Research Article

New Characterizations and Representations of the Bott–Duffin Inverse

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The paper focuses on the class of the Bott–Duffin inverses. Several original features of the class are identified and new properties are characterized. Some of the results available in the literature are recaptured in a more general form. BD matrices are also introduced and some properties are given.

1. Introduction

$C^n$ stands for the vector space of $n$-tuples over the field of complex numbers. The symbol $C^{m\times n}$ denotes the set of complex $m \times n$ matrices. The symbols $R(A)$, $N(A)$, $A^*$ and rank $(A)$ represent the range space, null space, conjugate transpose, and rank of $A \in C^{m\times n}$, respectively. The symbol Ind $(A)$ stands for the index of $A \in C^{m\times n}$ which is the smallest non-negative integer $k$ such that rank $(A^k) = \text{rank}(A^{k+1})$. The symbol $I_n$ means the identity matrix in $C^{m\times n}$. The symbol $O$ means the null matrix. If $L$ is a subspace of $C^n$, we use the notation $L \subseteq C^n$ while $L^\perp$ means the orthogonal complement subspace of $L$. The dimension of $L$ is denoted by dim $(L)$. $P_L$, $P_M$ stands for the oblique projector onto $L$ along $M$, where $L,M \subseteq C^n$ and $L \oplus M = C^n$. $P_L$ is the orthogonal projector onto $L$.

Additionally, the Moore–Penrose inverse $A^+ \in C^{m\times n}$ of $A \in C^{m\times n}$ is the unique matrix verifying $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+)^* = AA^+$, $(A^+A)^* = A^+A$ (see [1–4]).

A matrix $X \in C^{m\times n}$ that satisfies $XAX = X$ is called an outer inverse of $A$ and is denoted by $A^{(2)}$ (3). Let $T \subseteq C^n$, dim $(T) = t$ and $S \subseteq C^m$, dim $(S) = m - t$. There exists a unique outer inverse $X$ of $A$ such that $R(X) = T$ and $N(X) = S$ if and only if $AT \oplus S = C^n$. In case, there exists $X$, we call an outer inverse with prescribed range and null space and denote it by $A_{T,S}^{(2)}$ (see [1, 4]).

The symbol $A_D$ stands for the Drazin inverse of $A \in C^{m\times n}$ which is the unique matrix satisfying $A_DAA_D = A_D$, $AA_D = A_DA$, $A_D A^{k+1} = A_D$, where $k = \text{Ind}(A)$ (see [5]). Especially, if $\text{Ind}(A) = 1$, then the Drazin inverse of $A$ is called the group inverse of $A$ and is denoted by $A^g$.

Bott and Duffin, in their famous paper [6], introduced the “constrained inverse” of a square matrix as an important tool in the electrical network theory. This inverse is called in their honor the Bott–Duffin inverse (in short, BD-inverse).

Definition 1 (see [6]). Let $A \in C^{m\times n}$ and $L \subseteq C^n$. If $AP_L + P_L$ is nonsingular, then the BD-inverse of $A$ with respect to $L$, denoted by $A_{T}^{(1)}$, is defined by the following equation:

$$A_{T}^{(1)} = P_L^{(1)}(AP_L + P_L)^{-1}.$$  \hspace{1cm} (1)

There are huge literatures on the BD-inverse and here we will mention only the part. Some important applications of the BD-inverse can be found in the monograph [1]. Chen [7] presented several properties and different representations of the BD-inverse. Also, certain relationships between a class of nonsingular bordered matrices and the BD-inverse are given.
in [8]. In [9], Wei studied the various norm-wise relative condition numbers that measure the sensitivity of the BD-inverse and the solution of constrained linear systems. The perturbation theory for the BD-inverse was discussed in [10].

In [11], Chen defined the generalized BD-inverse of $A$ (denoted by $A_{(L)}^{(-1)}$). In terms of the form of definitions, $A_{(L)}^{(-1)} = P_L (AP_L + P_L^T)^{-1}$ is a natural extension of $A_{(L)}^{(-1)} = P_L (AP_L + P_L^T)^{-1}$. However, if $A \in \mathbb{C}^{m \times n}$ is arbitrary, we have $\mathcal{N}(A_{(L)}^{(-1)}) = \mathcal{N}(P_L (AP_L + P_L^T)) = L \cap (AL)^{-1} \oplus L \cap [L + (AL)^-1]$ by using Theorem A.1 (see Appendix A) and [5, Lemma 1]. It is not convenient or even difficult to study the properties of $A_{(L)}^{(-1)}$. In order to avoid this difficulty and obtain more interesting properties of $A_{(L)}^{(-1)}$, all of the theorems in [11, 12] restrict matrix $A$ to an $L$-p.s.d matrix, which satisfies the following three conditions:

(i) $A^* = A$
(ii) $x^* Ax \geq 0$ for all $x \in L$
(iii) $x^* Ax = 0$ for $x \in L$ implies $Ax = 0$

However, for $A_{(L)}^{(-1)}$, matrix $A$ does not need to satisfy these conditions, only $A_{(L)}^{(-1)}$ needs to exist. And the necessary and sufficient condition for the existence of $A_{(L)}^{(-1)}$ is given in Lemma 2. Therefore, for the differences in studying $A_{(L)}^{(-1)}$ and $A_{(L)}^{(-1)}$, it is meaningful to research the properties of $A_{(L)}^{(-1)}$.

