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Research Article

On the Convergence Result of the Fractional Pseudoparabolic Equation

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In this paper, we consider the nonlinear fractional Laplacian pseudoparabolic equation (NFLPPE). We mainly focus on the convergence of mild solutions with respect to the order of fractional Laplacian. By using many techniques, we obtain the result that the mild solution will converge when the fractional order of the Laplacian tends to 1^-. The proof of convergent result relies on sharp techniques of evaluating the exponential terms, the Sobolev embeddings, and weakly singular Gronwall inequalities.

1. Introduction

In this paper, we consider a pseudoparabolic equation with fractional Laplacian defined as follows:

$$\begin{cases} v_t - k\Delta v_t + (-\Delta)^s v = G(v(x,t)), & \text{in } \Omega \times (0,T], \\ v(x,t) = 0, & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where Ω denotes the domain of the spatial variable x, $\partial\Omega$ is the boundary of Ω, and $k > 0$ is a constant coefficient. The function G is the nonlinear source term which appears in some physical phenomena and $v(x,t)$ describes the state of the unknown function generally at position x and time t. Our main goal in this paper is to study the convergence of the mild solution of problem 1.1 with the initial condition as follows:

$$v(x,0) = f(x), \quad x \in \Omega. \tag{2}$$

In this paper, for simplicity, we only consider $\Omega = (0,\pi)$ and so $\partial\Omega$ can be understood by the collection of 2 discrete points $\partial\Omega = \{0,\pi\}$. The notation $(-\Delta)^s$ is called fractional Laplacian with order $0 < s \neq 1$. From the abovementioned reasons, we can call this equation the "1-dimensional nonlinear fractional Laplacian pseudoparabolic equation."

As for $k = 0$, we obtain the classical nonlinear parabolic equation. Moreover, when $s = 0$, we obtain the ordinary pseudoparabolic equation. Both of these equations had been carefully studied by many researchers recently [1–3].

Pseudoparabolic equations have been studied extensively in recent years. It describes a variety of physical phenomena and also has applications in many different fields. One of the debates that have taken place in relation to equation (1) is about the local fractional operator $(-\Delta)^s$, in which many researchers believe that many physical phenomena are better described compared to the classical integral differential equation. For more information on this regard and the properties of the operator $(-\Delta)^s$, refer to references [4–9].

The study of fractional pseudoparabolic equations has always attracted the attention of many researchers because of their various applications in different fields, such as unidirectional propagation of long waves in a nonlinear dispersed medium, homogeneous liquid permeability in fractured rock, and heat conduction involving two temperatures [10, 11]. Let us mention some previous results on fractional pseudoparabolic equations. In [4], the researchers carried out on the fractional parabolic equation by considering the Cauchy problem of this equation in the whole space $\mathbb{R}^n$. In this work, the authors have investigated the global existence and time-decay rates for small-amplitude solutions. Some researchers have also studied the semilinear pseudoparabolic equation with Caputo derivative [1, 12], a final boundary value problem for a class of fractional pseudoparabolic with a nonlinear reaction term [3], and a nonlinear Kirchhoff's model of the pseudoparabolic type in references [13, 14]. In these works, first, the existence, uniqueness, and regularity of the mild local solution have been investigated. Next, the stability and regularity of the solution are studied and discussed. The main techniques and methods frequently used are the modified Lavrentiev regularization method and the Fourier truncated regularization method. To the fractional pseudoparabolic equation, sometimes, the inverse source problem is also discussed [15–17].

In [13], the authors considered a nonlinear Kirchhoff's model of pseudoparabolic type. They obtained the results on the local existence and regularity of mild solution. The authors also showed that the ill-posed property in the sense of Hadamard of the problem when the fractional order is larger than 1. By using the Fourier truncation method to regularize the problem, they established some stability estimates on the H^p norm under some a-priori conditions on the sought solution.

