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Research Article

On New Solutions of Fuzzy Hybrid Differential Equations by Novel Approaches

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The goal of this paper is to find the best of two sixth-order methods, namely, RK-Huta and RK–Butcher methods for solving the fuzzy hybrid systems. We state a necessary definition and theorem in terms of consistency for convergence, and finally, we compare the obtained numerical results of two different methods with analytical solution using two different numerical examples. In addition to that, we generalize the solutions obtained by RK-6 Huta and RK-6 Butcher methods (same order different stage methods) for both the problems we handled. We are proposing these two methods in order to reduce the error in accuracy and to establish these two methods are better than any other existing numerical methods. The best of two sixth-order methods are found by the error analysis study for both the problems. Also, we show whether the change in number of stages of same order methods affects the accuracy of the approximation or not.

1. Introduction

The hybrid systems are the dynamic systems which involve both continuous and discrete actions. We shall discuss about the hybrid systems in detail. The term hybrid is not a new thing to the world. We are using the term hybrid everywhere knowingly or unknowingly. The botanist often used this term while some plants, fruits, and vegetables are produced by the technique of hybridization. This hybrid plantation is quite common in all the developing countries. The people are used to compare these hybrid products and original organic products though some of the hybrid products are even organic. The industrialists often use this term hybrid in making of innovative technologies. Nowadays, we are offered to use the hybrid cars which are making use of two different fuels as a combination of liquid fuel and electric motor. Now, in mathematics, it is used to call some functions as hybrid functions. The system which is involving two or more functions are termed as hybrid systems. The functions which are both continuous and discrete depending upon the interval of time being considered as hybrid functions. Sometimes, we have to call a function as a hybrid function when it exhibits continuous discontinuities such as modulus functions and trigonometric functions.

The hybrid system is often modeled with the aid of differential equations. Obviously, we can easily grasp that it should be a nonlinear equations. Since it is tedious to obtain the exact solutions, we prefer to apply the concept of numerical techniques to adopt the solutions. After achieving the approximate solutions, it is our prior most duty to assure the readers that our method is providing the better approximations. For that, we can take two different methods but one can easily say higher order method will automatically provide better approximation. So instead of taking two different order methods, we are using same sixth-order methods and going to compare them in order to find the method providing better approximations.

These dynamic behaviors with Zadeh's fuzzy theory [1] paved a way to fuzzy differential equations [2–5] and fuzzy hybrid differential equations (FHDEs) [5–15]. In the present paper, two sixth-order methods called RK-Huta and RK–Butcher, respectively, having eight stages and seven stages are used to obtain approximate solutions of FHDE. Since they are the higher-order methods, the solution converges rapidly to exact solutions than any other numerical methods. Numerical solutions of FHDE are studied over a period of time using various methods such as Euler and Runge–Kutta, by various authors like Pederson and Sambandham [16, 17] and Jayakumar and Kanagarajan [18, 19]. Other than them, Salahshour along with Allahviranloo and Ahmadian et al. made remarkable contributions in hybrid fuzzy differential equations [20, 21]. The readers are encouraged to go through the various applications of numerical methods to solve various types of differential equations through [4–15, 22–24].

We are eager to present one such study of FHDE for the benefit of the authors. Two different sixth-order methods such as RK-Huta and Bucher methods are presented and compared while solving these types of nonlinear differential equations.

The whole study was split as four sections in which FHDE and two RK-6 methods such as Huta and Butcher methods are presented in Section 2. Section 3 contains numerical examples, and finally, the study is concluded in Section 4.

2. Fuzzy Hybrid Differential Equations (FHDEs) and RK-6 Methods

The picture of the hybrid system is shown in Figure 1 in a way as P.B. Dhandapani et al. showed it in the discontinuity between continuity or continuity between discontinuity in [15].

Following the preliminaries of [16], the hybrid systems are treated via the continuous and discrete parameters.

E^n represents the set of $y_H\colon R^n \longrightarrow [0,1]$ in the following manner:

(i) y_H is always normal, as there exists an $t_0 \in R^n$ such that $y_H(t_0) = 1$

(ii) y_H is convex under fuzzy definitions, as for $t_1, t_2, y_H \in R^n$ and $0 \le \alpha \le 1$

$$y_H(\alpha t_1 + (1-\alpha)t_2) \ge \min\left[y_H(t_1), y_H(t_2)\right]. \tag{1}$$

(iii) y_H is always upper-semi continuous

(iv) $[y_H]^0 \equiv$ the closure of $[t \in R^n\colon y_H(t) > 0]$ is compact

For $0 < \alpha \le 1$, we define $[y_H(2t)]^\alpha = [t \in R^n\colon y_H(t) \ge \alpha]$. The $\alpha-$ level sets of $y_H(t)$ throughout the paper are given by the following equation:

$$y_H(t;\alpha) = \left[\underline{y}_H(t;\alpha), \overline{y}_H(t;\alpha)\right]. \tag{2}$$

Here, H is used to represent the association of hybrid systems in the system. Since we are dealing with fuzzy functions, we are defining below the minimum and maximum of $y_H(t)$, i.e.,

$$\begin{aligned}\underline{y}_H(t;\alpha) &= [0.75 + 0.25\alpha]y_H(t),\\ \overline{y}_H(t;\alpha) &= [1.125 - 0.125\alpha]y_H(t).\end{aligned} \tag{3}$$

Consider the FHDE, similar to [16, 18, 19]

$$\begin{cases}\Delta y_H(t) = \delta\left(t, y_H(t), \omega_k\left(y_{H_k}\right)\right), & t \in [t_k, t_{k+1}],\\ y_H(t_k) = y_{H_k},\end{cases}$$
$$\omega_k(u_k) = \begin{cases}\widehat{0}, & \text{if } k = 0,\\ u_k, & \text{if } k \in \{1,2,\ldots\},\end{cases} \tag{4}$$

where Δ denotes Seikkala's differentiation, $0 \le t_0 < t_1 < \cdots < t_r < \cdots, t_r \longrightarrow \infty$,

$$\delta \in C\left[R^+ \times E^1 \times E^1, E^1\right], \omega_r \in C\left[E^1, E^1\right]. \tag{5}$$

We may alter (4) by an equivalent system

$$\begin{cases}\Delta \underline{y}_H(t) = \underline{\delta}\left(t, y_H, \omega_k\left(y_{H_k}\right)\right) \equiv P_k\left(t, \underline{y_H}, \overline{y_H}\right), \underline{y_H}(t_k) = \underline{y_{H_k}},\\ \Delta \overline{y_H}(t) = \overline{\delta}\left(t, y_H, \omega_k\left(y_{H_k}\right)\right) \equiv Q_k\left(t, \underline{y_H}, \overline{y}_H\right), \overline{y_H}(t_k) = \overline{y_{H_k}},\end{cases} \tag{6}$$

$[\underline{y_H}(t;\alpha), \overline{y_H}(t;\alpha)]$ is a fuzzy number, and also the solutions of the parametric form given by the following equation:

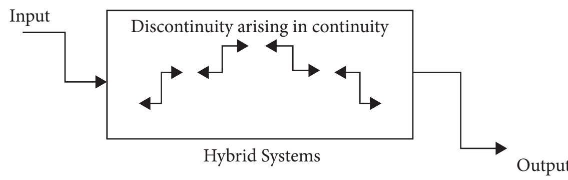

Figure 1: Hybrid systems.

$$\begin{cases} \Delta \underline{y_H}(t;\alpha) = P_k\left[t, \underline{y_{H_k}}(t;\alpha), \overline{y_H}(t;\alpha)\right], & \Delta \underline{y_H}(t_k;\alpha) = \underline{y_{H_k}}(\alpha), \\ \Delta \overline{y_H}(t;\alpha) = Q_k\left[t, \underline{y_{H_k}}(t;\alpha), \overline{y_H}(t;\alpha)\right], & \overline{y_H}(t_k;\alpha) = \overline{y_{H_k}}(\alpha), \alpha \in [0,1], \end{cases} \quad (7)$$

where $P_k[t, \underline{y_{H_k}}(t;\alpha), \overline{y_H}(t;\alpha)]$ and $Q_k[t, \underline{y_{H_k}}(t;\alpha), \overline{y_H}(t;\alpha)]$ are the parametric forms to represent the function.

2.1. Convergence. From the notes of [25], the general single-step method for (4) is given by $y_{m+1} = y_m + \phi(t_m, y_m, h)$, $m = 0, 1, \ldots, M-1$. Here, $\phi(t_m, y_m, h)$ is the increment function. The true value of $y(t_m)$ will satisfy $y(t_{m+1}) = y(t_m) + h\phi(t_m, y\ (t_m), h) + T_n, n = 0, 1, 2, \ldots, N-1$. Here, T_n is the truncation error.

