Hindawi
Journal of Mathematics
Volume 2023, Article ID 7963973, 6 pages
https://doi.org/10.1155/2023/7963973

Research Article

Natural Factorization of Linear Control Systems through Parallel Gathering of Simple Systems

M. V. Carriegos

Departamento de Matemáticas, Universidad de León, León, Spain

Correspondence should be addressed to M. V. Carriegos; miguel.carriegos@unileon.es

Received 12 August 2022; Revised 11 April 2023; Accepted 18 April 2023; Published 20 May 2023

Academic Editor: Firdous A. Shah

Linear systems over vector spaces and feedback morphisms form an additive category taking into account the parallel gathering of linear systems. This additive category has a minimal exact structure and thus a notion of simple systems as those systems have no subsystems apart from zero and themselves. The so-called single-input systems are proven to be exactly the simple systems in the category of reachable systems over vector spaces. The category is also proven to be semisimple in objects because every reachable linear system is decomposed in a finite parallel gathering of simple systems. Hence, decomposition result is fulfilled for linear systems and feedback morphisms, but category of reachable linear systems is not abelian semisimple because it is not balanced and hence fails to be abelian. Finally, it is conjectured that the category of linear systems and feedback actions is in fact semiabelian; some threads to find the result and consequences are also given.

1. Introduction

Mathematical study of control systems arises from engineering after seminal work of Maxwell [1] on flyball governors of steam engines. Linear systems are found almost everywhere in control theory [2] both as linear models or as linearizations. In particular, the algebraic study of linear systems in the state-space approach [3] deals with linear systems defined on algebras and modules over a commutative ring [4, 5]. This approach has been used recently in the field of convolutional codes [6–10]. Convolutional codes are in fact error-correcting codes over a finite field $\mathbb{F}$ defined as vector subspaces of $\mathbb{F}(z)^n$, where $\mathbb{F}(z)$ is the field of rationals which are realized as linear control systems over $\mathbb{F}$.

Feedback is the main tool in the state-space approach [11]: two linear systems are feedback equivalent if one can be transformed in the other by means of a feedback action. The feedback equivalence relation has been studied for decades from many different perspectives: Kronecker invariants associated to a linear system were found in [12] to characterize when two reachable linear systems (scalars in a field $\mathbb{K}$) are feedback equivalent; differential geometric tools were applied in [13] to study feedback equivalence of holomorphic pairs of matrices; linear systems over finitely generated R-modules were suggested as algebraic tool to study parametric systems and integer valued systems [4, 5]; and later R-module invariants were found [14–16]. On the other hand, if $\mathbb{K} = \mathbb{C}$, then (see [4], p. 276) a linear system gives rise to a vector bundle over the Riemann sphere $\mathbb{CP}^1$ and equivalence of linear systems is characterized by the isomorphism of their associated vector bundles. The reader can see Lomadze's article [17] as one main reference in the use of vector bundle decomposition results (mainly Grothendieck's Theorem) in order to classify linear systems over $\mathbb{C}$-vector spaces.

Several algebraic [18, 19] and geometric objects have been employed to study linear systems. Casti [4] (p. 292) quotes that each gadget, whether it be a $\mathbb{K}[z]$-module, a vector bundle, or a Grassmann variety, illuminates a different aspect of the overall category called linear systems. However, Casti defined the category of realizations of behaviours [4], p. 304, but not the category of linear systems itself. Another topic was studied using categories: Brewer and Klingler [20] proved that if R is a commutative ring containing a nonzero finitely generated maximal ideal $\mathfrak{m}$ containing its annihilator, such that every unit of $R/\mathfrak{m}$ lifts to a unit of R, then the category of reachable systems over R is "wild" in the sense of classical representation theory. From this it follows that a canonical form for a reachable system over a principal ideal domain is not likely to be found. More specifically, canonical forms are unlikely to be found for systems over $\mathbb{Z}$ and $\mathbb{K}[t]$. Precisely, the impossibility of finding out canonical forms for arbitrary linear systems over rings is one of the main motivations of introducing categories of linear systems as a way to detect feedback invariants of linear systems.