Recently, in [15], the authors focused on the source problem for the pseudoparabolic equation with fractional Laplacian. In this article, they also investigated the convergence of the source function when the fractional order tends to 1^-. There are not many results devoted to the convergence of mild solution $v(x,t)$ when the fractional order tends to 1^-. Motivated by the results in [15], we decided to study the fractional pseudoparabolic equation (1) and investigate the convergence of the state function $v(x,t)$ when the fractional order tends to 1^-.

In the following section, we present a brief overview of this work. The next section gives some preliminary knowledge on used notations, the spectral analysis of Laplacian the definition of fractional Laplacian, and some information about the functional space of interest. Section 3 is dedicated to the calculation of the explicit formula of mild solution to problem 1.1. Section 4 is to investigate the convergence of the mild solution concerning its fractional order in the Laplacian operator when s tends to 1^-. This result is important because of the relationship between the physical phenomena involved in equations when $s=1$ and $s<1$. By letting s tend to 1^-, we observe the association of subdiffusion phenomena with normal diffusion. The last section gives some discussion and proposes some directions for improving.

2. Preliminaries

2.1. The Less Than or Equivalent To Notation. Given two positive quantities y and z, we write $y \lesssim z$ if there exists a constant $C>0$ such that $y \leq Cz$.

2.2. Relevant Notation. Let us recall the following spectral problems:

$$\begin{cases} (-\Delta)e_j(x) = \lambda_j e_j(x), & \text{in } \Omega, \\ e_j(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

which admit a family of eigenvalues $0<\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots \nearrow \infty$.

We also notice that the collection of eigenfunction $e_j(x)$ could form an orthonormal basis of $L^2(\Omega)$. In this paper, the domain of the spatial variable is $\Omega = (0,\pi)$, and we can directly calculate the eigenvalues $\lambda_j = j^2$ for $j = 1,2,3,\ldots$ along with the eigenfunctions $e_j(x) = \sqrt{2/\pi}\sin(jx)$. But for more convenience, we sometimes reuse these symbols $e_j(x)$ and λ_j in next steps.

2.3. The Mittag–Leffler Function

Definition 1

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad z \in \mathbb{C}, \tag{4}$$

where $\alpha>0$ and $\beta \in \mathbb{R}$ are arbitrary constants and Γ is the Gamma function.

2.4. The Fractional Laplacian Operator and Inner Product. For $s \geq 0$, we define by $(-\Delta)^s$ the following operator:

$$(-\Delta)^s v := \sum_{j=1}^{\infty} \langle v, e_j\rangle \lambda_j^s e_j, \tag{5}$$

and the inner product is defined as follows:

$$\langle u(x), v(x)\rangle = \int_0^{\pi} u(x)v(x)\mathrm{d}x. \tag{6}$$

We recall the Hilbert scale space, which is given as follows:

$$H^s(\Omega) = \left\{ f \in L^2(\Omega) \middle| \sum_{j=1}^{\infty} \lambda_j^{2s}\left(\int_{\Omega} f(x)e_j(x)\mathrm{d}x\right)^2 < \infty \right\}, \tag{7}$$

for any $s \geq 0$. It is well known that $H^s(\Omega)$ is a Hilbert space corresponding to the norm as follows:

$$\|f\|_{H^s(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^{2s} \left(\int_{\Omega} f(x) e_j(x) \mathrm{d}x \right)^2 \right)^{1/2}, \quad f \in H^s(\Omega). \tag{8}$$

3. Mild Solution of the Problem 1.1 and Some Lemmas

Assume that the problem has a unique solution. Let

$$v(x,t) = \sum_{j=1}^{\infty} v_j(t) e_j(x), \text{with } v_j(t) = \langle v(.,t), e_j(\cdot)\rangle, \tag{9}$$

be the decomposition of $v(x,t)$ in $L^2(\Omega)$. From (3) and by taking the inner product with $e_j(x)$ to both sides of problem 1.1, we have

$$\begin{cases} \dfrac{\partial}{\partial t}\langle v(.,t), e_j\rangle + k.\lambda_j \dfrac{\partial}{\partial t}\langle v(.,t), e_j\rangle + \lambda_j^s \langle v(.,t), e_j\rangle = \langle G(.,t), e_j\rangle, \\ \langle v(.,0), e_j\rangle = \langle f, e_j\rangle. \end{cases} \tag{10}$$