Definition 1 (see [25]). The general single-step method $y_{m+1} = y_m + \phi(t_m, y_m, h), m = 0, 1, \ldots, M-1$ is said to be consistent if $\phi(t, y, 0) = \psi(t, y)$

Theorem 1 (see [25]). *A necessary and sufficient condition for the convergence of a single step method which is regular of order $p \geq 1$ is consistency.*

Proof 1. According to Jain [25],"there exist a unique solution $y(t)$ on $[t_0, h]$ where $a \leq t_0 \leq t \leq t_0 + h \leq b$ and also $y(t) \in C^{p+1}[t_0, b]$, for $p \geq 1$. The solution $y(t)$ can be expanded in a Taylor series about any point t_n.

$$\begin{aligned} y(t) &= y(t_n) + (t-t_n)\Delta y(t_n) + \frac{1}{2!}(t-t_n)^2\Delta^2 y(t_n) + \ldots \\ &\quad + \frac{1}{p!}(t-t_n)^p\Delta^p y(t_n) + \frac{1}{(p+1)}!(t-t_n)^2\Delta^{p+1}y(t_n)\epsilon_n f. \end{aligned} \quad (8)$$

This expansion holds good for $t \in [t_0, b], t_n < \epsilon < t$. Substituting $t = t_{n+1}$ in (8), we obtain the following equation:

$$y(t_{n+1}) = y(t_n) + h\Delta y(t_n) + \frac{h^2}{2}!\Delta^2 y(t_n) + \cdots + \frac{h^p}{p!}\Delta^p y(t_n), \quad (9)$$

and $h\phi(t_n, y(t_n), h)$ is to be obtained from $h\phi(t_n, y(t_n), h)$ by using an approximate value y_n in place of the exact value $y(t_n)$. We compute $y_{n+1} = y_n + h\phi(t_n, y_n, h), n = 0, 1, 2, \ldots, N-1$ to approximate $y(t_{n+1})$. This is called Taylor's series method of order p. When $p = 1$, the Taylor series method becomes Euler's method as $y_{n+1} = y_n + hf(t_n, y_n), n = 0, 1, 2, \ldots N-1$. The values of $\delta^2(y(t_n))$ and higher derivatives can be computed by substituting $t = t_n$. Therefore, we can compute $y(t_{n+1})$ with an error

$$\frac{h^{p+1}}{(p+1)!y^{(p+1)}(\epsilon_n)}, t_n < \epsilon_n < t_{n+1}. \quad (10)$$

By which consistency could be established since number of terms in Taylor's series is fixed by means of permissible error. From the theorem, the result ensures that the approximate solution converges to the exact solution since the convergence of Euler's method and Runge–Kutta method for hybrid fuzzy differential equations are proved by Pederson et al., [16, 17]. Also from the theorem of [16, 17], the point-to-point convergence for all k in α is fixed. □

2.2. Numerical Methods. In order to get better clarity about the terms involved in Runge–Kutta methods, we shall present here the fourth-order Runge–Kutta method. When the hybrid term involves, the representation will change accordingly which will be shown on following sixth-order RK-Huta method.

2.3. Fourth-Order Runge–Kutta Method for ODE. For non-fuzzy ODE,

$$\begin{cases} \Delta y_H(t) = \delta(t, y(t)), t \in [t_k, t_{k+1}], \\ y(t_k) = y_k, \end{cases} \quad (11)$$

Table 1: Sixth-order RK-Huta [25].

1/9	1/9							
1/6	1/24	3/24						
1/3	1/6	−3/6	4/6					
1/2	−5/8	27/8	−24/8	6/8				
2/3	221/9	−981/9	867/9	−102/9	1/9			
5/6	−183/48	678/48	−472/48	−66/48	80/48	3/48		
1	716/82	−2079/82	1002/82	834/82	−454/82	−9/82	72/82	
	41/840	0	216/840	27/840	272/840	27/840	216/840	41/840

Table 2: Sixth-order RK–Butcher [25].

1/3	1/3						
2/3	0	2/3					
1/3	1/12	1/3	−1/12				
1/2	−1/16	9/8	−3/16	−3/8			
1/2	0	9/8	−3/8	−3/4	1/2		
1	9/44	−9/11	63/44	18/11	0	−16/11	
	11/120	0	27/40	27/40	−4/15	−4/15	11/120

we have

$$\begin{aligned} k_1 &= h\delta(t, y_k),\\ k_2 &= h\delta\left(t+\frac{h}{2}, y_k+\frac{k_1}{2}\right),\\ k_3 &= h\delta\left(t+\frac{h}{2}, y_k+\frac{k_2}{2}\right),\\ k_3 &= h\delta(t+h, y_k+k_3),\\ y_{n+1} &= y_n+\frac{1}{6}(k_1+2k_2+2k_3+k_4), \end{aligned} \tag{12}$$

where $k_1, k_2, \ldots$ represents stages. y_n represents previous stage when y_{n+1} represents present iteration. The number of iterations to move from y_n to y_{n+1} is decided by the factor called step size h; i.e., if we take $h = 0.1.$, then to reach $y(1)$ from $y(0)$, we have 11 iterations such that $y(0), y(0.1), y(0.2), \ldots y(1)$.

Since it is too complex to show these two sixth-order methods in stage form involving terms like $y_{H_{k,n}}(\alpha)$, $y_{H_{k,n+1}}(\alpha)$, and k_i (which is explained previously), we are just giving the coefficients involved in these two sixth-order methods, namely, Huta method and Butcher method from the study of [25]. With the belief that the reader may be familiar with conversion from the coefficient form to the stage form of numerical methods, we present that coefficient terms involved in sixth-order RK–Butcher and RK-Huta methods. The readers may refer basic studies like [18, 25] given in reference. Tables 1 and 2 represent the Sixth Order RK-Huta and Butcher values, which can be expanded as the following equation:

$$\begin{aligned} \underline{y_{H_{k,n+1}}}(\alpha) - \underline{y_{H_{k,n}}}(\alpha) &= \sum_{i=1}^{8} v_i \underline{k}_i\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right),\\ \overline{y_{H_{k,n+1}}}(\alpha) - \overline{y_{H_{k,n}}}(\alpha) &= \sum_{i=1}^{8} v_i \overline{k}_i\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right), \end{aligned} \tag{13}$$

where $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7$, and ν_8 are constants and

$$
\begin{aligned}
&\underline{k}_1\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \min\Big\{h_k\delta\left(t_{k,n}, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), (\alpha)\right]\Big\},\\
&\overline{k}_1\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \max\Big\{h_k\delta\left(t_{k,n}, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\underline{k}_2\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \min\Big\{h_k\delta\left(t_{k,n} + \frac{1}{9}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_1}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_1}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\overline{k}_2\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \max\Big\{h_k\delta\left(t_{k,n} + \frac{1}{9}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_1}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_1}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\underline{k}_3\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \min\Big\{h_k\delta\left(t_{k,n} + \frac{1}{6}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_2}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_2}\left(t_{k,n}, y_{H_{k,n}}(r)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\overline{k}_3\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \max\Big\{h_k\delta\left(t_{k,n} + \frac{1}{6}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_2}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_2}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\underline{k}_4\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \min\Big\{h_k\delta\left(t_{k,n} + \frac{1}{3}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_3}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_3}\left(t_{k,n}, y_{H_{k,n}}(r)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\overline{k}_4\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \max\Big\{h_k\delta\left(t_{k,n} + \frac{1}{3}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_3}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_3}\left(t_{k,n}, y_{H_{k,n}}(r)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\underline{k}_5\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \min\Big\{h_k\delta\left(t_{k,n} + \frac{1}{2}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_4}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_4}\left(t_{k,n}, y_{H_{k,n}}(r)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\overline{k}_5\left(t_{k,n}; y_{H_{k,n}}(r)\right) = \max\Big\{h_k\delta\left(t_{k,n} + \frac{1}{2}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_4}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_4}\left(t_{k,n}, y_{H_{k,n}}(r)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\underline{k}_6\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \min\Big\{h_k\delta\left(t_{k,n} + \frac{2}{3}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_5}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_5}\left(t_{k,n}, y_{H_{k,n}}(r)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\underline{k}_6\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \min\Big\{h_k\delta\left(t_{k,n} + \frac{2}{3}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_5}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_5}\left(t_{k,n}, y_{H_{k,n}}(r)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\overline{k}_6\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \max\Big\{h_k\delta\left(t_{k,n} + \frac{2}{3}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_5}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_5}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\underline{k}_7\left(t_{k,n}; y_{H_{k,n}}(r)\right) = \min\Big\{h_k\delta\left(t_{k,n} + \frac{5}{6}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_6}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_6}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\qquad u \in \left[\underline{\zeta}_{k_6}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_6}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\overline{k}_7\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \max\Big\{h_k\delta\left(t_{k,n} + \frac{5}{6}h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_6}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_6}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\underline{k}_8\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \min\Big\{h_k\delta\left(t_{k,n} + h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_7}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_7}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},\\
&\overline{k}_8\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) = \max\Big\{h_k\delta\left(t_{k,n} + h_k, u, \omega_k(u_k)\right)\Big| \\
&\qquad u \in \left[\underline{\zeta}_{k_7}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \overline{\zeta}_{k_7}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\right], u_k \in \left[\underline{y_{H_{k,0}}}(\alpha), \overline{y_{H_{k,0}}}(\alpha)\right]\Big\},
\end{aligned}
\tag{14}
$$