The present paper provides a further contribution to the stream of works devoted to the BD-inverse. Several new characterizations of the BD-inverse are derived in terms of certain matrix equations and EP-property. Also, we give some new representations of the BD-inverse as well as the relationships between the BD-inverse and other generalized inverses. The definition and properties of the BD-matrices are also given. Several original features of the BD-inverses are identified and new properties are characterized. In some cases, the results available in the literature are recaptured therein in a more general form.

The rest of this paper is organised as follows. In Section 2, we introduce some lemmas and a matrix decomposition which will be used later in the paper. In Section 3, we present several characterizations of the BD-inverse in terms of certain matrix equations and EP matrix. In Section 4, we present several representations of the BD-inverse. We focus on the relationships between the BD-inverse and other generalized inverses within Section 5. In addition, we give the definition of the BD-matrices and present some of their properties.

Henceforth, the symbol $\mathbb{C}^{n \times n}_{EP}$ will stand for the set of $n \times n$ EP matrices, i.e.,

\[ \mathbb{C}^{n \times n}_{EP} = \{ A | A \in \mathbb{C}^{n \times n}, AA^* = A^*A \} \]

\[ \mathbb{C}^{n \times n}_{EP} = \{ A | A \in \mathbb{C}^{n \times n}, \mathcal{R}(A) = \mathcal{R}(A^*) \} \]

### 2. Preliminaries

Let $A \in \mathbb{C}^{m \times n}$ and $L \subseteq \mathbb{C}^n$. In order to discuss some properties of the BD-inverse, we will consider appropriate matrix decomposition of $A$ with respect to $L$. Since there exists a unitary matrix $U \in \mathbb{C}^{m \times m}$ such that

\[
P_L = U \begin{bmatrix} I_l & O \\ O & O \end{bmatrix} U^*,
\]

where $l = \dim(L)$, a matrix $A$ can be written as follows:

\[
A = U \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix} U^*,
\]

where $A_L \in \mathbb{C}^{l \times n}$, $B_L \in \mathbb{C}^{(n-l) \times n}$, $C_L \in \mathbb{C}^{(n-l) \times l}$, and $D_L \in \mathbb{C}^{(n-l) \times (n-l)}$.

Now, we are ready to give the necessary and sufficient condition for the existence of $A_{(L)}^{(-1)}$ as well as the representation of $A_{(L)}^{(-1)}$.

**Lemma 2.** Let $P_L$ and $A$ be given by equations (3) and (4), respectively. $A_{(L)}^{(-1)}$ exists if and only if $A_L$ is invertible. In this case,

\[
A_{(L)}^{(-1)} = U \begin{bmatrix} A_L^{-1} & O \\ O & O \end{bmatrix} U^*.
\]

**Proof.** By equations (3) and (4), we get the following equation:

\[
AP_L + P_L^T = U \begin{bmatrix} A_L & O \\ C_L & I_{n-l} \end{bmatrix} U^*.
\]

Evidently, $AP_L + P_L^T$ is invertible if and only if $A_L$ is invertible. In this case, from equations (1) and (6), we get that equation (5) is satisfied.

The next lemma gives some basic properties of the BD-inverse, for example, that it is an outer inverse of $A$ with range $L$ and null space $L^\perp$, etc.

**Lemma 3** (see [7]). Let $A \in \mathbb{C}^{m \times n}$ and $L \subseteq \mathbb{C}^n$. If $AP_L + P_L^T$ is invertible, then the following statements hold:

(i) $P_L = A_{(L)}^{(-1)} AP_L = P_L A_{(L)}^{(-1)}$
(ii) $A_{(L)}^{(-1)} = P_L A_{(L)}^{(-1)} = A_{(L)}^{(-1)} P_L$
(iii) $\mathcal{R}(A_{(L)}^{(-1)}) = L, \mathcal{N}(A_{(L)}^{(-1)}) = L^\perp$
(iv) $A_{(L)}^{(-1)} = A_{(L)}^{(-1)}$
(v) $AA_{(L)}^{(-1)} = P_{AL} L^\perp$
(vi) $A_{(L)}^{(-1)} A = P_{L(A^*)} L^\perp$

### 3. Some New Characterizations of the BD-Inverse

In this section, we provide several characterizations of the BD-inverse of $A \in \mathbb{C}^{m \times n}$ (in the case when it exists) mainly in terms of certain matrix equations and EP-property. By Lemma 3, we know that $A_{(L)}^{(-1)}$ is an outer inverse of $A$. Using this property, we present several new characterizations of the BD-inverse of $A$.