The first equation of (10) is a differential equation with the classical derivative as follows:

$$\frac{\partial}{\partial t}\langle v(.,t), e_j\rangle + \frac{\lambda_j^s}{1+k\lambda_j}\langle v(.,t), e_j\rangle = \frac{1}{1+k\lambda_j}\langle G(.,t), e_j\rangle. \tag{11}$$

It is easy to see that the solution of it as follows:

$$\langle v(.,t), e_j\rangle = \exp\left(-\frac{\lambda_j^s}{1+k\lambda_j} t\right)\langle f, e_j\rangle + \frac{1}{1+k\lambda_j}\int_0^t \exp\left(\frac{\lambda_j^s}{1+k\lambda_j}(r-t)\right)\langle G(.,t), e_j\rangle \mathrm{d}r. \tag{12}$$

For simplicity, we denote $f_j = \langle f, e_j\rangle$ and $G_j = \langle G(.,t), e_j\rangle$ and obtain the formula as follows:

$$v_j(t) = \exp\left(-\frac{\lambda_j^s}{1+k\lambda_j} t\right) f_j + \frac{1}{1+k\lambda_j}\int_0^t \exp\left(\frac{\lambda_j^s}{1+k\lambda_j}(r-t)\right) G_j \mathrm{d}r. \tag{13}$$

Lemma 2. *The mild solution to NFLPPE (1) and (2) is given by the following formula:*

$$v_s(x,t) = \sum_j \exp\left(-\frac{j^{2s} t}{1+kj^2}\right) f_j e_j(x) + \sum_j \left[\frac{1}{1+kj^2}\int_0^t \exp\left(-\frac{j^{2s}(t-r)}{1+kj^2}\right) G_j(v_s(r)) \mathrm{d}r\right] e_j(x), \tag{14}$$

where

$$\begin{aligned} f_j &= \int_0^\pi f(x)e_j(x)\mathrm{d}x, \\ G_j(v_s(r)) &= \int_0^\pi G(v_s(x,r))e_j(x)\mathrm{d}x. \end{aligned} \tag{15}$$

Lemma 3. *The mild solution to NFLPPE (1) and (2), when $s = 1$ is given by the following formula:*

$$v^*(x,t) = \sum_j \exp\left(-\frac{j^2 t}{1+kj^2}\right) f_j e_j(x) + \sum_j \left[\frac{1}{1+kj^2}\int_0^t \exp\left(-\frac{j^2(t-r)}{1+kj^2}\right) G_j(v^*(r))\mathrm{d}r\right] e_j(x). \tag{16}$$

Lemma 4 (Weakly singular Gronwall's inequality, see [18]). *Let $v \in L^1[0,T]$ and assume that $A, B, \beta', \gamma' \in (0,\infty)$ with $\beta' + \gamma' > 1$, we have*

$$v(t) \leq A + B\int_0^t (t-r)^{\beta'-1} r^{\gamma'-1} v(r)\mathrm{d}r. \tag{17}$$

Thus, for $0 < t \leq T$, we conclude

$$v(t) \leq AE_{\beta',\gamma'}\left(B\left(\Gamma\left(\beta'\right)\right)^{1/\beta'+\gamma'-1} t\right). \tag{18}$$

4. Main Results

The existence result of the mild solution to problem 1.1 is widely and carefully discussed in Theorem 4.1 of [19] when the nonlinear term G is the global Lipschitz and satisfied some particular conditions, so we ignore that part and focus only in investigating the convergence of the mild solution while $s \longrightarrow 1^-$. The obtained result is fully presented by the following theorem.