Next, we are defining

$$
\begin{aligned}
\underline{\zeta}_{k_1}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \underline{y_{H_{k,n}}}(\alpha) + \frac{1}{9}\underline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\overline{\zeta}_{k_1}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \overline{y_{H_{k,n}}}(\alpha) + \frac{1}{9}\overline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\underline{\zeta}_{k_2}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \underline{y_{H_{k,n}}}(\alpha) + \frac{1}{24}\underline{k}_1\left(t_{k,n}, y_{H_{k,n}}(r)\right) + \frac{3}{24}\underline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\overline{\zeta}_{k_2}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \overline{y_{H_{k,n}}}(r) + \frac{3}{24}\overline{k}_1\left(t_{k,n}, y_{H_{k,n}}(r)\right) + \frac{3}{24}\overline{k}_2\left(t_{k,n}, y_{H_{k,n}}(r)\right),\\
\underline{\zeta}_{k_3}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \underline{y_{H_{k,n}}}(r) + \frac{1}{6}\underline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad - \frac{3}{6}\underline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{4}{6}\underline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\overline{\zeta}_{k_3}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \overline{y_{H_{k,n}}}(\alpha) + \frac{1}{6}\overline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad - \frac{3}{6}\overline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{4}{6}\overline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\underline{\zeta}_{k_4}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \underline{y_{H_{k,n}}}(\alpha) - \frac{5}{8}\underline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad + \frac{27}{8}\underline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) - \frac{24}{8}\underline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{6}{8}\underline{k}_4\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\overline{\zeta}_{k_4}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \overline{y_{H_{k,n}}}(\alpha) - \frac{5}{8}\overline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad + \frac{27}{8}\overline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) - \frac{24}{8}\overline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{6}{8}\overline{k}_4\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right),\\
\underline{\zeta}_{k_5}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \underline{y_{H_{k,n}}}(\alpha) + \frac{221}{9}\underline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) - \frac{981}{9}\underline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad + \frac{867}{9}\underline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) - \frac{102}{9}\underline{k}_4\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{1}{9}\underline{k}_5\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\overline{\zeta}_{k_5}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \overline{y_{H_{k,n}}}(\alpha) + \frac{221}{9}\overline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{981}{9}\overline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad + \frac{867}{9}\overline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{102}{9}\overline{k}_4\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \frac{1}{9}\overline{k}_5\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\underline{\zeta}_{k_6}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \underline{y_{H_{k,n}}}(\alpha) - \frac{183}{48}\underline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{678}{48}\underline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad - \frac{472}{48}\underline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) - \frac{66}{48}\underline{k}_4\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) + \frac{80}{48}\underline{k}_5\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad + \frac{3}{48}\underline{k}_6\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\overline{\zeta}_{k_6}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \overline{y_{H_{k,n}}}(\alpha) + \frac{183}{48}\overline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{678}{48}\overline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad + \frac{472}{48}\overline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{66}{48}\overline{k}_4\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right), \frac{80}{48}\overline{k}_5\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad + \frac{3}{48}\overline{k}_6\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right),\\
\underline{\zeta}_{k_7}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \underline{y_{H_{k,n}}}(\alpha) - \frac{716}{82}\underline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad - \frac{2079}{82}\underline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{1002}{82}\underline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{834}{82}\underline{k}_4\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right)\\
&\quad - \frac{454}{82}\underline{k}_5\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) - \frac{9}{82}\underline{k}_6\left(t_{k,n}; y_{H_{k,n}}(r)\right) + \frac{72}{82}\underline{k}_7\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right),\\
\overline{\zeta}_{k_7}\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) &= \overline{y_{H_{k,n}}}(\alpha) + \frac{716}{82}\overline{k}_1\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right)\\
&\quad - \frac{2079}{82}\overline{k}_2\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{1002}{82}\overline{k}_3\left(t_{k,n}, y_{H_{k,n}}(\alpha)\right) + \frac{834}{82}\overline{k}_4\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right)\\
&\quad - \frac{454}{82}\overline{k}_5\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) - \frac{9}{82}\overline{k}_6\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right) + \frac{72}{82}\overline{k}_6\left(t_{k,n}; y_{H_{k,n}}(\alpha)\right).
\end{aligned}
\tag{15}
$$

Next, we define

$$S_k\left[t_{k,n}, \underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right] = 41\underline{k}_1 + 0\underline{k}_2 + 216\underline{k}_3 + 27\underline{k}_4 + 272\underline{k}_5 + 27\underline{k}_6 + 216\underline{k}_7 + 41\underline{k}_8,$$
$$T_k\left[t_{k,n}, \underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right] = 41\overline{k}_1 + 0\overline{k}_2 + 216\overline{k}_3 + 27\overline{k}_4 + 272\overline{k}_5 + 27\overline{k}_6 + 216\overline{k}_7 + 41\overline{k}_8, \tag{16}$$

where $k_1, k_2, \ldots$ represents the stages involved in Runge–Kutta methods. The exact solution at $t_{k,n+1}$ is given by

$$\begin{cases} \underline{Y_{H_{k,n+1}}}(\alpha) \approx \underline{y_{H_{k,n}}}(\alpha) + \dfrac{1}{840} S_k\left[t_{k,n}, \underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right], \\ \overline{Y_{H_{k,n+1}}}(\alpha) \approx \overline{y_{H_{k,n}}}(\alpha) + \dfrac{1}{840} T_k\left[t_{k,n}, \underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right]. \end{cases} \tag{17}$$

The approximate solution is given by

$$\begin{cases} \underline{y_{H_{k,n+1}}}(\alpha) \approx \underline{y_{H_{k,n}}}(\alpha) + \dfrac{1}{840} S_k\left[t_{k,n}, \underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right], \\ \overline{y_{H_{k,n+1}}}(\alpha) \approx \overline{y_{H_{k,n}}}(\alpha) + \dfrac{1}{840} T_k\left[t_{k,n}, \underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right]. \end{cases} \tag{18}$$

In the similar fashion of RK-Huta expansion, we can also expand RK–Butcher method. As it is similar, we are not providing the expansion here. These methods are fuzzified and treated for fuzzy hybrid differential equations as followed by Pederson and Sambandham in [16, 17]. Then, from theory of [18, 19], we solve a numerical example from which these two RK-6 methods can be easily understood.

3. Numerical Example

Example 1. Similar to [16], the fuzzy hybrid IVP is taken.

$$\begin{cases} \Delta y_H(t) = y_H(t) + \xi_H(t)\omega_k y_H(t_k), \quad t \in [t_k, t_{k+1}], t_k = k, k = 0, 1, 2, 3, \ldots, \\ y_H(0; \alpha) = \left[\left(\dfrac{750}{1000} + \dfrac{250\alpha}{1000}\right), \left(\dfrac{1125}{1000} - \dfrac{125\alpha}{1000}\right)\right], \quad 0 \le \alpha \le 1, \end{cases} \tag{19}$$

where

$$\xi_H(t) = \begin{cases} \dfrac{10}{5}(1 - t(\text{mod } 1)), & \text{for } t(\text{mod } 1) > \dfrac{5}{10}, \\ \dfrac{10}{5}(t(\text{mod } 1)), & \text{for } t(\text{mod } 1) \le \dfrac{5}{10}, \end{cases}$$
$$\omega_k(u_k) = \begin{cases} \widehat{0}, & \text{if } k = 0, \\ u_k, & \text{if } k \in \{1, 2, \ldots\}. \end{cases} \tag{20}$$

In (19), $y_H(t_k) + \xi_H(t_k)\omega_k(y_H(t_k)$ is continuous function of t, y and $\omega_k(y_H(t_k)$.

$$\begin{cases} \Delta y_H(t) = y_H(t) + \xi_H(t)\omega_k(y(t_k)), \quad t \in [t_k, t_{k+1}], t_k = k, \\ y_H(t_k) = y_{Ht_k}, \end{cases} \tag{21}$$

$\Delta y_H(t)$ has a continuous solution on $[t_k, t_{k+1}]$.