**Theorem 4.** Let $A \in \mathbb{C}^{m \times n}$ and $L \subseteq \mathbb{C}^n$ be such that $A_{(L)}^{(-1)}$ exists and let $X \in \mathbb{C}^{m \times n}$. The following statements are equivalent:

(a) $X = A_{(L)}^{(-1)}$
(b) $XAX = X, XP_L = X,$ and $XA = P_L(A^\perp L)^*.$
(c) $XAX = X, XAP_L = P_L,$ and $AX = P_{ALL}.$
(d) $XAX = X, P_1X = X,$ and $AX = P_{ALL}.$
(e) $XAX = X, P_1AX = P_L,$ and $XA = P_L(A^\perp L)^*.$
(f) $XAX = X, P_1XP_L = X,$ and $AX = P_{ALL}.$

Proof

(a) $\implies$ (b): This follows directly by (ii), (iv), and (vi) of Lemma 3.

(b) $\implies$ (c): From $XA = P_L(A^\perp L)^*$, we have $XAP_L = P_L.$ Since $XAX = X,$ it follows that $(X) \equiv (X)$ and by $XA = P_L(A^\perp L)^*,$ we get $(X) \equiv L.$ Hence, $(X) \equiv AL.$ By $XP_L = X,$ we have $L \equiv M(X)$ which together with $(X) \equiv L$ gives $N(L) \equiv L.$ Thus, $AX = P_{(X)}(X) \equiv P_{ALL}.$

(c) $\implies$ (d): Since $XAX = X$ and $AX = P_{ALL},$ we have $M(A) \equiv (A)(X) \equiv L,$ which implies $\dim(X) \equiv \dim(L).$ Since $XAP_L = P_L$ and $XAX = X,$ it follows that $L \equiv (X) \equiv (X).$ Thus, $(X) \equiv L,$ which implies $P_1X = X.$

(d) $\implies$ (e): The proof is similar to the implication (b) $\implies$ (c).

(e) $\implies$ (f): By $XAX = X$ and $XAX = P_L(A^\perp L)^*,$ we have $(X) \equiv (X) \equiv L,$ which implies $\dim(X) \equiv \dim(L).$ Since $XP_L = X,$ it follows that $X_1$ is invertible and $X_1 = A_L^{-1}.$ Thus, by equation (5), we have $X = A_L^{(-1)}.$

(f) $\implies$ (a): Suppose that $X$ is given by the following equation:

\[
X = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} U^*,
\]

where $X_1 \in C^{1,4},$ $X_2 \in C^{n-n},$ $X_3 \in C^{n-n},$ and $X_4 \in C^{n-n}.$

In the following theorem, we present different characterizations of the BD-inverse in terms of two matrix equations.

**Theorem 5.** Let $A \in C^{n,n}$ and $L \subseteq C^n$ be such that $A_L^{(-1)}$ exists and let $X \in C^{n,n}.$ The following statements are equivalent:

(a) $X = A_L^{(-1)}$
(b) $XAP_L = P_L$ and $XP_L = X$
(c) $P_1AX = P_L$ and $P_1X = X$
(d) $XAP_L = P_L$ and $P_1XP_L = X$
(e) $P_1AX = P_L$ and $P_1XP_L = X$
(f) $AX = P_{ALL}$ and $P_1X = X$

(g) $XAX = P_L(A^\perp L)^*$ and $XP_L = X$

Proof. Item (a) implies any of the assertions (b) – (g) which follows directly by Lemma 3 and Theorem 3. For the converse implications, we will only give the proof that (b) implies (a) since other proofs are similar.

(b) $\implies$ (a): Let $P_L, A,$ and $X$ be given by equations (3), (4), and (7), respectively. The condition $XP_L = X$ implies $X_2 = O$ and $X_4 = O.$ By $XAP_L = P_L,$ we have that $X_1 = A_L^{-1}$ and $X_3 = O.$ Hence, from equation (5), we have $X = A_L^{(-1)}.$

Using the representation of $A_L^{(-1)}$ given in Lemma 2, we can easily conclude that $A_L^{(-1)} \in C^{n,n}.$ In the following theorem, we discuss other characterizations of the BD-inverse using this fact.

**Theorem 6.** Let $A \in C^{n,n}$ and $L \subseteq C^n$ be such that $A_L^{(-1)}$ exists and let $X \in C^{n,n}.$ The following statements are equivalent:

(a) $X = A_L^{(-1)}$
(b) $X \in C^{n,n},$ $XA = P_L(A^\perp L)^*,$ and $P_1X = X$
(c) $X \in C^{n,n},$ $XAP_L = P_L,$ and $P_1X = X$
(d) $X \in C^{n,n},$ $AX = P_{ALL},$ and $XP_L = X$
(e) $X \in C^{n,n},$ $P_1AX = P_L,$ and $XP_L = X$

Proof

(a) $\implies$ (b): It is obvious by (ii), (iii), and (vi) of Lemma 3.