Category S_R of linear systems and feedback morphisms was defined in [21] as the category whose objects are linear systems over a fixed commutative ring R and whose morphisms are the feedback morphisms. A feedback morphism is a linear map between state spaces preserving the dynamics up to feedback actions. From this point of view, feedback actions are exactly the isomorphisms in the category (no matter the scalar ring R). Moreover a collection of feedback invariants is found in [12]. This set of invariants generalizes both Kronecker's invariants over fields and R-module invariants found in [14] to a complete set of feedback invariants, not in the case of arbitrary linear systems but in the case of regular systems [12]. Hence, an answer to Brewer and Klingler's negative result would be as follows: though trying to classify all reachable linear systems is a wild problem, the classification of all regular linear systems is given over any commutative ring [21]. Moreover, both feedback equivalence and so-called dynamic feedback equivalence [22, 23] can be studied by taking into account the K-theory groups of the symmetric monoidal category of regular systems [21].

The notion of feedback morphism was introduced in [21] to circumscribe the problem of feedback classification of linear systems over R-modules as a linear map preserving both the dynamics and controls of systems (see [24] as an early article in the use of dynamorphisms). But the notion of feedback morphism is interesting by itself in terms of linear systems over $\mathbb{K}$-vector spaces [25]. Kernels and cokernels of feedback morphisms of linear systems were introduced in [25] and showed that the category $S_{\mathbb{K}}$ has all cokernels. It was conjectured that the category has all kernels as well, and hence $S_{\mathbb{K}}$ will be preabelian. But despite this conjecture, it is proved that $S_{\mathbb{K}}$ is not abelian because it is not balanced (i.e., there are bimorphisms that are not isomorphisms).

This article focuses exact structures [26, 27] onto the additive category $(S_{\mathbb{K}}, \oplus)$ of linear systems over vector spaces and parallel gathering (biproduct) of linear systems. The goal of this article is to find out the simple objects in that category. The results are that the simple objects, for the minimal exact structure [27], are exactly the classical canonical controller forms [2] adapted to this framework. On the other hand, the category is semisimple in objects because Brunovsky's theorem [12] states, in our framework, that every reachable linear system is parallel gathering (biproduct) of a finite number of simple systems. Hence, a solution of decomposition problems [24] is given in terms of feedback actions instead only for dynamorphisms. Finally, note that the category itself fails to be semisimple because $S_{\mathbb{K}}$ is not abelian and Schur's lemma does not hold in this category [28].

The article is structured as follows. Main definitions: linear system, feedback morphism, reachable system, parallel gathering, and decomposition are found in the second section "categories of linear systems". Then, third section "simple systems over vector spaces" is devoted to give the main results, the characterization of simple systems over a vector space, and that every reachable system decomposes as finite parallel gathering of simple systems. Minimal exact structure is introduced as well. Canonical controller forms and Brunovsky's theorem are stated in our categorical framework. Finally, we give some concluding remarks, where some results are highlighted and some threads to develop our results are given.

2. Categories of Linear Systems

A linear system is a triple $\sigma = (V, f, B)$ where V is a finite-dimensional $\mathbb{K}$-vector space, $f\colon V \longrightarrow V$ is a linear map, and $B \leq V$ is a vector subspace (we will use $\leq$ to denote vector subspace). The category of $\mathbb{K}$-vector spaces and linear maps is denoted by $V_{\mathbb{K}}$. The category $S_{\mathbb{K}}$ of linear systems over finite dimensional $\mathbb{K}$-vector spaces gathers linear systems $\sigma = (V, f, B)$ as objects in the category and feedback morphisms $a\colon (V, f, B) \longrightarrow (V', f', B')$ as morphisms in the category.

Definition 1. (see [21], Defintion 3.2.) A feedback morphism $a\colon \sigma = (V, f, B) \longrightarrow (V', f', B') = \sigma'$ is given by a linear map $F(a) = a\colon V \longrightarrow V'$ satisfying the following properties:

(i) $a(B) \subseteq B'$

(ii) $im(f'^{\circ} a - a^{\circ} f) \subset B'$.