3.1. Numerical Solution by RK–Butcher. For numerically solving the Fuzzy Hybrid IVP (19), let δ: $[0, \infty) \times R \times R \longrightarrow R$ be given by the following equation:

$$\delta(t, y_H, \omega_k(y_H(t_k))) = y_H(t) + \xi_H(t)\omega_k(y_H(t_k)),$$
$$t_k = k, k = 0, 1, 2, \ldots, \tag{22}$$

where ω_k: $R \longrightarrow R$ is given by the following equation:

$$\omega_k(y_H)(t_k) = \begin{cases} 0, & \text{if } k = 0, \\ y_H(t_k), & \text{if } k \in \{1, 2, \ldots\}. \end{cases} \tag{23}$$

By example 3 of [2], (19) gives

$$y_{H_1}\left(\frac{10}{10}, \alpha\right) = \left[\left(\frac{750}{1000} + \frac{250\alpha}{1000}\right)(D_{1,0})^{10}, \left(\frac{1125}{1000} - \frac{125\alpha}{1000}\right)(D_{1,0})^{10}\right]. \tag{24}$$

Now, we define

$$
\begin{aligned}
L_k\left[t_{k,n}, \underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right] &= 11\underline{k}_1 + 0\underline{k}_2 + 810\underline{k}_3 - 32\underline{k}_4 + 11\underline{k}_5,\\
M_k\left[t_{k,n}, \underline{y_{H_{k,n}}}(\alpha), \overline{y_{H_{k,n}}}(\alpha)\right] &= 11\overline{k}_1 + 0\overline{k}_2 + 810\overline{k}_3 - 32\overline{k}_4 + 11\overline{k}_5.
\end{aligned} \tag{25}
$$

From which, we obtain

$$
\begin{aligned}
\underline{y_{H_1}}\left(\frac{11}{10};\alpha\right) &= \underline{y_{H_1}}\left(\frac{10}{10};\alpha\right) + \frac{1}{120}L_k\left[\frac{10}{10}, \underline{y_{H_1}}\left(\frac{10}{10};\alpha\right), \overline{y_{H_1}}\left(\frac{10}{10};\alpha\right)\right],\\
\overline{y_{H_1}}\left(\frac{11}{10};\alpha\right) &= \overline{y_{H_1}}\left(\frac{10}{10};\alpha\right) + \frac{1}{120}M_k\left[\frac{10}{10}, \underline{y_{H_1}}\left(\frac{10}{10};\alpha\right), \overline{y_{H_1}}\left(\frac{10}{10};\alpha\right)\right].
\end{aligned} \tag{26}
$$

To obtain $y_{H_1}((20/10);\alpha)$,

$$
\begin{aligned}
D_{1,0} &= 1 + d + d^2 + \frac{d^3}{3} + \frac{d^4}{24} + \frac{d^5}{120} + \frac{d^6}{720} - \frac{d^7}{2160},\\
D_1 &= d^2 + \frac{d^3}{3} + \frac{d^4}{12} + \frac{d^5}{60} + \frac{d^6}{360} + \frac{d^7}{1080},\\
D_2 &= 3d^2 + \frac{4d^3}{3} + \frac{5d^4}{12} + \frac{6d^5}{60} + \frac{7d^6}{360} + \frac{8h^7}{540} - \frac{d^8}{1080},\\
D_3 &= 5d^2 + \frac{7d^3}{3} + \frac{3d^4}{4} + \frac{11d^5}{60} + \frac{13d^6}{360} + \frac{d^7}{216} - \frac{d^8}{540},\\
D_4 &= 7d^2 + \frac{10d^3}{3} + \frac{13d^4}{12} + \frac{4d^5}{60} + \frac{19d^6}{360} + \frac{d^7}{135} - \frac{d^8}{360},\\
D_5 &= 9d^2 + \frac{13d^3}{3} + \frac{17d^4}{12} + \frac{7d^5}{20} + \frac{5d^6}{72} + \frac{11d^7}{1080} - \frac{d^8}{270},\\
D_6 &= 2d - 10d^2 - 5d^3 - \frac{5d^4}{3} - \frac{5d^5}{12} - \frac{d^6}{12} - \frac{d^7}{72} + \frac{d^8}{216},\\
D_7 &= 2d - 12d^2 - 6d^3 - 2d^4 - \frac{d^5}{2} - \frac{d^6}{10} - \frac{d^7}{60} + \frac{d^8}{180},\\
D_8 &= 2d - 14d^2 - 7d^3 - \frac{7d^4}{3} - \frac{7d^5}{12} - \frac{7d^6}{60} - \frac{7d^7}{360} + \frac{7d^8}{1080},\\
D_9 &= 2d - 16d^2 - 8d^3 - \frac{8d^4}{3} - \frac{5d^5}{12} - \frac{d^6}{12} - \frac{d^7}{72} + \frac{d^8}{216},\\
D_{10} &= 2d - 10d^2 - 5d^3 - \frac{5d^4}{3} - \frac{5d^5}{12} - \frac{d^6}{12} - \frac{d^7}{72} + \frac{d^8}{216},\\
y_{H_1}\left(\frac{i}{10};\alpha\right) &= \left[\left(\frac{750}{1000} + \frac{250\alpha}{1000}\right)\left(D_{1,0}\right)^i, \left(\frac{1125}{1000} - \frac{125\alpha}{1000}\right)\left(D_{1,0}\right)^i\right], i = 1, 2, \ldots, 10,\\
y_{H_1}\left(\frac{i}{10};\alpha\right) &= \left(D_{1,0}\right)^i + \left(D_{1,0}\right)^{10}\left(D_{i-10}\right), i = 11, 12, \ldots, 20.
\end{aligned} \tag{27}
$$

3.2. Numerical Solution by RK-Huta. To numerically solve the Fuzzy Hybrid IVP (19),

$$\delta(t, y_H, \omega_r(y_H(t_k))) = y_H(t) + \xi_H(t)\omega_k(y_H(t_k)),\quad t_k = k, k = 0, 1, 2, \ldots, \tag{28}$$

where $\omega_k\colon R \longrightarrow R$ is given by the following equation:

$$\omega_k(u_k) = \begin{cases} 0, & \text{if } k = 0, \\ u_k, & \text{if } k \in \{1, 2, \ldots\}. \end{cases} \tag{29}$$

By example 3 of [2], (19) gives

$$y_{H_1}\left(\frac{10}{10}, \alpha\right) = \left[\left(\frac{750}{1000} + \frac{250\alpha}{1000}\right)(D_{1,0})^{10}, \left(\frac{1125}{1000} - \frac{125\alpha}{1000}\right)(D_{1,0})^{10}\right], \tag{30}$$

where

$$\begin{aligned} D_{1,0} &= 1 + d + \frac{d^2}{2} + \frac{d^3}{6} + \frac{d^4}{24} + \frac{d^5}{120} + \frac{d^6}{720} + \frac{d^7}{4480} + \frac{d^8}{483840}, \\ \underline{y_{H_1}}\left(\frac{11}{10}; \alpha\right) &= \underline{y_{H_1}}\left(\frac{10}{10}; \alpha\right) + \frac{1}{840} S_k\left[\frac{10}{10}, \underline{y_{H_1}}\left(\frac{10}{10}; \alpha\right), \overline{y_{H_1}}\left(\frac{10}{10}; \alpha\right)\right], \\ \overline{y_{H_1}}\left(\frac{11}{10}; \alpha\right) &= \overline{y_{H_1}}\left(\frac{10}{10}; \alpha\right) + \frac{1}{840} T_K\left[\frac{10}{10}; \underline{y_{H_1}}\left(\frac{10}{10}; \alpha\right), \overline{y_{H_1}}\left(\frac{10}{10}; \alpha\right)\right]. \end{aligned} \tag{31}$$

To obtain $y_{H_1}((20/10); \alpha)$, $i = 1, 2, 3, 4, 5$

$$\begin{aligned} \underline{y}_H\left(1 + \frac{i}{10}; \alpha\right) &= \underline{y}_H\left(1 + \frac{i-1}{10}; \alpha\right) D_{1,0} + \left[(2i-1)d^2 + \frac{(3i-2)h^3}{3} + \frac{(4i-3)d^4}{12} \right. \\ &\quad + \frac{(5i-4)d^5}{60} + \frac{(6i-1)d^6}{360} + \frac{(56i-47)d^7}{20160} + \frac{(108i-107)d^8}{241920} \\ &\quad \left. + \frac{(5i-5)d^9}{120960}\right] \underline{y}_H\left(\frac{10}{10}; \alpha\right), \\ \overline{y}_H\left(1 + \frac{i}{10}; \alpha\right) &= \overline{y}_H\left(1 + \frac{i-1}{10}; \alpha\right) D_{1,0} + \left[(2i-1)d^2 + \frac{(3i-2)d^3}{3} + \frac{(4i-3)d^4}{12} \right. \\ &\quad + \frac{(5i-4)d^5}{60} + \frac{(6i-1)d^6}{360} + \frac{(56i-47)d^7}{20160} + \frac{(108i-107)d^8}{241920} \\ &\quad \left. + \frac{(5i-5)d^9}{120960}\right] \overline{y}_H\left(\frac{10}{10}; \alpha\right). \end{aligned} \tag{32}$$

Then, for $i = 6, 7, 8, 9, 10$,

$$\underline{y}_H\left(1+\frac{i}{10};\alpha\right)=\underline{y}_H\left(1+\frac{i-1}{10};\alpha\right)D_{1,0}+\left[\frac{1}{5}-\left((2i-2)d^2+(i-1)d^3+\frac{(i-1)d^4}{3}+\frac{(i-1)d^5}{12}+\frac{(i-1)d^6}{60}+\frac{(i-1)d^7}{360}+\frac{(i-1)d^8}{2240}+\frac{(i-1)d^9}{241920}\right)\right]\underline{y}_H(10/10;\alpha),$$
$$\overline{y}_H\left(1+\frac{i}{10};\alpha\right)=\overline{y}_H\left(1+\frac{i-1}{10};\alpha\right)D_{1,0}+\left[\frac{1}{5}-\left((2i-2)h^2+(i-1)h^3+\frac{(i-1)h^4}{3}+\frac{(i-1)h^5}{12}+\frac{(i-1)h^6}{60}+\frac{(i-1)h^7}{360}+\frac{(i-1)h^8}{2240}+\frac{(i-1)h^9}{241920}\right)\right]\overline{y}_H\left(\frac{10}{10};\alpha\right). \tag{33}$$

3.3. Exact Solution. The analytically obtained solution of (19) for $t\in[1,1.5]$ is

$$y_H(t;\alpha)=y_H(1;\alpha)\left(3e^{t-1}-2t\right),0\le\alpha\le 1,$$
$$y_H(1.5;\alpha)=y_H(1;\alpha)(3\sqrt{e}-3),0\le\alpha\le 1. \tag{34}$$

Then, $y_H(1.5;1)$ is nearly approximate to 5.290221725637059, and $y_{H_1}(1.5,1)$ is approximately nearer to 5.290221725881617.