Theorem 5. *Let G be the source function such that for any $K_g > 0$ and $v_1, v_2 \in L^2(\Omega)$,*

$$\left\|G(v_1) - G(v_2)\right\|_{L^2(\Omega)} \leq K_g \left\|v_1 - v_2\right\|_{L^2(\Omega)}. \tag{19}$$

We assume that $v^* \in L^\infty(0,T;L^2(\Omega))$ and $f \in \mathbb{H}^{(s+\theta-2)\mu+\rho}(\Omega)$, where $s, \theta, \mu, \rho > 0$ and $(s+\theta-2)\mu + \rho \leq 2$, where f is a Cauchy data. Then, we have

$$\left\|v_s(.,t) - v^*(.,t)\right\|_{\mathbb{H}^\rho(\Omega)} \lesssim \left(T^{1+2\mu} + k^{-\mu}\right)(1-s)^{\mu\theta}\left(\|f\|_{\mathbb{H}^{(s+\theta-2)\mu+\rho}(\Omega)} + \left\|v^*\right\|_{L^\infty(0,T;L^2(\Omega))}\right). \tag{20}$$

Proof. To start the proof, we consider (14) and (16). By subtracting these two, we have

$$\begin{aligned} &v_s(x,t) - v^*(x,t) \\ &= \sum_j \left(\exp\left(-\frac{j^{2s}t}{1+kj^2}\right) - \exp\left(-\frac{j^2 t}{1+kj^2}\right)\right) f_j e_j(x) \\ &+ \sum_j \frac{1}{1+kj^2}\left[\int_0^t \exp\left(-\frac{j^{2s}(t-r)}{1+kj^2}\right)\left(G_j(v_s(r)) - G_j(v^*(r))\right)\mathrm{d}r\right] e_j(x) \\ &+ \sum_j \frac{1}{1+kj^2}\left[\int_0^t \left(\exp\left(-\frac{j^{2s}(t-r)}{1+kj^2}\right) - \exp\left(-\frac{j^2(t-r)}{1+kj^2}\right)\right) G_j(v^*(r))\mathrm{d}r\right] e_j(x) \\ &= \mathscr{M}_1(x,t) + \mathscr{M}_2(x,t) + \mathscr{M}_3(x,t). \end{aligned} \tag{21}$$

In the sequel, we prove estimates for $\mathscr{M}_1$, $\mathscr{M}_2$, and $\mathscr{M}_3$, respectively. Therefore, first, we go to prove the boundedness of $\mathscr{M}_1$.

($\mathscr{M}_1$): from [15], we get that if $j \geq 1$, then we find that $j^{2s} - j^2 \leq C_\theta j^{s+\theta}(1-s)^\theta$, for any $\theta > 0$ and C_θ is the constant which depends on θ. For any $\mu > 0$, in view of the inequality $|e^{-m} - e^{-n}| \leq C_\mu |m-n|^\mu$, we obtain

$$\begin{aligned}\left|\exp\left(-\frac{j^{2s}t}{1+kj^2}\right) - \exp\left(-\frac{j^2 t}{1+kj^2}\right)\right| &\leq C_\mu t^\mu \left|j^{2s} - j^2\right|^\mu \left(1+kj^2\right)^{-\mu} \\ &\leq C(\mu,\theta) t^\mu j^{(s+\theta)\mu}(1-s)^{\mu\theta}\left(1+kj^2\right)^{-\mu} \\ &\leq C(\mu,\theta) k^{-\mu}(1-s)^{\mu\theta} j^{(s+\theta-2)\mu} t^\mu.\end{aligned} \tag{22}$$