The pair $S_{\mathbb{K}} = (\text{linear systems}, \text{feedb ack morphisms})$ is a category. In fact, it is a $\mathbb{K}$-linear category (i.e., enriched on the category $V_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces and linear maps). The functor forget-the-dynamics $F\colon S_{\mathbb{K}} \longrightarrow V_{\mathbb{K}}$ given by $F(V, f, B) = V$ in objects and by $F(a) = a$ in morphisms is obviously injective on morphisms, hence F is a faithful functor. F is also a dense functor because every vector space V occurs as $V = F(V, 0, 0)$. But functor F is not full because not every linear map arises as a feedback morphism, i.e., the induced map $F_{\sigma,\sigma'}\colon \hom_{S_{\mathbb{K}}}(\sigma, \sigma') \longrightarrow \hom_{\mathbb{K}}(F(\sigma), F(\sigma'))$ is injective, but it is not surjective in general. A linear combination of feedback morphisms is a feedback morphism hence the set of feedback morphisms between two linear systems is a vector subspace of linear maps between state spaces. Feedback morphisms generalize the notion of feedback equivalence ([21], Proposition 3.3.). In fact, two linear systems σ and σ' are feedback equivalent (in the classical sense) exactly when they are isomorphic in $S_{\mathbb{K}}$, i.e., when there exists a feedback morphism $a\colon \sigma \longrightarrow \sigma'$ such that its inverse $a^{-1}\colon \sigma' \longrightarrow \sigma$ is also a feedback morphism. Note that the inverse of a feedback morphism is not a feedback morphism in general even in the case of underlying linear map happens to be invertible: the morphism $f\colon (\mathbb{K}, 0, 0) \longrightarrow (\mathbb{K}, 0, \mathbb{K})$ given by $F(f) = (1)$ does not admit an inverse as feedback morphism [25], and hence f is not an isomorphism in $S_{\mathbb{K}}$, but however f is both monic and epic in the category, and thus a bimorphism in $S_{\mathbb{K}}$, while $F(f)$ is a isomorphism in $V_{\mathbb{K}}$.

2.1. Linear System Decomposition. Categories of linear systems are additive. The parallel gathering of systems (V, f, B) and (W, g, D) given by the following equation:

$$(V, f, B) \oplus (W, g, D) = \left(V \oplus W, f \oplus g = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}, B \oplus D \right), \tag{1}$$

is a biproduct (both product and coproduct) in $S_{\mathbb{K}}$ ([21], Lemma 3.5.). Hence, the category is additive, in fact symmetric monoidal. The zero object is linear system $0 = (0, 0, 0)$.

A linear system σ is indecomposable if whenever one has $\sigma \cong \sigma' \oplus \sigma''$, then one has that either $\sigma' = 0$ or $\sigma'' = 0$. Next section is devoted to prove the main result of this article which is the canonical decomposition of reachable linear systems over vector spaces. First of all, we recall the definition and some key properties of reachable systems.

Consider a linear system $\sigma = (V, f, B)$ and the subspaces of V given recursively by $N_0^\sigma = 0$, and in general $N_k^\sigma = B + f(N_{k-1}^\sigma)$. This sequence of subspaces is an ascending chain (i.e., $N_k^\sigma \leq N_{k+1}^\sigma$). The chain is strict up to an index (the degree of the linear system) $s = \deg(\sigma) \leq \dim V$, and from this index, the chain stabilizes forever $N_0 < N_1 < \cdots < N_{s-1} < N_s = N_{s+1} = \cdots$

Since V is finite dimensional, it follows that $\deg\,(V, f, B) \leq \dim V$. We will often use the notation $f^*(B)$ to denote $f^*(B) = B + f(B) + f^2(B) + \cdots + f^k(B) + \cdots = B + f(B) + f^2(B) + \cdots + f^{(\dim V - 1)}(B) \leq V$. Linear system $\sigma = (V, f, B)$ is called reachable if $N_{\deg \sigma}^\sigma = f^*(B) = V$. Note that the zero system $0 = (0, 0, 0)$ is reachable of degree 0.