Since the exact solution of (19) for $t\in[1.5,2]$ is

$$y_H(t;\alpha)=y_H(1;\alpha)\left(2t-2+e^{t-1.5}(3\sqrt{e}-4)\right),0\le\alpha\le 1. \tag{35}$$

Therefore, $y_H(2.0;\alpha)=y_H(1;\alpha)(2+3e-4\sqrt{e})$.

Then, $y_H(2.0;1)$ is nearly approximate to 9.67697567235778, and $y_{H_1}(2.0;1)$ is nearly approximate to 9.676975672823584.

The approximate solution by RK-Huta is plotted at $t\in[0,2]$, $\alpha\in[0,1]$ (see Figure 2), and the error analysis has also been shown (see Table 3). The comparison of approximately obtained solutions by sixth-order methods and exact solutions are plotted at $t=2$, $\alpha\in[0,1]$ (see Tables 4 and 5 and Figure 3).

Next, consider the following hybrid fuzzy IVP.

Example 2

$$\begin{cases}\Delta y_H(t)=y(t)+\xi(t)\omega_k y_H(t_k), & t\in[t_k,t_{k+1}],t_k=k,k=0,1,2,3,\ldots,\\ y_H(0,\alpha)=[(0.75+0.25\alpha),(1.125-0.125\alpha)], & 0\le\alpha\le 1,\end{cases} \tag{36}$$

where $\xi(t)=|\sin(\pi t)|$, $\quad k=0,1,2,\ldots,$

$$\omega_k(u_k)=\begin{cases}0, & \text{if } k=0,\\ u_k, & \text{if } k\in\{1,2,\ldots\}.\end{cases} \tag{37}$$

Then, $y_H(t)+\xi_H(t)\omega_k(y_H(t_k)$ is continuous function of t, y and $\omega_k(y(t_k)$. Therefore, by example 6.1 of Kaleva [3], for each $k=0,1,2,\ldots$, the fuzzy IVP

$$\begin{cases}\Delta y_H(t)=y_H(t)+\xi_H(t)\omega_k(y_H(t_k),t\in[t_k,t_{k+1}],t_k=k,\\ y(t_k)=y_{Ht_k},\end{cases} \tag{38}$$

has a unique solution on $[t_k,t_{k+1}]$.

3.4. Numerical Solution by RK–Butcher. For numerically solving the hybrid fuzzy IVP (36), we will apply the RK–Butcher method of order six for hybrid fuzzy differential equations with $N=10$. To obtain $y_{H_1}(2.0,\alpha)$, $y_{H_1}(2.0;\alpha)$ is approximated. Let δ: $[0,\infty)\times R\times R\longrightarrow R$ be given by the following equation:

$$\delta(t,y_H,\omega_k(y_H(t_k)))=y_H(t)+\xi_H(t)\omega_k(y_H(t_k)),$$
$$t_k=k,k=0,1,2,\ldots, \tag{39}$$

where ω_k: $R\longrightarrow R$ is given by the following equation:

$$\omega_k(u_k)=\begin{cases}0, & \text{if } k=0,\\ u_k, & \text{if } k\in\{1,2,\ldots\}.\end{cases} \tag{40}$$

By example 3 of [2], (36) gives

$$y_{H_1}(1.0,\alpha)=\left[(0.75+0.25\alpha)\left(D_{1,0}\right)^{10},(1.125-0.125\alpha)\left(D_{1,0}\right)^{10}\right]. \tag{41}$$

Then,

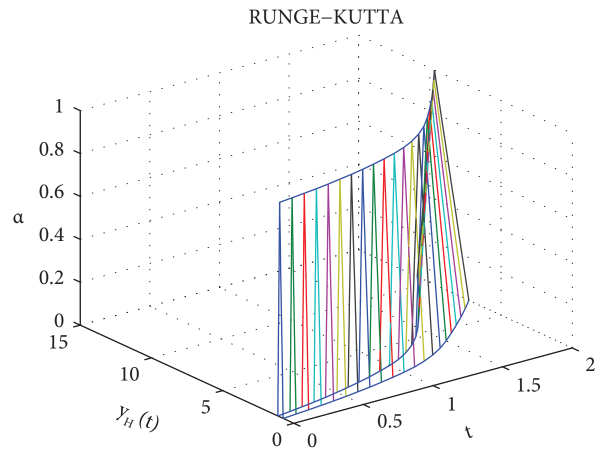

Figure 2: Approximate solution by sixth-order RK-Huta method (for $h = 0.1$) in Example 1.

Table 3: Approximate solutions by RK-Butcher and RK-Huta in Example 1.

α	Sixth-order RK–Butcher method		Sixth-order RK-Huta method	
	$\underline{y}_H(t_i;\alpha)$	$\overline{y}_H(t_i;\alpha)$	$\underline{y}_H(t_i;\alpha)$	$\overline{y}_H(t_i;\alpha)$
0	7.25773174395989	10.8865976159398	7.25773175461769	10.8865976319265
0.1	7.49965613542523	10.7656354202072	7.49965614643828	10.7656354360162
0.2	7.74158052689056	10.6446732244745	7.74158053825887	10.6446732401059
0.3	7.98350491835588	10.5237110287418	7.98350493007946	10.5237110441956
0.4	8.22542930982121	10.4027488330092	8.22542932190005	10.4027488482854
0.5	8.46735370128655	10.2817866372765	8.46735371372064	10.2817866523751
0.6	8.70927809275187	10.1608244415439	8.70927810554123	10.1608244564648
0.7	8.95120248421720	10.0398622458112	8.95120249736182	10.0398622605545
0.8	9.19312687568253	9.91890005007852	9.19312688918240	9.91890006464417
0.9	9.43505126714786	9.79793785434586	9.43505128100300	9.79793786873388
1.0	9.67697565861319	9.67697565861319	9.67697567282358	9.67697567282358

Table 4: Exact solution in Example 1.

α	Exact solution	
	$\underline{y}_H(t_i;\alpha)$	$\overline{y}_H(t_i;\alpha)$
0	7.25773175426834	10.8865976314025
0.1	7.49965614607728	10.7656354354980
0.2	7.74158053788623	10.6446732395936
0.3	7.98350492969517	10.5237110436891
0.4	8.22542932150411	10.4027488477846
0.5	8.46735371331306	10.2817866518801
0.6	8.70927810512201	10.1608244559757
0.7	8.95120249693095	10.0398622600712
0.8	9.19312688873989	9.91890006416673
0.9	9.43505128054884	9.79793786826225
1.0	9.67697567235778	9.67697567235778

Table 5: Error in sixth-order RK-Butcher method and sixth-order RK-Huta method in Example 1.

α	Sixth-order RK-Butcher method		Sixth-order RK-Huta method	
	$\underline{Y}_H(t_i;\alpha)$	$\overline{Y}_H(t_i;\alpha)$	$\underline{Y}_H(t_i;\alpha)$	$\overline{Y}_H(t_i;\alpha)$
0	1.03085×10^{-8}	1.54627×10^{-8}	3.49350×10^{-10}	5.24000×10^{-10}
0.1	1.06520×10^{-8}	1.52908×10^{-8}	3.61000×10^{-10}	5.18201×10^{-10}
0.2	1.09957×10^{-8}	1.51191×10^{-8}	3.72641×10^{-10}	5.12301×10^{-10}
0.3	1.13393×10^{-8}	1.49473×10^{-8}	3.84290×10^{-10}	5.06500×10^{-10}
0.4	1.16829×10^{-8}	1.47754×10^{-8}	3.95939×10^{-10}	5.00799×10^{-10}

Table 5: Continued.