(b) $\implies$ (c): Evidently, $XA = P_L(A^\perp L)^*$ implies $XAP_L = P_L.$

(c) $\implies$ (d): Since $P_1X = X,$ multiplying $XAP_L = P_L$ by $X$ from the right, we get $XAX = X.$ Then, $(X) = (X), (X) \equiv (X),$ and $AX$ is idempotent. From $XAX = X$ and $AX = (X),$ we get $L \equiv (X) \equiv (X),$ respectively, which implies $(X) = L.$ Hence, $(X) \equiv AL.$ Since $X \in C^{n,n}$ and $(X) \equiv L,$ we have $(X) \equiv L.$ So $XP_L = X$ and $(X) \equiv L.$ Thus, $AX = P_{ALL}.$

(d) $\implies$ (e): From $AX = P_{ALL},$ we easily get $P_1AX = P_L.$

(e) $\implies$ (a): Let $P_L, A,$ and $X$ be given by equations (3), (4), and (7), respectively. From $XP_L = X$ and $X \in C^{n,n},$ we have $X_3 = O,$ $X_3 = O,$ and $X_4 = O.$ By $P_1AX = P_L,$ it follows that $X_1 = A_L^{-1}$ if $X_4 = 0.$ Now, by equation (5) from Lemma 2, we get $X = A_L^{(-1)}.$

4. Different Representation of the BD-inverse

**Theorem 7.** Let $A \in C^{n,n}$ and $L \subseteq C^n$ and let $a, b, c, d \in C$ be such that $a + b \neq 0$ and $cd \neq 0.$ If $A_L^{(-1)}$ exists, then...
\[
A_{(\bar{L})}^{(-1)} = (a + b)P_L (aAP_L + dP_L + bP_LAP_L)^{-1} \\
= (a + b)(aP_LA + dP_L + bP_LAP_L)^{-1}P_L \\
= c(cP_LAP_L + dP_L)^{-1} - \frac{c}{d}P_L.
\]

**Proof.** Let \( P_L \) and \( A \) be given by equations (3) and (4), respectively. We have the following equation:

\[
aAP_L + dP_L + bP_LAP_L = U \begin{bmatrix} (a + b)A_L & O \\ aC_L & dI_{n-I} \end{bmatrix} U^*. \tag{9}
\]

Evidently, \( aAP_L + dP_L + bP_LAP_L \) is nonsingular and using equation (5) and the facts that \( a + b \neq 0 \) and \( cd \neq 0 \), we get the following equation:

\[
(a + b)P_L (aAP_L + dP_L + bP_LAP_L)^{-1} = U \begin{bmatrix} (a + b)A_L & O \\ aC_L & dI_{n-I} \end{bmatrix} U^* \\
= U \begin{bmatrix} (a + b)I \ O \\ O \ O \end{bmatrix} \begin{bmatrix} \frac{1}{(a + b)}A_L^{-1} & O \\ \frac{-a}{(a + b)d}C_L A_L^{-1} & \frac{1}{d}I_{n-I} \end{bmatrix} U^* \\
= U \begin{bmatrix} A_L^{-1} & O \\ O & O \end{bmatrix} U^* = A_{(\bar{L})}^{(-1)}.
\]

Then,

\[
\text{Similarly, we have the following equation:} \\
aP_LA + dP_L + bP_LAP_L = U \begin{bmatrix} (a + b)A_L & aB_L \\ O & dI_{n-I} \end{bmatrix} U^*. \tag{11}
\]

\[
(a + b)(aP_LA + dP_L + bP_LAP_L)^{-1}P_L = U \begin{bmatrix} (a + b)I \ O \\ O \ (a + b)I_{n-I} \end{bmatrix} \begin{bmatrix} (a + b)A_L & aB_L \\ O & dI_{n-I} \end{bmatrix}^{-1} P_L \\
= U \begin{bmatrix} (a + b)I \ O \\ O \ (a + b)I_{n-I} \end{bmatrix} \begin{bmatrix} \frac{1}{(a + b)}A_L^{-1} & \frac{-a}{(a + b)d}B_L A_L^{-1} \\ O & dI_{n-I} \end{bmatrix} \begin{bmatrix} I \ O \\ O \ O \end{bmatrix} U^* \\
= U \begin{bmatrix} A_L^{-1} & O \\ O & O \end{bmatrix} U^* \\
= A_{(\bar{L})}^{(-1)}.
\]

The rest of the proof follows similarly. □

**Remark 8.** Under the hypotheses of Theorem 7 and additional assumption \( a = 0 \), we have the following equation:

\[
A_{(\bar{L})}^{(-1)} = b(bP_LAP_L + dP_L)^{-1}P_L \\
= bP_L(bP_LAP_L + dP_L)^{-1}, \tag{13}
\]

while if \( b = 0 \), we have the following equation:

\[
A_{(\bar{L})}^{(-1)} = aP_L(aAP_L + dP_L)^{-1}P_L \\
= a(aP_LA + dP_L)^{-1}P_L. \tag{14}
\]

If we take \( a = 1 \) and \( d = 1 \) or \( b = 1 \) and \( d = 1 \) in Theorem 7, we get results from the paper of Chen [6, Lemma 4 (a)].

In [13], Yuan and Zuo presented several limit expressions for the BD-inverse. Motivated by this result, in the following theorem we give some similar expressions.