Denote by $A_{\mathbb{K}}$ the full subcategory of $S_{\mathbb{K}}$ collecting all reachable linear systems and all feedback morphisms in $S_{\mathbb{K}}$ between them. If linear systems σ, σ' are reachable then parallel gathering system $\sigma \oplus \sigma'$ is also reachable. Hence, $\oplus$ is internal to $A_{\mathbb{K}}$ and reachable systems is also an additive category, in fact it is also symmetric monoidal. Consider the restriction of forget-the-dynamics functor F to reachable systems $G\colon A_{\mathbb{K}} \longrightarrow V_{\mathbb{K}}$. Functor G is newly injective on morphisms, hence G is faithful. Functor G is also dense because every vector space V occurs as $V = G(V, 0, V)$, and $(V, 0, V)$ is trivially a reachable linear system. But G is not full because the induced liner map between $\mathbb{K}$-vector spaces $G_{\sigma,\sigma'}\colon \hom_{A_{\mathbb{K}}}(\sigma, \sigma') \longrightarrow \hom_{\mathbb{K}}(G(\sigma), G(\sigma'))$ is not surjective in general.

3. Simple Systems over Vector Spaces

This section contains the main result of this article (Theorem 5), which states that simple reachable linear systems over a vector space are exactly those linear systems (V, f, B), where $\dim B = 1$. This result will be proven by using exact structures on the category of reachable linear systems. Finally, simple systems will arise as those nonzero reachable linear systems have no strict subsystems.

3.1. Exact Structure on the Category of Linear Systems. Bühler's systematic elementary expository article [26] is followed in the sequel in order to recall exact categories from additive ones and to state the minimal exact structure $\mathscr{E} = \mathscr{E}_{\min}$ on additive category $A_{\mathbb{K}}$. The exact structure $\mathscr{E}$, though minimal, will be enough to find out simple linear systems.

Definition 2. (see [26], Definition 2.1.) A kernel-cokernel pair in $A_{\mathbb{K}}$ is a pair of composable feedback morphisms (i, p) such that i is the kernel of p, and p is the cokernel of i. This fact is denoted by $\sigma' \overset{i}{\longrightarrow} \sigma \twoheadrightarrow \sigma''$. If a class $\mathscr{E}$ of kernel-cokernel pairs is fixed, an admissible monic is a morphism i such that there exists a morphism p such that $(i, p) \in \mathscr{E}$. An admissible epic is defined dually. An exact structure is a class $\mathscr{E}$ of kernel-cokernel pairs which is closed under isomorphisms of linear systems and satisfies the following axioms:

(i) The identity of the zero object $\mathbf{1}_{(0,0,0)}$ is an admissible epic.

(ii) The class of admissible monics and the class of admissible epics are closed under composition.

(iii) The push-out of admissible monic $\sigma \overset{i}{\longrightarrow} \tau$ along feedback map $\sigma \overset{a}{\longrightarrow} \sigma'$ yields an admissible monic.

$$\begin{array}{ccc} \sigma & \overset{i}{\hookrightarrow} & \tau \\ {\scriptstyle a}\downarrow & & \downarrow \\ \sigma' & \hookrightarrow & \sigma' \sqcup_\sigma \tau \end{array} \tag{2}$$

(iv) The pull-back of admissible epic $\sigma \twoheadrightarrow \tau$ along feedback map $\tau' \overset{a}{\longrightarrow} \tau$ yields an admissible epic.

$$\begin{array}{ccc} \sigma \times_\tau \tau' & \twoheadrightarrow & \tau' \\ \downarrow & & \downarrow{\scriptstyle a} \\ \sigma & \overset{p}{\twoheadrightarrow} & \tau \end{array} \tag{3}$$

There would be several exact structures on $A_{\mathbb{K}}$. The next result remarks that we have at least a minimal exact structure which gathers all kernel-cokernel feedback pairs isomorphic to a splitting pair.

Theorem 3. *The kernel-cokernel pairs isomorphic to* $\sigma \xrightarrow{\begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix}} \sigma \oplus \tau \xrightarrow{(0\ \mathbf{1})} \tau$ *form a exact structure* $\mathscr{E}_{\min}$, *and every other exact structure contains* $\mathscr{E}_{\min}$.

Proof (see [27], Proposition 2.12). □

3.2. Simple Linear Systems. In the sequel, we consider the exact structure $\mathscr{E}_{\min} = \mathscr{E}$ and the exact category $(A_{\mathbb{K}}, \mathscr{E})$. Next, we define the admissible subsystem in terms of the concept of admissible subobject in an exact category.

