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Research Article

Strong z°-Submodules of a Module

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Let R be a commutative ring with an identity. The purpose of this paper is to introduce and investigate the notion of strong z°-submodules of an R-module as an extension of z°-submodules and quasi z°-submodules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

For an ideal I of R, let $\mathscr{M}(I)$ be the set of maximal ideals of R containing I and $\operatorname{Max}(R)$ be the set of all maximal ideals of R. A proper ideal I of R is called a *z-ideal* if whenever any two elements of R are contained in the same set of maximal ideals and I contains one of them, it also contains the other one [1]. Intersections of maximal ideals are z-ideals and they are called strong z-ideals. It is easy to check that a proper ideal I of R is a strong z-ideal if and only if for ideals J, K of R with $\mathscr{M}(K)\subseteq\mathscr{M}(J)$ and $K\subseteq I$, then $J\subseteq I$. In general, the strong z-ideals are not the only z-ideals [[1], p. 281].

For each $a \in R$, let $\mathfrak{P}_a$, be the intersection of all minimal prime ideals containing a. A proper ideal I of R is called a z°-ideal if for each $a \in I$, we have $\mathfrak{P}_a\subseteq I$ [2]. z°-ideals have been studied in [3] under the name of d-ideals.

Let M be an R-module. A proper submodule P of M is said to be prime if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P:_R M)$ [4, 5]. The intersection of all prime submodules of M containing a submodule N of M is said to be the prime radical of N and denoted by $\operatorname{rad}_N M$. In case N does not contain in any prime submodule, the prime radical of N is defined to be M [6]. A prime submodule P is a minimal prime submodule over N if P is a minimal element of the set of all prime submodules of M that contain N [7]. A minimal prime submodule of M means a minimal prime submodule over the 0 submodules of M. The set of all minimal prime submodules of M will be denoted by $\operatorname{Min}^p(M)$. The intersection of all minimal prime submodules of M containing a submodule K of M is denoted by $\mathfrak{P}_K$. In case K does not contain in any minimal prime submodule of M, $\mathfrak{P}_K$ is defined to be M. Also, the intersection of all minimal prime submodules of M containing $x \in M$ is denoted by $\mathfrak{P}_x$. In case x does not contain in any minimal prime submodule of M, $\mathfrak{P}_x$ is defined to be M. If N is a submodule of M, define $V(N) = \{P \in \operatorname{Min}^p(M) | N\subseteq P\}$.

An R-module M is said to be a multiplication module if for every submodule N of M, there exists an ideal I of R such that $N = IM$ [8]. An R-module M is said to be *reduced* if the intersection of all the prime submodules of M is equal to zero [9].

In [10, 11], the notions of z°-submodules and quasi z°-submodules of an R-module M as an extension of z°-ideals were introduced and some of their properties were investigated when M is a reduced multiplication R-module. A proper submodule N of an R-module M is said to be a z°-submodule of M if $\mathfrak{P}_x\subseteq N$ for all $x \in N$ [10]. A proper submodule N of an R-module M is said to be a quasi z°-submodule of M if $\mathfrak{P}_{aM}\subseteq N$ for all $a \in (N:_R M)$ [11]. In this paper, we define the notion of strong z°-submodules of an R-module M as a generalization of z°-submodules and quasi z°-submodules. We say that a proper submodule N of an R-module M is a strong z°-submodule of M if $\mathfrak{P}_K\subseteq N$ for all submodules K of N. Among other results, we give some characterizations for strong z°-submodules of an R-module M, in particular, when M is a Noetherian reduced multiplication module.

2. Main Results

We begin Section 2 by introducing the concept of the strong z°-submodule of a module.

Definition 1. We say that a proper submodule N of an R-module M is a strong z°-submodule of M if $\mathfrak{P}_K \subseteq N$ for all submodules K of N. Also, we say that a proper ideal I of R is a strong z°-ideal if I is a strong z°-submodule of an R-module R.

Remark 1. Let M be an R-module. Clearly, if N is a strong z°-submodule of M, then N is a z°-submodule of M. The converse holds when every submodule of M is a cyclic R-module. However, as we see (Example 4.2 in [12]), the converse is not true in general.

Remark 2. If N is a strong z°-submodule of M, then for every submodule K of N, we have $\mathfrak{P}_K \neq M$. Clearly, every minimal prime submodule of M is a strong z°-submodule of M. Also, the family of strong z°-submodules of M is closed under the intersection. Therefore, if $\mathrm{rad}_M(M) \neq M$, then $\mathrm{rad}_M(M)$ is a strong z°-submodule of M and it is contained in every strong z°-submodule of M.

Proposition 1. *Let N be a submodule of an R-module M. Then, N is a strong z°-submodule of M if and only if N is an intersection of minimal prime submodules of M.*

Proof. Since $N \subseteq N$, we have $\mathfrak{P}_N \subseteq N$; therefore, $\mathfrak{P}_N = N$. The converse is clear. □

Lemma 1. *Let M be an R-module. A submodule N of M is a strong z°-submodule if and only if $N = \sum_{K \in \Lambda} \mathfrak{P}_K$, where Λ is the collection of all submodules of N.*

Proof. This is straightforward.

A submodule N of an R-module M is said to be a multiple of M, provided that $N = rM$ for some $r \in R$. If every submodule of M is a multiple of M, then M is said to be a principal ideal multiplication module [13]. □

Remark 3. Clearly, if N is a strong z°-submodule of M, then N is a quasi z°-submodule of M. The converse holds when M is a principal ideal multiplication module.