**Theorem 9.** Let \( A \in C^{m \times n} \) and \( L \subseteq S^n \) such that \( AP_L + P_L \) is nonsingular. Then,
\begin{align*}
(a) \ A_{(L)}^{(-1)} &= \lim_{\lambda \to 0} -p_L(\lambda I_n + (P_L A)^* (AP_L))^{-1} A^* P_L \\
(b) \ A_{(L)}^{(-1)} &= \lim_{\lambda \to 0} -p_L(\lambda I_n + (AP_L A)^* (PA_L))^{-1} P_L A^* P_L \\
(c) \ A_{(L)}^{(-1)} &= \lim_{\lambda \to 0} -p_L(\lambda I_n + (P_L A) (AP_L)^*)^{-1} P_L \\
(d) \ A_{(L)}^{(-1)} &= \lim_{\lambda \to 0} -p_L(\lambda I_n + (AP_L) (PL A)^*)^{-1} P_L
\end{align*}

Proof. (a): Let \( P_L \) and \( A \) be given by equations (3) and (4), respectively. By Lemma 2, we know that \( A_L \) is invertible, so \( A_L^* A_L \) is positive definite. For \( \lambda \) small enough, we have that \((\lambda I_n + A_L^* A_L)^{-1}\) is invertible; so,

\begin{equation}
A_L^{-1} = \lim_{\lambda \to 0} (\lambda I_n + A_L^* A_L)^{-1} A_L^*.
\end{equation}

Let \( S = P_L(\lambda I_n + (P_L A)^* AP_L)^{-1} A^* P_L \). By equations (3) and (4), we have the following equation:

\begin{equation}
\begin{aligned}
S &= U \begin{bmatrix}
I_L & O \\
O & O
\end{bmatrix}
\begin{bmatrix}
\lambda I + A_L^* A_L & O \\
B_L^* A_L & \lambda I_n - I
\end{bmatrix}^{-1}
\begin{bmatrix}
A_L^* & O \\
B_L^* & O
\end{bmatrix} U^* \\
&= U \begin{bmatrix}
I_L & O \\
O & O
\end{bmatrix}
\begin{bmatrix}
(\lambda I + A_L^* A_L)^{-1} O \\
\frac{1}{\lambda} B_L^* A_L (\lambda I + A_L^* A_L)^{-1} 1 - I
\end{bmatrix}
\begin{bmatrix}
A_L^* & O \\
B_L^* & O
\end{bmatrix} U^* \\
&= U \begin{bmatrix}
(\lambda I + A_L^* A_L)^{-1} A_L^* O \\
O & O
\end{bmatrix} U^*.
\end{aligned}
\end{equation}

Thus,

\begin{equation}
\begin{bmatrix}
\lambda + 1 \\
\lambda^2 + 3\lambda + 1
\end{bmatrix}
\begin{bmatrix}
\lambda \\
\lambda^2 + 3\lambda + 1
\end{bmatrix}
= A_L^{(-1)}.
\end{equation}

In the next theorem, we present representations for the BD-inverse, using the projectors \( P = P_{(A^*)^*} \) and \( Q = P_{(A^*)^*} \).

**Theorem 10.** Let \( A \in C^{m \times n} \) and \( L \subseteq C^n \) be such that \( AP_L + P_L \) is nonsingular and let \( P = P_{(A^*)^*} \) and \( Q = P_{(A^*)^*} \). For any \( a, b, c, d \in C \) such that \( cd \neq 0 \) and \( a + b \neq 0 \), the following statements hold:

\begin{align*}
(a) \quad A_{(L)}^{(-1)} &= (aAP_L + bP_L AP_L + cP_L A)^{-1} (a(I_n - Q) + bP_L) \\
(b) \quad A_{(L)}^{(-1)} &= (a(I_n - P) + bP_L) (aP_L A + bP_L AP_L + dQ)^{-1} \\
(c) \quad A_{(L)}^{(-1)} &= c(cP_L AP_L + dP)^{-1} P_L \\
(d) \quad A_{(L)}^{(-1)} &= cP_L (cP_L AP_L + dQ)^{-1}
\end{align*}

Proof

(a) Let \( B = aAP_L + bP_L AP_L + cP_L A \). In terms of Lemma 3, we have the following equation:

\begin{equation}
BA_{(L)}^{(-1)} = (aAP_L + bP_L AP_L + cP_L A)(I_n - A_{(L)}^{-1} A_{(L)})
\end{equation}

\begin{equation}
= aAA_{(L)}^{(-1)} + bP_L
\end{equation}

\begin{equation}
= a(I_n - Q) + bP_L.
\end{equation}

Then, we only need to prove that \( B \) is invertible. From (iii) and (vi) in Lemma 3, it is easy to derive the following equation:

\begin{equation}
P_L P = P_L (I_n - A_{(L)}^{(-1)} A)
\end{equation}

\begin{equation}
= P_L\cdot
\end{equation}

Let \( P_L \) and \( A \) be given by equations (3) and (4), respectively. Then,

\begin{equation}
B = U \begin{bmatrix}
(a + b)A_L O \\
ac_L cI_w - L
\end{bmatrix} U^*.
\end{equation}
From \(a + b \neq 0, c \neq 0\), and Lemma 2, we can verify the invertibility of \(B\).