Definition 4. [27], 3.1.). System σ' is an admissible subsystem of σ (denoted by $\sigma' \subset_{\mathscr{E}} \sigma$) if there exists an admissible section $\sigma' \overset{i}{\longrightarrow} \sigma$ or equivalently if one has $\sigma' \overset{i}{\longrightarrow} \sigma \twoheadrightarrow \sigma''$ for some σ'' in $A_{\mathbb{K}}$. A nonzero system σ is $\mathscr{E}$-simple if every subsystem $\sigma' \subset_{\mathscr{E}} \sigma$ verifies either $\sigma' = 0$ or $\sigma' \cong \sigma$.

Now, we are ready to deal with the main result of the article.

Theorem 5. *Simple reachable systems are exactly those linear systems $\sigma = (V, f, B)$ such that* $\dim B = 1$

Proof. The proof involves the categorical version of some classical results which we will prove later (Lemmas 7 and 9) together with a dimension computation result Lemma 8. The proof of Theorem 5, up to these results, works as follows:

Set a field $\mathbb{K}$ and consider the exact category of reachable linear systems $(A_{\mathbb{K}}, \mathscr{E}_{\min})$. Consider a reachable linear system $\sigma = (V, f, B)$, then by Lemma 9, we have that $\sigma \cong \Sigma^{k_1} \oplus \cdots \oplus \Sigma^{k_s}$ where $s = \dim B$, and systems $\Sigma^{k_i} = (V_i, f_i, B_i) \neq (0,0,0)$ are not zero, and $\dim B_i = 1$. It follows that $\Sigma^{k_1} \longrightarrow^{(1,0,\ldots,0)^t} (\Sigma^{k_1} \oplus \cdots \oplus \Sigma^{k_s}) \cong \sigma$ is an admissible section, hence Σ^{k_1} is a nonzero subsystem of σ. Therefore, in order to σ being simple, it is necessary that $s = 1$, $\sigma \cong \Sigma^{k_1}$, and $\dim B = 1$.

Conversely, consider $\sigma = (V, f, B)$ a reachable subsystem where dim $B = 1$ and let us prove that σ is simple. The proof is performed by contradiction. Assume that dim $B = 1$ and that σ is not simple. Then, there exists a nonzero subsystem σ', that is to say, there exists an admissible section $\sigma' \longrightarrow^{i} \sigma$. Newly by Lemma 9, we can suppose that $\sigma' \cong \Sigma^{k_1} \oplus \cdots \oplus \Sigma^{k_s}$, where $s = \dim B$, $\Sigma^{k_i} = (V_i, f_i, B_i) \neq (0,0,0)$, and $\dim B_i = 1$. Put $\Sigma^{k_1} \longrightarrow^{(1,0,\ldots,0)^t} \sigma' \longrightarrow^{i} \sigma$, then $j = (1,0,\ldots,0)^{t\circ} i$ is also an admissible section, and therefore, we may assume without loss that $\sigma' = \Sigma^{k_1}$. On the other hand, $\sigma = (V, f, B)$ verifies itself that dim $B = 1$ and hence, by Lemma 7, one has $\sigma \cong \Sigma^{\dim V}$.

Therefore, the situation we have reached is that $\Sigma^{k_1} \longrightarrow^{j} \Sigma^{\dim V}$ is an admissible section. Since $j \neq 0$, it follows by Lemma 8 that $k_1 \geq \dim V$. On the other hand, every functor preserves sections, then $G(j)\colon V_1 \longrightarrow V$ is a section and therefore $k_1 = \dim V_1 \leq \dim V$. Consequently, $k_1 = \dim V$ and newly by Lemma 8, because of $j \neq 0$, one has that $\sigma' \cong \Sigma^{k_1} \cong \Sigma^{\dim V} \cong \sigma$ and that σ is simple because every nonzero subsystem of σ must be isomorphic to σ

Now, the next paragraphs are devoted to prove the results we used above. □

Definition 6. (canonical controller form). A linear system $\sigma = (V, f, B)$, where $\dim B = 1$ is called a single-input system. The linear system $\Sigma^n = (\mathbb{K}^n, J_n(0), [e_1])$, where $J_n(0)$ is the Jordan block of size n and eigenvalue 0, and $[e_1] = \langle e_1 \rangle$ is the subspace spanned by the first vector of standard basis of $\mathbb{K}^n$ will be called canonical controller form of size n. In other words,

$$\Sigma^n = \left(\mathbb{K}^n, \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \right). \quad (4)$$