Let us choose s, θ such that $s + \theta \leq 2$. Then, we obtain

$$\begin{aligned}&\left|\exp\left(-\frac{j^{2s}t}{1+kj^2}\right) - \exp\left(-\frac{j^2 t}{1+kj^2}\right)\right| \\ &\quad\leq C(\mu,\theta) k^{-\mu}(1-s)^{\mu\theta} t^\mu.\end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned}\left\|\mathscr{M}_1(.,t)\right\|^2_{\mathbb{H}^\rho(\Omega)} &= \sum_j j^{2\rho}\left(\exp\left(-\frac{j^{2s}t}{1+kj^2}\right) - \exp\left(-\frac{j^2 t}{1+kj^2}\right)\right)^2 \left|f_j\right|^2 \\ &\leq |C(\mu,\theta)|^2 k^{-2\mu}(1-s)^{2\mu\theta}\sum_j j^{2(s+\theta-2)\mu+2\rho}\left|f_j\right|^2 \\ &= |C(\mu,\theta)|^2 k^{-2\mu}(1-s)^{2\mu\theta}\|f\|^2_{\mathbb{H}^{(s+\theta-2)\mu+\rho}(\Omega)},\end{aligned} \tag{24}$$

and this means that $\mathscr{M}_1$ is bounded.

($\mathscr{M}_2$): Using the Hölder inequality, we find that

$$\begin{aligned}&\left\|\mathscr{M}_2(.,t)\right\|^2_{\mathbb{H}^\rho(\Omega)} \\ &\quad= \sum_j \frac{j^{2\rho}}{\left(1+kj^2\right)^2}\left[\int_0^t \exp\left(-\frac{j^{2s}(t-r)}{1+kj^2}\right)\left(G_j\left(v_s(r)\right) - G_j\left(v^*(r)\right)\right)\mathrm{d}r\right]^2 \\ &\quad\leq \sum_j \frac{j^{2\rho}}{\left(1+kj^2\right)^2}\int_0^t \exp\left(-\frac{2j^{2s}(t-r)}{1+kj^2}\right)\left(G_j\left(v_s(r)\right) - G_j\left(v^*(r)\right)\right)^2\mathrm{d}r.\end{aligned} \tag{25}$$

Due to the $e^{-z} \leq C_\varepsilon z^{-\varepsilon}$, we get that

$$\exp\left(-\frac{2j^{2s}(t-r)}{1+kj^2}\right) \leq \left|C_\varepsilon\right|^2\left(\frac{j^{2s}}{1+kj^2}\right)^{-2\varepsilon}(t-r)^{-2\varepsilon} = \left|C_\varepsilon\right|^2 j^{-4s\varepsilon}\left(1+kj^2\right)^{2\varepsilon}(t-r)^{-2\varepsilon}. \tag{26}$$

This implies that

$$\frac{j^{2\rho}}{\left(1+kj^2\right)^2}\exp\left(-\frac{2j^{2s}(t-r)}{1+kj^2}\right) \le \left|C_\varepsilon\right|^2 j^{2\rho-4s\varepsilon}\left(1+kj^2\right)^{2\varepsilon-2}(t-r)^{-2\varepsilon}. \tag{27}$$

Since $\rho \le 2\,s\varepsilon$ and $\varepsilon \le 1$, we obtain that

$$\begin{aligned}
&\left\|\mathscr{M}_2(.,t)\right\|^2_{\mathbb{H}^\rho(\Omega)}\\
&\le \sum_j \frac{j^{2\rho}}{\left(1+kj^2\right)^2}\int_0^t \exp\left(-\frac{2j^{2s}(t-r)}{1+kj^2}\right)\left(G_j\left(v_s(r)\right)-G_j\left(v^*(r)\right)\right)^2 \mathrm{d}r\\
&\le \left|C_\varepsilon\right|^2\int_0^t (t-r)^{-2\varepsilon}\left\|G\left(v_s(r)\right)-G\left(v^*(r)\right)\right\|^2_{L^2(\Omega)}\mathrm{d}r.
\end{aligned} \tag{28}$$