(b) The proof can be given as for item (a).

(c) By Lemma 3, we have the following equation:

\[
(cP_LAP_L + dP)A_{(L)}^{(-1)} = (cP_LAP_L + d(I_n - A_{(L)}^{(-1)}A))A_{(L)}^{(-1)}
\]

\[
= cP_L.
\]

(24)

Next, we need to prove the invertibility of \(cP_LAP_L + dP\). Let \(P_L\) and \(A\) be given by equations (3) and (4), respectively. Then,

\[
cP_LAP_L + dP = U \begin{bmatrix} cA_L & -dA_L^{-1}B_L \\ O & dI_{n-l} \end{bmatrix} U^*,
\]

(25)

so \(cP_LAP_L + dP\) is invertible.

(d) The proof can be given as for item (c). \qed

We provide the following example to calculate \(A_{(L)}^{(-1)}\) by using Theorem 10 (a).

Example 2. Let the matrix \(A\) and the subspace \(L\) be given as in Example 1. Then,

\[
P = I_4 - P_{L(A^{*}L)},
\]

(26)

\[
Q = I_4 - P_{ALL}.
\]

By direct calculation,

\[
\begin{bmatrix}
1 \\
-1 \\
0 \\
0
\end{bmatrix}
\]

5. Relations of the BD-inverse with Other Generalized Inverses

First of all, we will present the connection between BD-inverse and \((B, C)\)-inverse. Recall that Drazin [14] introduced the \((b, c)\)-inverse in a semigroup. In [15], the \((B, C)\)-inverse of matrices was studied by Benítez et al.
Theorem 13. Let $A \in \mathbb{C}^{m \times n}$ and $B, C \in \mathbb{C}^{n \times m}$. If there exist a matrix $X \in \mathbb{C}^{n \times n}$ satisfying the following equation:

$$
XAB = B, \\
CAX = C, \\
\mathcal{R}(X) = \mathcal{R}(B), \\
\mathcal{N}(X) = \mathcal{N}(C),
$$

(29)

then $X$ is called the $(B, C)$-inverse of $A$, denoted by $A^{(B,C)}$.

The next theorem shows that the BD-inverse of a matrix $A \in \mathbb{C}^{m \times n}$ is a special case of $(B, C)$-inverse.

Theorem 14. Let $A \in \mathbb{C}^{n \times n}$ and $L \subseteq \mathbb{C}^n$ such that $AP_L + P_L$ is nonsingular. Then,

$$
A_{(L)}^{(-1)} = A^{(P_L, P_L)}.
$$

(30)

Proof. Let $B = C = P_L$. By (i) and (iii) of Lemma 3, we have the following equation:

$$
\mathcal{R}(A_{(L)}^{(-1)}) = \mathcal{R}(B), \\
\mathcal{N}(A_{(L)}^{(-1)}) = \mathcal{N}(C), \\
A_{(L)}^{(-1)} AB = B, \\
CAA_{(L)}^{(-1)} = C.
$$

(31)

Thus, $A_{(L)}^{(-1)} = A^{(P_L, P_L)}$. \hfill \Box

From (v) and (vi) of Lemma 3, we have that $AA_{(L)}^{(-1)}$ and $A_{(L)}^{(-1)} A$ are oblique projectors. Next, we will discuss the necessary and sufficient conditions for $AA_{(L)}^{(-1)}$ and $A_{(L)}^{(-1)} A$ to be the orthogonal projector onto $L$, which can easily be derived by Lemma 3.

Theorem 15. Let $A \in \mathbb{C}^{n \times n}$, and $(A) = k$, and $L \subseteq \mathbb{C}^n$ be such that $A_{(L)}^{(-1)}$ exists. The following statements hold:

(a) $A_{(L)}^{(-1)} A = P_L$ if and only if $A^* L = L$

(b) $AA_{(L)}^{(-1)} = P_L$ if and only if $AL = L$

Proof. Evidently.

In the following theorem, we give the relationships between the BD-inverse and other generalized inverses such as Moore–Penrose inverse $A^*$, Drazin inverse $A^D$, core–EP inverse $A^C$, DMP inverse $A^{D,M}$, generalized Moore–Penrose inverse $A^G$, dual DMP inverse $A^{D,D}$, BT-inverse $A^S$, and weak group inverse $A^Ω$.

Theorem 16. Let $A \in \mathbb{C}^{n \times n}$, and $(A) = k$, and $L \subseteq \mathbb{C}^n$ be such that $A_{(L)}^{(-1)}$ exists. The following statements hold:

(a) $A_{(L)}^{(-1)} A = A^* L = \mathcal{R}(A)$ and $A \in \mathbb{C}^{n \times n}$

(b) $A_{(L)}^{(-1)} = A^D L = \mathcal{R}(A^D)$ and $A^D \in \mathbb{C}^{n \times n}$

(c) $A_{(L)}^{(-1)} = A^Ω L = \mathcal{R}(A^Ω)$

(d) $A_{(L)}^{(-1)} = A^{D,M} L = \mathcal{R}(A^{D,M})$ and $A^{D,M} \in \mathbb{C}^{n \times n}$

(e) $A_{(L)}^{(-1)} = A^S L = \mathcal{R}(A^S)$

(f) $A \in \mathbb{C}^{n \times n}$, and $(A) = k$, and $L \subseteq \mathbb{C}^n$ be such that $A_{(L)}^{(-1)}$ exists.

