) = 2, when \varepsilon = 0; \zeta_{c_0}(\varepsilon) = 2+\varepsilon, when 0 \le \varepsilon \le 2$

(b) $\zeta_{l_1}(\varepsilon) = 2, when \varepsilon = 0; \zeta_{l_1}(\varepsilon) = 2+\varepsilon, when 0 \le \varepsilon \le 2$

(c) If H is a Hilbert space, $\zeta_H(\varepsilon) = 2\sin^{-1}\varepsilon^2/8$, when $0 \le \varepsilon \le 2$

Theorem 7 (See [6]). *If $\zeta_X(\varepsilon) < 1+\varepsilon$, for any $0 \le \varepsilon < 1$, or $\zeta_X(\varepsilon) < 2\varepsilon$, for any $1 \le \varepsilon \le 2$. Then, X is uniformly nonsquare.*

Theorem 8 (See [6]). *If $\zeta_X(\varepsilon) < 1+\varepsilon$, for any $0 \le \varepsilon \le 1$, then X has a normal structure.*

Theorem 9. *If $B(X^*)$ is weak$*$ sequentially compact and for any $0 \le \varepsilon \le 2$, $\zeta_{X^*}(\varepsilon) < 1+\varepsilon$, then space X has a normal structure.*

Proof. Since $\zeta_{X^*}(\varepsilon) < 1+\varepsilon$ for any $0 \le \varepsilon \le 2$, we have either $\zeta_{X^*}(\varepsilon) < 1+\varepsilon$, for any $0 \le \varepsilon < 1$, or $\zeta_{X^*}(\varepsilon) < 2\varepsilon$, for any $1 \le \varepsilon < 2$, so X^* is uniformly nonsquare. Therefore, both X and X^* are super reflexive. So the weak normal structure and the normal structure coincide.

If X fails to have a weak normal structure, then for any $\varepsilon > 0$, there are a sequence $\{x_n\} \subseteq S(X)$ and a sequence $\{f_n\} \subseteq S(X^*)$ such that it satisfies 4 conditions of Lemma 2.

Suppose $\theta(t) > 0$, and $f = (t(-f_1)+(1-t)\ (-f_2)) + \theta(t)(-f_2-f_1)) = (t+\theta(t))(-f_1) + \ (1-t+\theta(t))(-f_2) \in S(X^*)$, we have $0 \le \theta(t) \le t$.

Since $\|-f_2+f_1/2\| \ge 1-\varepsilon$, from Lemma 3, $\|t(f_1)+(1-t)(-f_2)\| \ge 1-2\varepsilon$, for $0 \le t \le 1$.

Then, $\|f-(-f_2)\| = \|(t+\theta(t))(-f_1)+(1-t+\theta(t))(-f_2)-(-f_2)\| = \|(t+\theta(t))\ (-f_1)+(-t+\theta(t))(-f_2)\| = \|(t+\theta(t))\ (f_1)+(t-\theta(t))(-f_2)\| = 2t\|(t+\theta(t))\ (f_1)+(t-\theta\ (t))(-f_2)/2t\| \ge 2t-4\varepsilon$.

Let $\langle x_2, f_1\rangle = \eta$. Then, $|\eta| \le \varepsilon$ and $\langle 1/1+2\varepsilon((1+\eta)x_1-x_2), f_1\rangle = 1$. By using the Bishop–Phelps–Bollobás theorem, there exist $y \in X$, and $g \in X^*$ with $\|y\| \le \varepsilon$, and $\|g\| \le \varepsilon$ such that $1/1+2\varepsilon((1+\eta)x_1-x_2)+y \in \nabla_{f_1+g} \in S(X^{**})$.