α	Sixth-order RK-Butcher method		Sixth-order RK-Huta method	
	$\underline{Y}_H(t_i;\alpha)$	$\overline{Y}_H(t_i;\alpha)$	$\underline{Y}_H(t_i;\alpha)$	$\overline{Y}_H(t_i;\alpha)$
0.5	1.20265×10^{-8}	1.46036×10^{-8}	4.07580×10^{-10}	4.95000×10^{-10}
0.6	1.23701×10^{-8}	1.44318×10^{-8}	4.19220×10^{-10}	4.89100×10^{-10}
0.7	1.27138×10^{-8}	1.42600×10^{-8}	4.30870×10^{-10}	4.83301×10^{-10}
0.8	1.30574×10^{-8}	1.40882×10^{-8}	4.42510×10^{-10}	4.77440×10^{-10}
0.9	1.34010×10^{-8}	1.39164×10^{-8}	4.54159×10^{-10}	4.71630×10^{-10}
1.0	1.37446×10^{-8}	1.37446×10^{-8}	4.65800×10^{-10}	4.65800×10^{-10}

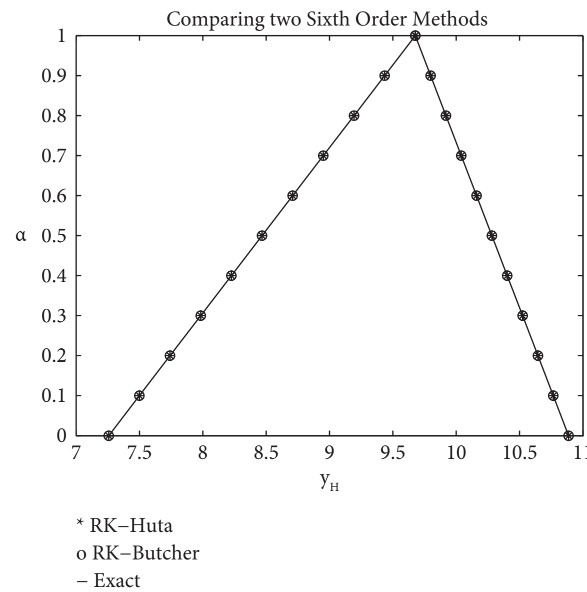

Figure 3: Comparison of approximate solution with exact solution (for $h = 0.1$) in Example 1.

$$\begin{aligned}
\underline{y_{H_1}}(1.1;\alpha) &= \underline{y_{H_1}}(1.0;\alpha) + \frac{1}{120}L_k\left[1.0, \underline{y_{H_1}}(1.0;\alpha), \overline{y_{H_1}}(1.0;\alpha)\right],\\
\overline{y_{H_1}}(1.1;\alpha) &= \overline{y_{H_1}}(1.0;\alpha) + \frac{1}{120}M_k\left[1.0; \underline{y_{H_1}}(1.0;\alpha), \overline{y_{H_1}}(1.0;\alpha)\right].
\end{aligned} \tag{42}$$

To obtain $y_{H_1}(2.0;\alpha)$ for $h = 0.1$, let

$$\begin{aligned}
&y_{H_1}\left(\frac{i}{10};\alpha\right) = \left(D_{1,0}\right), i = 1, 2, \ldots, 10\\
&y_{H_1}\left(\frac{i}{10};\alpha\right) = \left(D_{1,0}\right)^i + \left(D_{1,0}\right)^{10}\left(\sin\left(1 + \frac{h}{3}\right)\pi\right)(A_1) - \left(\sin\left(1 + \frac{h}{2}\right)\pi\right)(A_2) + \left(\sin\left(1 + \frac{2h}{3}\right)\pi\right)(A_3) + (\sin(1 + h)\pi)(A_4) i = 11\\
&y_1\left(\frac{i}{10};\alpha\right) = \left(D_{1,0}\right)^i + \left(\left(D_{1,0}\right)^{10}\right)(D_{i-10}) i = 12, 13, \ldots, 20\\
&\quad D_{i-10} = \sin((1 + (i - 11)h)\pi)B_1 + \sin\left(\left(1 + \frac{((i - 8 + 2j)h)}{3}\right)\pi\right)B_2 + \sin\left(\left(1 + \frac{((i - 9 + j)h)}{3}\right)\pi\right)B_3 +\\
&\qquad \sin\left(\left(1 + \frac{((i - 8 + 2j)h)}{3}\right)\pi\right)A_3 + \sin((1 + (i - 10)h)\pi)A_4 \text{ for } i = i = 11, 12, \ldots, 20, j = i - 12,
\end{aligned} \tag{43}$$

for $i = 11, 12, \ldots, 20$, $j = i - 12$ where

$$A_1 = \frac{27h}{40} + \frac{9h^2}{20} + \frac{3h^3}{20} + \frac{h^4}{40} + \frac{h^5}{120} + \frac{h^6}{720}$$
$$A_2 = \frac{8h}{15} + \frac{4h^2}{15} + \frac{h^3}{15}$$
$$A_3 = \frac{27h}{40} + \frac{9h^2}{40} + \frac{3h^3}{80} - \frac{h^5}{480}$$
$$A_4 = \frac{11h}{120} \tag{44}$$
$$B_1 = \frac{11h}{120} + \frac{11h^2}{120} + \frac{11h^3}{240} + \frac{h^4}{60} + \frac{h^5}{480} + \frac{h^6}{360} + \frac{h^7}{2160}$$
$$B_2 = \frac{27h}{40} + \frac{9h^2}{40} + \frac{3h^3}{80} + \frac{h^4}{40} + \frac{h^5}{120} - \frac{h^6}{720}$$
$$B_3 = \frac{-8h}{15} - \frac{4h^2}{15} - \frac{h^3}{15}$$

3.5. Numerical Solution by RK-Huta. For numerically solving the hybrid fuzzy IVP (36), we will apply the RK-Huta method of order six for hybrid fuzzy differential equations with $N = 10$. To obtain $y_1(2.0;\alpha)$, $y(2.0;\alpha)$ is approximated. Let δ: $[0,\infty) \times R \times R \longrightarrow R$ be given by the following equation:

$$\delta(t, y_H, \omega_k(y(t_k))) = y(t) + \xi(t)\omega_k(y(t_k)),\quad t_k = k, k = 0, 1, 2, \ldots, \tag{45}$$

where ω_k: $R \longrightarrow R$ is given by the following equation:

$$\omega_k(u_k) = \begin{cases} 0, & \text{if } k = 0,\\ u_k, & \text{if } k \in \{1, 2, \ldots\}. \end{cases} \tag{46}$$

By example 3 of [2], (36) gives

$$y_{H_1}(1.0,\alpha) = \left[(0.75 + 0.25\alpha)\left(D_{1,0}\right)^{10}, (1.125 - 0.125\alpha)\left(D_{1,0}\right)^{10}\right], \tag{47}$$

where

$$D_{1,0} = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{720} + \frac{h^7}{4480} + \frac{h^8}{483840},$$
$$\underline{y_{H_1}}(1.1;\alpha) = \underline{y}_1(1.0;\alpha) + \frac{1}{840} S_k\left[1.0, \underline{y}_1(1.0;\alpha), \overline{y_{H_1}}(1.0;\alpha)\right], \tag{48}$$
$$\overline{y}_1(1.1;\alpha) = \overline{y}_1(1.0;\alpha) + \frac{1}{840} T_k\left[1.0; \underline{y}_1(1.0;\alpha), \overline{y}_1(1.0;\alpha)\right].$$

To obtain $y_1(2.0;\alpha)$ for $h = 0.1$, let

$$\begin{aligned}
D_1 &= 23616h + 23616h^2 + 5328h^3 + 7920h^4 + 1872h5 - 264h6 + 99h^7 + h^8,\\
D_2 &= -5292h^5 + 864h^6 + 99h^7,\\
D_3 &= 124416h + 103680h^2 + 67392h^3 + 432h^4 + 7344h^5 + 72h^6,\\
D_4 &= 15552h + 103680h^2 - 28512h^3 + 11664h^4 + 108h^5,\\
D_5 &= 156672h + 78336h^2 + 35136h^3 + 144h^4,\\
D_6 &= 15552h + 5184h^2 + 1296h^3,\\
D_7 &= 124416h + 20736h^2,\\
D_8 &= 23616h,\\
\underline{y}(1.1;\alpha) &= \underline{y}(1.0;\alpha)\left(D_{1,0}\right) + \frac{1}{483840}\left[D_2 \sin\frac{\pi}{90} + D_3 \sin\frac{\pi}{60} + D_4 \sin\frac{\pi}{30},\right.\\
&\quad \left. + D_5 \sin\frac{\pi}{20} + D_6 \sin\frac{2\pi}{30} + D_7 \sin\frac{5\pi}{60} + D_8 \sin\frac{\pi}{10}\right]\underline{y}(1.0;\alpha),\\
\overline{y}(1.1;\alpha) &= \overline{y}(1.0;\alpha)\left(D_{1,0}\right) + \frac{1}{483840}\left[D_2 \sin\frac{\pi}{90} + D_3 \sin\frac{\pi}{60} + D_4 \sin\frac{\pi}{30},\right.\\
&\quad \left. + D_5 \sin\frac{\pi}{20} + D_6 \sin\frac{2\pi}{30} + D_7 \sin\frac{5\pi}{60} + D_8 \sin\frac{\pi}{10}\right]\overline{y}(1.0;\alpha).
\end{aligned} \tag{49}$$