A classical result in control theory shows that feedback classification is trivial for single-input reachable systems. Next, we state this result in our categorical framework:

Lemma 7. *Let $\sigma = (V, f, B)$ be a reachable system and* dim $B = 1$, *then* $\sigma \cong \Sigma^{\dim V}$

Proof. Let $B = [b]$. Since σ is reachable, it follows that $V = f^*(B) = b + fb + \cdots + f^{n-1}b$, and hence $\{b, fb, \ldots, f^{n-1}b\}$ is a basis of $\mathbb{K}^n$. Let $\alpha_0 + \alpha_1 z + \cdots + \alpha_{n-1} z^{n-1} + z^n$ be the characteristic polynomial of f. Now, consider the basis of V given by the following equation:

$$\mathscr{B} = \left\{ b, fb - \alpha_{n-1} b, f^2 b - \alpha_{n-1} fb - \alpha_{n-2} b \ldots, f^{n-1} b - \alpha_{n-1} f^{n-2} b - \cdots - \alpha_1 b \right\}. \quad (5)$$

Then, linear system σ is written in above basis $\mathscr{B}$ as follows:

$$\sigma' = \left(\mathbb{K}^n, A', [e_1] \right) = \left(\mathbb{K}^n, \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \right), \quad (6)$$

which is feedback isomorphic to Σ^n by means of identity morphism because $im(A' - J_n(0)) \subseteq [e_1]$.

Morphisms between single-input reachable systems are studied in [29]. The dimension of the vector space of feedback morphisms between two single-input reachable systems is computed as follows: □

Lemma 8 (see [29], 6.3.). *The dimension of the space of feedback morphisms between canonical controller forms is as follows:*

$$\dim(\hom(\Sigma^n,\Sigma^m)) = \begin{cases} 0 & if \quad n<m, \\ 1 & if \quad n=m, \\ n-m+1 & if \quad n>m. \end{cases} \tag{7}$$

Proof. Let $a\colon \Sigma^n = (\mathbb{K}^n, J_n(0), [e_1]) \longrightarrow (\mathbb{K}^m, J_m(0), [\varepsilon_1]) = \Sigma^m$ be any feedback morphism. Consider $A = (a_{ij}) \in \mathbb{K}^{m\times n}$ the matrix of $G(a)$ in the standard bases $\{e_j\}$ and $\{\varepsilon_i\}$. Since a is a feedback morphism, it follows:

(1) $Ae_1 \in [\varepsilon_1]$ and therefore $a_{21} = \cdots = a_{m1} = 0$

(2) $AJ_n(0) - J_m(0)A \subset [\varepsilon_1]$ and therefore

  (i) $a_{ij} = a_{i+1,j+1}$ for all $1 \le i \le m-1$ and all $1 \le j \le n-1$

  (ii) $a_{1n} = \cdots = a_{m-1,n} = 0$

Now consider the abovementioned restrictions in the three cases:

$(n>m)$: from (i), we have that A is on the form

$$A = \begin{pmatrix} a_{11} & a_{m,m+1} & a_{m,m+2} & \cdots & a_{mn} & a_{m-1,n} & \cdots & a_{1n} \\ a_{21} & a_{11} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \ddots & a_{m-1,n} \\ a_{m1} & \cdots & a_{21} & a_{11} & a_{m,m+1} & a_{m,m+2} & \cdots & a_{mn} \end{pmatrix}. \tag{8}$$

Now, from (i) and (ii), we have that matrix

$$A = \begin{pmatrix} a_{11} & a_{m,m+1} & a_{m,m+2} & \cdots & a_{mn} & 0 & \cdots & 0 \\ 0 & a_{11} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{11} & a_{m,m+1} & a_{m,m+2} & \cdots & a_{mn} \end{pmatrix}, \tag{9}$$

depends exactly on $n-m+1$ free parameters $(n=m)$.

In these case restrictions, (i) and (ii) yield that A is on the form

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{11} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{11} \end{pmatrix}, \tag{10}$$

$(n<m)$. Finally in this case, the restrictions yield that the only possibility is $A = (0)$.