Then,

(a) $A_{(L)}^{(-1)} = A^D L = \mathcal{R}(A)$

(b) $A_{(L)}^{(-1)} = A^Ω L = \mathcal{R}(A)$

(c) $A_{(L)}^{(-1)} = A^{D,M} L = \mathcal{R}(A^{D,M})$ and $A^{D,M} \in \mathbb{C}^{n \times n}$

(d) $A_{(L)}^{(-1)} = A^S L = \mathcal{R}(A^S)$

(f) $A_{(L)}^{(-1)} = A^* L = \mathcal{R}(A^*)$ and $A^* \in \mathbb{C}^{n \times n}$

Remark 17. Let $A \in \mathbb{C}^{n \times n}$, and $(A) = k$, and $L \subseteq \mathbb{C}^n$ be such that $A_{(L)}^{(-1)}$ exists.

Then,

(a) $A_{(L)}^{(-1)} = A^D L = \mathcal{R}(A)$

(b) $A_{(L)}^{(-1)} = A^Ω L = \mathcal{R}(A)$

(c) $A_{(L)}^{(-1)} = A^{D,M} L = \mathcal{R}(A^{D,M})$ and $A^{D,M} \in \mathbb{C}^{n \times n}$

(d) $A_{(L)}^{(-1)} = A^S L = \mathcal{R}(A^S)$

(f) $A_{(L)}^{(-1)} = A^* L = \mathcal{R}(A^*)$ and $A^* \in \mathbb{C}^{n \times n}$

In [22], we have investigated the properties of rank$(AA_{(L)}^{(-1)} - A_{(L)}^{(-1)} A) = n$. Naturally, we investigate the properties of rank$(AA_{(L)}^{(-1)} - A_{(L)}^{(-1)} A) = 0$ (i.e.,
Theorem 18. Let $A \in \mathbb{C}^{m \times n}$ and $L \subseteq \mathbb{C}^n$ be such that $A_{(L)}^{(-1)}$ exists. Then, $A$ is called a BD-matrix with respect to $L$ if and only if $AA_{(L)}^{(-1)} = A_{(L)}^{(-1)}A$. The set of all BD-matrices with respect to $L$ is denoted by

$$C_n^{BD}(L) = \left\{ A \in \mathbb{C}^{m \times n} \mid A_{(L)}^{(-1)} \text{ exists, } AA_{(L)}^{(-1)} = A_{(L)}^{(-1)}A \right\}. \quad (33)$$

In the following theorem, we give some characterizations of the BD-matrices.

Theorem 19. Let $A \in \mathbb{C}^{m \times n}$ be given in equation (4), $L \subseteq \mathbb{C}^n$, and $\dim(L) = 1$ be such that $A_{(L)}^{(-1)}$ exists. Let $P = AA_{(L)}^{(-1)}$ and $Q = A_{(L)}^{(-1)}A$. Then, the following statements are equivalent:

(a) $A \in C_n^{BD}(L)$
(b) $L = AL = A^*L$
(c) $A$ is given by equation (4), where $B_L = O$ and $C_L = O$;
(d) $AP_L = P_L A$;
(e) $P_L AP_L = P_L A P_L$;
(f) $A^* P_L = P_L A^*$;
(g) $P_L A P_L = P_L A^* P_L$;
(h) $A^* P_L = P_L A^*$;
(i) $A^* P_L = P_L A^*$.

Proof

(a)$\Rightarrow$(b): By Lemma 3, we have the following equation:

$$AA_{(L)}^{(-1)} = A_{(L)}^{(-1)} A$$

$$\Leftarrow P_{ALL^*} = P_{L(A^*L)^*}$$

$$\Leftarrow AL = L \text{ and } L^* = (A^*L)^*$$

$$\Leftarrow L = AL = A^*L. \quad (34)$$

(a)$\Rightarrow$(c): Let $A$ and $A_{(L)}^{(-1)}$ be given by equations (4) and (5). Then,

$$AA_{(L)}^{(-1)} = A_{(L)}^{(-1)} A$$

$$\Rightarrow U \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix} \begin{bmatrix} A_{(L)}^{(-1)} \\ O \end{bmatrix} U^* = U \begin{bmatrix} A_{(L)}^{(-1)} \\ O \end{bmatrix} \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix} U^*$$

$$\Rightarrow U \begin{bmatrix} I_L & A_{(L)}^{(-1)} B_L \\ C_L & A_{(L)}^{(-1)} \end{bmatrix} U^* = U \begin{bmatrix} I_L & A_{(L)}^{(-1)} B_L \\ C_L & A_{(L)}^{(-1)} \end{bmatrix} U^*$$

$$\Rightarrow B_L = O \text{ and } C_L = O. \quad (35)$$

(c)$\Rightarrow$(d): This follows directly from equations (4) and (5).