Since the global Lipschitz is G, we derive that

$$\begin{aligned}
&\int_0^t (t-r)^{-2\varepsilon}\left\|G\left(v_s(r)\right)-G\left(v^*(r)\right)\right\|^2_{L^2(\Omega)}\mathrm{d}r\\
&\le K_g\int_0^t (t-r)^{-2\varepsilon}\left\|v_s(r)-v^*(r)\right\|^2_{L^2(\Omega)}\mathrm{d}r\\
&\le K_g C_\rho\int_0^t (t-r)^{-2\varepsilon}\left\|v_s(r)-v^*(r)\right\|^2_{\mathbb{H}^\rho(\Omega)}\mathrm{d}r,
\end{aligned} \tag{29}$$

where we have used the Sobolev embedding $\mathbb{H}^\rho(\Omega)\hookrightarrow L^2(\Omega)$.

($\mathscr{M}_3$): applying the Hölder inequality, we infer that

$$\begin{aligned}
\left\|\mathscr{M}_3(.,t)\right\|^2_{\mathbb{H}^\rho(\Omega)} &= \sum_j \frac{j^{2\rho}}{\left(1+kj^2\right)^2}\times\left[\int_0^t\left(\exp\left(-\frac{j^{2s}(t-r)}{1+kj^2}\right)-\exp\left(-\frac{j^2(t-r)}{1+kj^2}\right)\right)G_j\left(v^*(r)\right)\mathrm{d}r\right]^2\\
&\le \sum_j \frac{j^{2\rho}}{\left(1+kj^2\right)^2}\times\int_0^t\left(\exp\left(-\frac{j^{2s}(t-r)}{1+kj^2}\right)-\exp\left(-\frac{j^2(t-r)}{1+kj^2}\right)\right)^2\left(G_j\left(v^*(r)\right)\right)^2\mathrm{d}r,
\end{aligned} \tag{30}$$

and using (22), we obtain that

$$\begin{aligned}
&\left|\exp\left(-\frac{j^{2s}(t-r)}{1+kj^2}\right)-\exp\left(-\frac{j^2(t-r)}{1+kj^2}\right)\right|\\
&\le C(\mu,\theta)k^{-\mu}(1-s)^{\mu\theta}j^{(s+\theta-2)\mu}(t-r)^\mu.
\end{aligned} \tag{31}$$

Thus, we obtain

$$\begin{aligned}
&\frac{j^{2\rho}}{\left(1+kj^2\right)^2}\left(\exp\left(-\frac{j^{2s}(t-r)}{1+kj^2}\right)-\exp\left(-\frac{j^2(t-r)}{1+kj^2}\right)\right)^2\left(G_j\left(v^*(r)\right)\right)^2\\
&\le C(\mu,\theta)k^{-2\mu-2}(1-s)^{2\mu\theta}j^{2(s+\theta-2)\mu+2\rho-4}(t-r)^{2\mu}.
\end{aligned} \tag{32}$$

Since $(s+\theta-2)\mu+\rho\le 2$, we obtain

$$\frac{j^{2\rho}}{\left(1+kj^2\right)^2}\left(\exp\left(-\frac{j^{2s}(t-r)}{1+kj^2}\right)-\exp\left(-\frac{j^2(t-r)}{1+kj^2}\right)\right)^2\left(G_j\left(v^*(r)\right)\right)^2$$
$$\leq C(\mu,\theta)k^{-2\mu-2}(1-s)^{2\mu\theta}(t-r)^{2\mu}. \tag{33}$$