Then, for $i = 1, 2, 3, \ldots, 10$,

$$\begin{aligned}\underline{y}\left(1+\frac{i}{10};\alpha\right) &= \underline{y}\left(1+\frac{i-1}{10};\alpha\right)\left(D_{1,0}\right)+\frac{1}{483840}\left[D_1\sin\frac{(i-1)\pi}{10}+D_2\sin\frac{(9i-8)\pi}{90}\right.\\ &\quad +D_3\sin\frac{(6i-5)\pi}{60}+D_4\sin\frac{(3i-2)\pi}{30}+D_5\sin\frac{(2i-1)\pi}{20}\\ &\quad \left.+D_6\sin\frac{(3i-1)\pi}{30}+D_7\sin\frac{(6i-1)\pi}{60}+D_8\sin\frac{(i)\pi}{10}\right]\underline{y}\,(1.0;\alpha),\\ \overline{y}\left(1+\frac{i}{10};\alpha\right) &= \overline{y}\left(1+\frac{i-1}{10};\alpha\right)\left(D_{1,0}\right)+\frac{1}{483840}\left[D_1\sin\frac{(i-1)\pi}{10}+D_2\sin\frac{(9i-8)\pi}{90}\right.\\ &\quad +D_3\sin\frac{(6i-5)\pi}{60}+D_4\sin\frac{(3i-2)\pi}{30}+D_5\sin\frac{(2i-1)\pi}{20}\\ &\quad \left.+D_6\sin\frac{(3i-1)\pi}{30}+D_7\sin\frac{(6i-1)\pi}{60}+D_8\sin\frac{(i)\pi}{10}\right]\overline{y}\,(1.0;\alpha).\end{aligned} \tag{50}$$

Let

$$\begin{aligned}D_{2,0} &= \left(D_{1,0}\right)^{10}+\sum_{k=1}^{10}\left(D_{1,0}\right)^{10-k}\frac{1}{483840}\left[D_1\sin\frac{(k-1)\pi}{10}+D_2\sin\frac{(9k-8)\pi}{90}\right.\\ &\quad +D_3\sin\frac{(6k-5)\pi}{60}+D_4\sin\frac{(3k-2)\pi}{30}+D_5\sin\frac{(2k-1)\pi}{20}\\ &\quad \left.+D_6\sin\frac{(3k-1)\pi}{30}+D_7\sin\frac{(6k-1)\pi}{60}+D_8\sin\frac{(k)\pi}{10}\right].\end{aligned} \tag{51}$$

Then,

$$\begin{aligned}y_{2.0;\alpha} &= D_{2,0}\,y_1\,(1.0;\alpha)\\ &= \left[D_{2,0}\,(0.75+0.25\alpha)\left(D_{1,0}\right)^{10}, D_{2,0}\,(1.125-0.125\alpha)\left(D_{1,0}\right)^{10}\right], 0\le\alpha\le 1,\end{aligned} \tag{52}$$

for $t \in [1, 2]$.

3.6. Exact Solution. The exact solution of (36) satisfies

$$\begin{aligned}\underline{y}_H\,(t;\alpha) &= \underline{y}_H\,(1;\alpha)\left[\frac{\pi\cos(\pi t)+\sin(\pi t)}{\pi^2+1}+e^{t-1}\left(1+\frac{\pi}{\pi^2+1}\right)\right],\\ \overline{y}_H\,(t;\alpha) &= \overline{y}_H\,(1;\alpha)\left[\frac{\pi\cos(\pi t)+\sin(\pi t)}{\pi^2+1}+e^{t-1}\left(1+\frac{\pi}{\pi^2+1}\right)\right].\end{aligned} \tag{53}$$

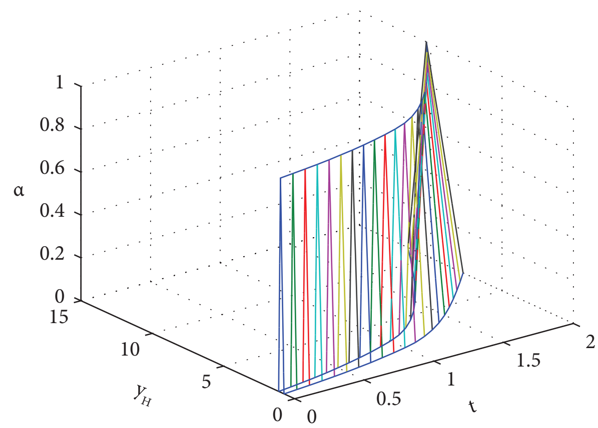

Figure 4: Approximate solution by sixth-order RK-Huta method (for $h = 0.1$) in Example 2.

Table 6: Approximate solutions by RK–Butcher and RK-Huta in Example 2.

α	Sixth-order RK–Butcher method		Sixth-order RK-Huta method	
	$\underline{y}_H(t_i;\alpha)$	$\overline{y}_H(t_i;\alpha)$	$\underline{y}_H(t_i;\alpha)$	$\overline{y}_H(t_i;\alpha)$
0	7.73275073110805	11.5991260966621	7.73275073776977	11.5991261066547
0.1	7.99050908881165	11.4702469178103	7.99050909569543	11.4702469276918
0.2	8.24826744651526	11.3413677389585	8.24826745362109	11.3413677487290
0.3	8.50602580421886	11.2124885601067	8.50602581154675	11.2124885697662
0.4	8.76378416192246	11.0836093812549	8.76378416947241	11.0836093908033
0.5	9.02154251962606	10.9547302024031	9.02154252739807	10.9547302118405
0.6	9.27930087732966	10.8258510235513	9.27930088532373	10.8258510328777
0.7	9.53705923503326	10.6969718446995	9.53705924324938	10.6969718539149
0.8	9.79481759273686	10.5680926658477	9.79481760117504	10.5680926749520
0.9	10.0525759504405	10.4392134869959	10.0525759591007	10.4392134959892
1.0	10.3103343081441	10.3103343081441	10.3103343170264	10.3103343170264

Table 7: Exact solution in Example 2.

α	Exact solution	
	$\underline{Y}_H(t_i;\alpha)$	$\overline{Y}_h(t_i;\alpha)$
0	7.73275073803317	11.5991261070497
0.1	7.99050909596760	11.4702469280825
0.2	8.24826745390204	11.3413677491153
0.3	8.50602581183648	11.2124885701481
0.4	8.76378416977092	11.0836093911809
0.5	9.02154252770536	10.9547302122137
0.6	9.27930088563980	10.8258510332464
0.7	9.53705924357424	10.6969718542792
0.8	9.79481760150867	10.5680926753120
0.9	10.0525759594431	10.4392134963448
1.0	10.3103343173776	10.3103343173776

Table 8: Error in sixth-order RK-Butcher method and sixth-order RK-Huta method in Example 2.

α	Sixth-order RK–Butcher method		Sixth-order RK-Huta method	
	$\underline{Y}_H(t_i;\alpha)$	$\overline{Y}_H(t_i;\alpha)$	$\underline{Y}_H(t_i;\alpha)$	$\overline{Y}_h(t_i;\alpha)$
0	6.92512×10^{-9}	1.03876×10^{-8}	2.63400×10^{-10}	3.95001×10^{-10}
0.1	7.15595×10^{-9}	1.02722×10^{-8}	2.72171×10^{-10}	3.90699×10^{-10}
0.2	7.38678×10^{-9}	1.01568×10^{-8}	2.80950×10^{-10}	3.86299×10^{-10}
0.3	7.61762×10^{-9}	1.00414×10^{-8}	2.89729×10^{-10}	3.81910×10^{-10}

Table 8: Continued.

α	Sixth-order RK–Butcher method		Sixth-order RK-Huta method	
	$\underline{Y}_H(t_i;\alpha)$	$\overline{Y}_H(t_i;\alpha)$	$\underline{Y}_H(t_i;\alpha)$	$\overline{Y}_h(t_i;\alpha)$
0.4	7.84846×10^{-9}	9.92600×10^{-9}	2.98510×10^{-10}	3.77600×10^{-10}
0.5	8.07930×10^{-9}	9.81060×10^{-9}	3.07290×10^{-10}	3.73200×10^{-10}
0.6	8.31014×10^{-9}	9.69510×10^{-9}	3.16071×10^{-10}	3.68699×10^{-10}
0.7	8.54098×10^{-9}	9.57970×10^{-9}	3.24858×10^{-10}	3.64300×10^{-10}
0.8	8.77181×10^{-9}	9.46430×10^{-9}	3.33630×10^{-10}	3.60000×10^{-10}
0.9	9.00260×10^{-9}	9.34890×10^{-9}	3.42400×10^{-10}	3.55600×10^{-10}
1.0	9.23350×10^{-9}	9.23350×10^{-9}	3.51200×10^{-10}	3.51200×10^{-10}

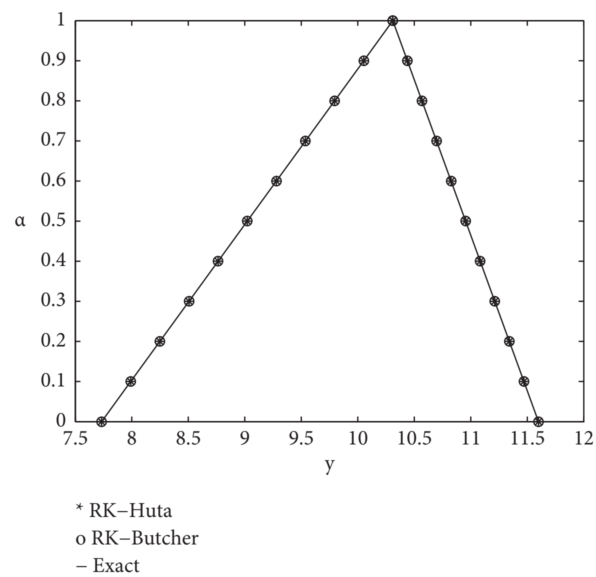

Figure 5: Comparison of approximate solution with exact solution (for $h = 0.1$) in Example 2.