We have $\langle 1/1+2\varepsilon((1+\eta)x_1-x_2)+y, f_1+g-f\rangle$

$$\begin{aligned} &= \langle \frac{1}{1+2\varepsilon}((1+\eta)x_1-x_2), f_1-f\rangle + \langle \frac{1}{1+2\varepsilon}((1+\eta)x_1-x_2)+y, g\rangle + \langle y, f_1-f\rangle \\ &= \langle \frac{1}{1+2\varepsilon}((1+\eta)x_1-x_2), (1+t+\theta(t))(f_1)+(1-t+\theta(t))(f_2)\rangle \\ &\quad + \langle \frac{1}{1+2\varepsilon}((1+\eta)x_1-x_2)+y, g\rangle + \langle y, f_1-f\rangle \le \frac{(1+\eta)}{1+2\varepsilon}((1+t+\theta(t))-(1-t+\theta(t))) \\ &\quad + 4\varepsilon = \frac{2t(1+\eta)}{1+2\varepsilon}+4\varepsilon \le 2t+6\varepsilon \text{ for } 0 \le t \le 1. \end{aligned} \tag{6}$$

From Theorem 6, we also have

$$l\left(f_1 \widetilde{+g}, f\right) \geq l\left(f_1, \widetilde{-f_2}\right) + l\left(\widetilde{-f_2}, f\right) - l\left(f_1 \widetilde{+g}, f_1\right) \geq \|f_1 - (-f_2)\| + \|-f_2 - f\| - 8\varepsilon \geq \|f_1 + f_2\| + \|f + f_2\| - 8\varepsilon \geq \left\langle \frac{x_1 + x_2}{2}, f_1 + f_2 \right\rangle + 2t - 4\varepsilon - 8\varepsilon \geq 1 + 2t - 12\varepsilon. \quad (7)$$

We have $\zeta_{X^*}(2t + 6\varepsilon) \geq 1 + 2t - 12\varepsilon$.

Since ε can be arbitrarily small, we have $\zeta_{X^*}(2t) \geq 1 + 2t$, for any $0 \leq t \leq 1$.

Let $2t = \varepsilon$. Then, $0 \leq \varepsilon \leq 2$.

This is equivalent to $\zeta_{X^*}(\varepsilon) \geq 1 + \varepsilon$, for all $0 \leq \varepsilon \leq 2$. □

4. Uniform Normal Structure

Let $\mathbb{N}$ be the set of all natural numbers, and let $X_i = X$ for all $i \in \mathbb{N}$ be a Banach space X. For more properties of an $X_{\mathscr{U}}$, please see [17–19].

We proved that

Theorem 10 (See [6]). *If $\zeta_X(\varepsilon) < 1 + \varepsilon$, then for all $0 \leq \varepsilon \leq 1$, for all nontrivial ultrafilter U on $\mathbb{N}$, we have $\zeta_{X_{\mathscr{U}}}(\varepsilon) = \zeta_X(\varepsilon)$.*

Similarly, we have

Theorem 11. *If $W_{1X}(\varepsilon) < \varepsilon/2$, then for all $0 \leq \varepsilon < 2$, and for all nontrivial ultrafilter U on $\mathbb{N}$, we have $W_{1X_{\mathscr{U}}}(\varepsilon) = W_{1X}(\varepsilon)$.*

Theorem 12 (See [20]). *For a super-reflexive Banach space X, $X_{\mathscr{U}}$ has a uniform normal structure if and only if X has a normal structure.*

Theorem 13. *Let $W_{1X^*}(\varepsilon) < \varepsilon/2$, for all $0 \leq \varepsilon \leq 2$, then both X and its dual X^* have a uniform normal structure.*

Proof. From Remark 2.1, Theorem 2 and Theorem 11, we have $W_{1X^*_{\mathscr{U}}}(\varepsilon) < \varepsilon/2$, for any $0 \leq \varepsilon \leq 2$, then from Theorem 3, Theorem 5, and Theorem 12, both X and its dual X^* have a uniform normal structure. □

Theorem 14. *Let $\zeta_{X^*}(\varepsilon) < 1 + \varepsilon$, for all $0 \leq \varepsilon \leq 1$, then both X and its dual X^* have a uniform normal structure.*

Proof. From Theorem 7, Remark 2.1 and Theorem 10, we have $\zeta_{X^*_{\mathscr{U}}}(\varepsilon) < 1 + \varepsilon$ for all $0 \leq \varepsilon \leq 1$. Then, from Theorem 8, Theorem 9, and Theorem 12, we have both space X and its dual space X^* have the uniform normal structure. □

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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