From (30), we have

$$\begin{aligned}\left\|\mathscr{M}_3(.,t)\right\|^2_{\mathbb{H}^\rho(\Omega)} &\leq C(\mu,\theta)k^{-2\mu-2}(1-s)^{2\mu\theta}\int_0^t (t-r)^{2\mu}\left\|G\left(v^*(r)\right)\right\|^2_{L^2(\Omega)}\mathrm{d}r\\ &\leq C(\mu,\theta)K_g k^{-2\mu-2}(1-s)^{2\mu\theta}\int_0^t (t-r)^{2\mu}\left\|v^*(r)\right\|^2_{L^2(\Omega)}\mathrm{d}r\\ &\leq C\left(\mu,\theta,K_g,k\right)(1-s)^{2\mu\theta}\left\|v^*\right\|^2_{L^\infty(0,T;L^2(\Omega))}\left(\int_0^t (t-r)^{2\mu}\mathrm{d}r\right),\end{aligned} \tag{34}$$

where $C(\mu,\theta,K_g,k)$ indicates the constant which depends on μ,θ,K_g,k. It is easy to see that $\int_0^t (t-r)^{2\mu}\mathrm{d}r = t^{1+2\mu}/1+2\mu$. Hence, we infer that

$$\left\|\mathscr{M}_3(.,t)\right\|_{\mathbb{H}^\rho(\Omega)} \leq C\left(\mu,\theta,K_g,k\right)T^{1+2\mu}(1-s)^{\mu\theta}\left\|v^*\right\|_{L^\infty(0,T;L^2(\Omega))}. \tag{35}$$

Combining three steps as mentioned earlier, we derive that

$$\begin{aligned}\left\|v_s(.,t)-v^*(.,t)\right\|_{\mathbb{H}^\rho(\Omega)} &\leq \left\|\mathscr{M}_1(.,t)\right\|_{\mathbb{H}^\rho(\Omega)}+\left\|\mathscr{M}_2(.,t)\right\|_{\mathbb{H}^\rho(\Omega)}+\left\|\mathscr{M}_3(.,t)\right\|_{\mathbb{H}^\rho(\Omega)}\\ &\leq C\left(\mu,\theta,K_g,k\right)\left(T^{1+2\mu}+k^{-\mu}\right)(1-s)^{\mu\theta}\times\left(\|f\|_{\mathbb{H}^{(s+\theta-2)\mu+\rho}(\Omega)}+\left\|v^*\right\|_{L^\infty(0,T;L^2(\Omega))}\right)\\ &\quad+K_g C_\rho\int_0^t (t-r)^{-2\varepsilon}\left\|v_s(r)-v^*(r)\right\|^2_{\mathbb{H}^\rho(\Omega)}\mathrm{d}r.\end{aligned} \tag{36}$$

By Lemma 4 and equation (36), we infer that

$$\begin{aligned}v(t) &= \left\|v_s(.,t)-v^*(.,t)\right\|_{\mathbb{H}^\rho(\Omega)},\\ A &= C\left(\mu,\theta,K_g,k\right)\left(T^{1+2\mu}+k^{-\mu}\right)(1-s)^{\mu\theta}\times\left(\|f\|_{\mathbb{H}^{(s+\theta-2)\mu+\rho}(\Omega)}+\left\|v^*\right\|_{L^\infty(0,T;L^2(\Omega))}\right),\end{aligned} \tag{37}$$

and $B=K_gC_\rho,\beta'=1-2\varepsilon$ and $\gamma'=1$. Next, by applying Lemma 4, we derive that

$$\left\|v_s(.,t)-v^*(.,t)\right\|_{\mathbb{H}^\rho(\Omega)}\lesssim\left(T^{1+2\mu}+k^{-\mu}\right)(1-s)^{\mu\theta}\left(\|f\|_{\mathbb{H}^{(s+\theta-2)\mu+\rho}(\Omega)}+\left\|v^*\right\|_{L^\infty(0,T;L^2(\Omega))}\right). \tag{38}$$

□

5. Conclusion

Motivated by the result of [15], this work has considered the 1-dimensional fractional Laplacian pseudoparabolic equation with the nonlinear source term. By considering the problem included with an initial condition and some conditions in the source function, we showed the continuous dependence of mild solution to the fractional-order parameter.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest. The corresponding author is a full-time faculty member at the Iran University of science and technology.

Authors' Contributions

All the authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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