The standard decomposition result on reachable linear systems is the Brunovsky's theorem [12]. The result in our categorical setting is as follows. □

Lemma 9. *(Brunovsky's theorem). Let $\sigma = (V, f, B)$ be a reachable linear system (in $A_{\mathbb{K}}$) and set $n = \dim V$. Then, there exists a partition $\kappa_1 + \cdots + \kappa_p = n$ of integer n such that $\sigma = \Sigma^{\kappa_1} \oplus \cdots \oplus \Sigma^{\kappa_p}$*

Proof. By recursion in n, consider a nonzero vector $v_1 \in B$ and linear system $\sigma_1 = (f^*([v_1]), f, [v_1])$. Natural inclusion $\iota_1\colon f^*([v_1]) \longrightarrow V$ gives raise to an exact sequence in $A_{\mathbb{K}}$

$$\sigma_1 = \left(f^*\left(\left[\overrightarrow{v}_1\right]\right), f, \left[\overrightarrow{v}_1\right]\right) \xrightarrow{\ i\ } \sigma \xrightarrow{\ p\ } \left(\frac{V}{f^*([v_1])}, \overline{f}, G(p)(B)\right). \tag{11}$$

Now, $\sigma \cong \sigma_1 \oplus (V/f^*([v_1]), \overline{f}, G(p)(B))$. System $(V/f^*([v_1]), \overline{f}, G(p)(B))$ is reachable, and note that $\dim(V/f^*([v_1])) < n$. Hence, by recursion, $\sigma \cong \sigma_1 \oplus \cdots \oplus \sigma_p$ where all σ_i are in $A_{\mathbb{K}}$.

Because $\dim[v_i] = 1$, it follows by Lemma 7 that every system $\sigma_i = (V_i, f, [v_i])$ is isomorphic to a system on the form Σ^{κ_i}. This concludes the proof. □

4. Conclusion

The category $A_{\mathbb{K}}$ of linear systems over vector spaces and feedback actions is studied. The parallel gathering $\oplus$ of linear systems is biproduct and thus category $(A_{\mathbb{K}}, \oplus)$ is additive and has the minimal natural exact structure $\mathscr{E}_{\min}$ given by split exact sequences. Single-input systems are shown to be the simple objects in $(A_{\mathbb{K}}, \mathscr{E}_{\min})$, and on the other hand, every object is a parallel gathering of simple objects. Thus, the following result is obtained.

Corollary 10. *Category $A_{\mathbb{K}}$ is object semisimple, i.e., every nonzero object is isomorphic to a finite biproduct of simple objects.*

Because category $A_{\mathbb{K}}$ is not balanced, it follows that it is not abelian and a fortiori and it is not abelian semisimple, that is to say, though simple systems are bricks in the sense of Enomoto's article ([28], Definition 2.1), the second statement of Schur's lemma does not hold in $A_{\mathbb{K}}$ because morphisms between nonisomorphic bricks are not zero. Note for instance that dim $\hom_{A_{\mathbb{K}}}(\Sigma^2, \Sigma^1) = 2$, thus we do not have a matrix-like representation for morphisms of linear systems, at least in the sense of Schur's lemma.

To conclude, we would like to point out some lines of further work. It is known that categories of linear systems have cokernels ([29], Theorem 3.4.) and that it is conjectured that categories of linear systems have kernels as well. First task is to compute effectively kernels of feedback morphisms. Once this is fulfilled, the category would be proven to be preabelian ([27], Definition 2.5.), and because every feedback morphism would have kernel and cokernel, it follows that image and coimage of every feedback morphism are obtained.

The second task is to check that canonical morphism from the coimage to the image of a feedback morphism is always a bimorphism. We conjecture that this is true and hence categories of linear systems are semiabelian ([27], Definition 2.5.) ([26], 4.10).

Even more, we conjectured that kernels and cokernels are stable ([26], Definition 4.1.). Hence, because kernels and cokernels are stable in $A_{\mathbb{K}}$, it would follow by Schneiders' result ([26], Proposition 4.4.) ([30], 1.1.7.) that the class of all kernel-cokernel pairs is the maximal and thus the natural, exact structure on $A_{\mathbb{K}}$. The third task is to prove the former and to show that our decomposition results are also the decomposition results taking the natural exact structure.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

Acknowledgments

The author would like to thank the CAFE (Ciberseguridad, Aplicaciones, Fundamentos y Educación) research group for its support.

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