(d)$\Rightarrow$(f): It is clear that

$$AP_L = P_L A \iff (I_n - P_L) A = A (I_n - P_L) \iff AP_L = P_L A. \quad (36)$$

(c)$\iff$(g): If $A$ is given by equation (4), where $B_L = O$ and $C_L = O$, we have the following equation:

$$A^* = U \begin{bmatrix} A_L^{(-1)} & O \\ O & D_L^{(†)} \end{bmatrix} U^*. \quad (37)$$

Therefore, it is easy to verify $A^* P_L = P_L A^*$. On the converse, if $A^* P_L = P_L A^*$, then $A^* = U \begin{bmatrix} A_L^{(-1)} & O \\ O & A_L^{(†)} \end{bmatrix} U^*$, where $A_L^{(†)} = A_L^{(-1)}$ and $A_L \in \mathbb{C}^{l \times d}$ and $A_L \in \mathbb{C}^{(n-l) \times (n-l)}$. Evidently, $A = (A^*)^* = U \begin{bmatrix} A_L^{(-1)} & O \\ O & A_L^{(†)} \end{bmatrix} U^*$. Hence, $A_L = A_L^{(†)}$, $B_L = O$, $C_L = O$, and $D_L = A_L^{(†)}$.

Equivalences (d)$\iff$(e), (g)$\iff$(h), and (g)$\iff$(i) can easily be verified.

From Definition 17, we use $AA_{(L)}^{(-1)} = AA_{(L)}^{(-1)}$, to characterize BD-matrices, and in the following theorem, we provide other equivalent characterizations of BD-matrices by $AA_{(L)}^{(-1)}$ and $AA_{(L)}^{(-1)}$.

Theorem 18. Let $A \in \mathbb{C}^{m \times n}$ be given in equation (4), $L \subseteq \mathbb{C}^n$, and $\dim(L) = 1$ be such that $A_{(L)}^{(-1)}$ exists. Let $P = AA_{(L)}^{(-1)}$ and $Q = A_{(L)}^{(-1)} A$. Then, the following statements are equivalent:

(a) $A \in \mathbb{C}_n^{BD}(L)$
(b) $PA_{(L)}^{(-1)} = A_{(L)}^{(-1)} Q$
(c) $PQ = QP$
(d) $P - Q$ is idempotent

Proof

(a)$\Rightarrow$(b). Since $P = Q$, it directly follows from (vi) in Lemma 3.

(b)$\Rightarrow$(a). From equations (4) and (5), it is easy to obtain $B_L = O$ and $C_L = O$ by simple calculation. In terms of the equivalence between (a) and (c) in Theorem 18, item (a) holds.

(a)$\Rightarrow$(c). Since $P = Q$ and $P$ and $Q$ are idempotent, it follows that $PQ = Q^2 = QP$.

(c)$\Rightarrow$(d). Using (i) in Lemma 3 and multiplying $QP = P^2$ by $P$, we have $Q = QP = P^2$. Hence, we have $(P - Q)^2 = P - Q$.

(d)$\Rightarrow$(a). From equations (4) and (5), we have the following equation:

$$P - Q = U \begin{bmatrix} O & -A_L^{(-1)} B_L \\ C_L A_{(L)}^{(-1)} & O \end{bmatrix} U^*. \quad (38)$$

Thus, if $(P - Q)^2 = P - Q$, by simple calculation, we can verify $B_L = O$ and $C_L = O$. By the equivalence between (a) and (c) in Theorem 18, item (a) holds.
6. Conclusion

In this paper, some characterizations of the BD-inverse are derived from certain matrices and EP matrix. Some representations of the BD-inverse are also given. Finally, we show the relationships between BD-inverse and other generalized inverse and give the definition of the BD-matrix. It is interesting to remark that analogous results can also be given in the case of generalized BD-inverse (see [11]) as well as in the setting of bounded linear operators. On a basis of the current research background, there are many topics on the BD-inverse which can be discussed. Some ideas are given as follows:

1. The solution of the restricted matrix equation
2. The iterative algorithm for computing the BD-inverse according to [25]
3. The perturbation analysis for the solution of restricted linear systems

Appendix

A Theorem Used in the Proof

Theorem A.1. Let A, B ∈ C^mn. Then,
\[ \mathcal{N}(AB) = B^t(\mathcal{R}(B) \cap \mathcal{N}(A)) \oplus \mathcal{N}(B), \] (A.1)

Proof. It is clear that B\mathcal{N}(AB) ⊂ \mathcal{R}(B) \cap \mathcal{N}(A). Let \forall q ∈ \mathcal{R}(B) \cap \mathcal{N}(A). Then, there exists k such that Bk = q. Since ABk = 0, then we have k ∈ \mathcal{N}(AB), which means \mathcal{R}(B) \cap \mathcal{N}(A) ⊂ B\mathcal{N}(AB). This prove that B\mathcal{N}(AB) = \mathcal{R}(B) \cap \mathcal{N}(A).

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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