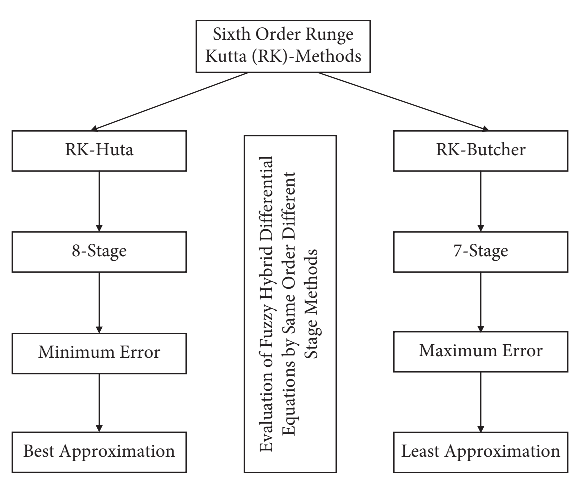

Figure 6: Flowchart of Conclusion: maximum stage maximum accuracy.

Therefore,

$$y(1;\alpha) = [(0.75 + 0.25\alpha)e, (1.125 - 0.125\alpha)e],$$
$$y(2;\alpha) = \left(\frac{\pi}{\pi^2 + 1} + e\left(1 + \frac{\pi}{\pi^2 + 1}\right)\right)y(1;\alpha). \quad (54)$$

Then, $y(2.0;1)$ is nearly approximate to 10.310334317377553, whereas $y_1(2.0;1)$ is nearly approximate to 10.310334317026362. The approximate solution by RK-Huta is plotted at $t \in [0,2], \alpha \in [0,1]$ (see Figure 4), and the error analysis has also been shown (see Table 6). The comparison of approximately obtained solutions by sixth order methods and exact solutions are plotted at $t = 2, \alpha \in [0,1]$ (see Tables 7 and 8 and Figure 5).

4. Conclusion

We can show our conclusion simply via the flow chart in Figure 6.

For clarifying the readers about convergence of numerical results, we stated the theorem by means of consistency. We solved famous-two problems of fuzzy hybrid systems and found numerical solution by sixth order eight stage RK-Huta method and sixth order seven stage RK–Butcher method and generalized them for both the problems by which the future readers can extend the numerical solution to next stage even without solving the problem. Comparison of solutions shows that sixth order RK-Huta method gives better results than sixth order RK–Butcher method for solving any fuzzy hybrid differential equations by the application of error analysis study (see Tables 3 and Table 6). As a part of our study, we are also arriving at the following results:

(1) When comparing two numerical methods of different order, the higher order will give better accuracy. For example, fourth order Runge–Kutta method will give better accuracy of approximation than Euler method.

(2) When comparing two numerical methods of same order, the higher stage will give better accuracy. The result was obtained by our research in this paper as we had shown sixth order 8 stage method (RK-Huta method) gives better accuracy than sixth order 7 stage method (RK–Butcher method) in numerical solutions.

(3) Previously many authors did their work on fuzzy hybrid systems found big errors in accuracy [16–21]. These errors are highly reduced by the above-given two sixth-order methods.

(4) As far as the numerical solution is concerned, these two methods are better than any other existing numerical methods, in fact RK-Huta of order 6-stage 8 is still best from our study but the thing may change when compare them with approximate analytical methods which will be our future work.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this paper and typed, read, and approved the final manuscript.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] M. Ma, M. Friedman, and A. Kandel, "Numerical solutions of fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 105, no. 1, pp. 133–138, 1999.

[3] O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 301–317, 1987.

[4] R. Goetschel and W. Voxman, "Elementary fuzzy calculus," *Fuzzy Sets and Systems*, vol. 18, no. 1, pp. 31–43, 1986.

[5] A. Kandel, "Fuzzy dynamical systems and the nature of their solutions," in *Fuzzy Sets: Theory and Application to Policy Analysis and Information Systems*, P. P. Wang and S. K. Chang, Eds., pp. 93–122, Plenum Press, New York, NY, USA, 1980.

[6] D. Prasantha Bharathi, T. Jayakumar, and S. Vinoth, "Numerical solutions of fuzzy multiple hybrid single neutral delay differential equations," *International Journal of Scientific &Technology Research*, vol. 8, no. 9, pp. 520–523, 2019.

[7] V. Lakshmikantham and X. Z. Liu, "Impulsive hybrid systems and stability theory," *International Journal of Nonlinear Differential Equations*, vol. 5, pp. 9–17, 1999.

[8] V. Lakshmikantham and R. N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor & Francis, Oxfordshire, UK, 2003.

[9] M. Paripour, E. Hajilou, A. Hajilou, and H. Heidari, "Application of Adomian decomposition method to solve hybrid fuzzy differential equations," *Journal of Taibah University for Science*, vol. 9, no. 1, pp. 95–103, 2015.

[10] P. B. Dhandapani, D. Baleanu, J. Thippan, and V. Sivakumar, "Fuzzy type RK4 solutions to fuzzy hybrid retarded delay differential equations," *Frontiers in Physics*, vol. 7, no. 168, pp. 1–6, 2019.

[11] S. Pederson and M. Sambandham, "Numerical solution of hybrid fuzzy differential equation IVPs by a characterization theorem," *Information Sciences*, vol. 179, no. 3, pp. 319–328, 2009.

[12] D. P. Bharathi, T. Jayakumar, and S. Vinoth, "Numerical solution of hybrid fuzzy mixed delay differential equation by fourth order Runge-Kutta method," *The interdisciplinary journal of Discontinuity, Nonlinearity, and Complexity*, vol. 10, no. 1, pp. 77–86, 2021.

[13] P. Prakash and V. Kalaiselvi, "Numerical solution of hybrid fuzzy differential equations by predictor-corrector method," *International Journal of Computer Mathematics*, vol. 86, no. 1, pp. 121–134, 2009.

[14] D. P. Bharathi, T. Jayakumar, T. Muthukumar, and S. Vinoth, "Existence and uniqueness of solutions for fuzzy mixed type of delay differential equations," *Journal of Applied Nonlinear Dynamics*, vol. 10, no. 1, pp. 187–196, 2021.

[15] P. B. Dhandapani, J. Thippan, C. Martin-Barreiro, V. Leiva, and C. Chesneau, "Numerical solutions of a differential system considering a pure hybrid fuzzy neutral delay theory," *Electronics*, vol. 11, no. 9, p. 1478, 2022.

[16] S. Pederson and M. Sambandham, "Numerical solution to hybrid fuzzy systems," *Mathematical and Computer Modelling*, vol. 45, pp. 1133–1144, 2007.
[17] S. Pederson and M. Sambandham, "The Runge-Kutta method for hybrid fuzzy differential equations," *Nonlinear Analysis Hybrid Systems*, vol. 2, pp. 626–634, 2008.
[18] T. Jayakumar and K. Kanagarajan, "Numerical solution for hybrid fuzzy system by Runge-Kutta method of order five," *International Journal of Applied Mathematical Sciences*, vol. 6, pp. 3591–3606, 2012.
[19] T. Jayakumar and K. Kanagarajan, "Numerical solution for hybrid fuzzy system by Runge-Kutta Fehlberg method," *International Journal of Mathematical Analysics*, vol. 6, pp. 2619–2632, 2012.
[20] A. Ahmadian, S. Salahshour, and C. S. Chan, "A Runge-Kutta method with reduced number of function evaluations to solve hybrid fuzzy differential equations," *Soft Computing*, vol. 19, no. 4, pp. 1051–1062, 2015.
[21] T. Allahviranloo and S. Salahshour, "Euler method for solving hybrid fuzzy differential equation," *Soft Computing*, vol. 15, no. 7, pp. 1247–1253, 2011.
[22] F. Saberiad, S. M. Karbassi, M. Heydari, and S. M. M. Hosseini, "Application of homotopy perturbation method for solving hybrid fuzzy differential equations," *Journal of Mathematical Extension*, vol. 12, no. 1, pp. 113–145, 2018.
[23] S. Seikkala, "On the fuzzy initial value problem," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 319–330, 1987.
[24] A. Sepahvandzadeh and B. Ghazanfari, "Variational釘iternation method for solving hybri fuzzy differential equations," *Journal of Mathematical Extension*, vol. 10, no. 4, pp. 75–85, 2016.
[25] M. K. Jain, *Numerical Solution of Differential Equations*, Wiley Eastern, Hoboken, NJ, USA, 1984.