Proposition 2. *Let N be a proper submodule of an R-module M. Then, N as an R-submodule is a strong z°-submodule if and only if as an $R/\mathrm{Ann}_R(M)$-submodule is a strong z°-submodule.*

Proof. This is straightforward. □

Remark 4. Let M be a faithful multiplication R-module. Then, $\mathfrak{P}_I M \subseteq \mathfrak{P}_{IM}$ for each ideal I of R. The reverse inclusion holds when M is a finitely generated R-module (see Theorem 2.8 in [11]).

Theorem 1. *Let N be a strong z°-submodule of a faithful multiplication R-module M. Then, $(N:_R M)$ is a strong z°-ideal of R. The converse holds when M is a finitely generated R-module.*

Proof. Let I be an ideal of R such that $I \subseteq (N:_R M)$. Then, $IM \subseteq N$ and so by assumption, $\mathfrak{P}_{IM} \subseteq N$. Hence, $\mathfrak{P}_I M \subseteq N$ by Remark 4. It follows that $\mathfrak{P}_I \subseteq (N:_R M)$ and so $(N:_R M)$ is a strong z°-ideal of R. For the converse, let M be a finitely generated R-module and K be a submodule of M such that $K \subseteq N$. Thus, $(K:_R M) \subseteq (N:_R M)$. Now, by assumption, $\mathfrak{P}_{(K:_R M)} \subseteq (N:_R M)$. By Remark 4, $\mathfrak{P}_{(K:_R M)M} = \mathfrak{P}_{(K:_R M)} M$. Therefore, $\mathfrak{P}_K = \mathfrak{P}_{(K:_R M)M} \subseteq (N:_R M)M = N$, as needed. □

Corollary 1. *Let M be a finitely generated faithful multiplication R-module. If I is a strong z°-ideal of R, then IM is a strong z°-submodule of M.*

Proof. By Theorem 10 in [14], $I = (IM:_R M)$. Now, the result follows from Theorem 1. □

Theorem 2. *Let N be a proper submodule of an R-module M. Then, the following are equivalent:*

(a) *N is a strong z°-submodule of M*
(b) *For submodules K, H of M, $\mathfrak{P}_K = \mathfrak{P}_H$ and $K \subseteq N$ imply that $H \subseteq N$*
(c) *For submodules K, H of M, $V(K) = V(H)$ and $K \subseteq N$ imply that $H \subseteq N$*
(d) *If K is a submodule of N, H a submodule of M, and $V(K) \subseteq V(H)$, then $H \subseteq N$*

Proof

$(a) \Rightarrow (b)$ Let K, H be submodules of M such that $\mathfrak{P}_K = \mathfrak{P}_H$ and $K \subseteq N$. By part (a), $\mathfrak{P}_K \subseteq N$. Thus, $H \subseteq \mathfrak{P}_H \subseteq N$

$(b) \Rightarrow (c)$ Let K, H be submodules of M such that $V(K) = V(H)$ and $K \subseteq N$. Then, $\mathfrak{P}_K = \mathfrak{P}_H$. Thus, by part (b), $H \subseteq N$

$(c) \Rightarrow (a)$ Let K be a submodule of N. One can see that $V(K) = V(\mathfrak{P}_K)$. Now, by part (c), $\mathfrak{P}_K \subseteq N$

$(a) \Rightarrow (d)$ Let K be a submodule of N, H a submodule of M, and $V(K) \subseteq V(H)$. Then, $\mathfrak{P}_H \subseteq \mathfrak{P}_K$. Hence, by part (a), $H \subseteq N$

$(d) \Rightarrow (c)$ This is clear. □

Lemma 2. *Let M be a reduced multiplication R-module and P be a minimal prime submodule of M. If K is a finitely generated submodule of M such that $K \subseteq P$, then* $\mathrm{Ann}_R(K) \subseteq (P:_R M)$.

Proof. This follows from the proof of (Theorem 3.6 in [9]). □

Proposition 3. *Let M be a reduced multiplication R-module. Then, the following are equivalent:*

(a) *For submodules K, H of M, $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(H)$ and $K \subseteq N$ imply that $H \subseteq N$*

(b) *For submodules K, H of M, $\mathrm{Ann}_R(K) \subseteq \mathrm{Ann}_R(H)$ and $K \subseteq N$ imply that $H \subseteq N$*

Proof

$(a) \Rightarrow (b)$ Let K, H be submodules of M. As M is a multiplication R-module, there exist ideals I and J of R such that $K = IM$ and $H = JM$. Assume that $\mathrm{Ann}_R(K) \subseteq \mathrm{Ann}_R(H)$ and $K \subseteq N$. Then, $(0\colon {}_M \mathrm{Ann}_R(JM)) \subseteq (0\colon {}_M \mathrm{Ann}_R(IM))$. Hence, $(0\colon {}_M \mathrm{Ann}_R(\mathrm{IJM})) = (0\colon {}_M \mathrm{Ann}_R(JM))$ by Theorem 2.5 in [11]. This implies that $\mathrm{Ann}_R(IJM) = \mathrm{Ann}_R(JM)$. Now, as $\mathrm{IJM} \subseteq N$, we have $H = JM \subseteq N$ by part (a).

$(b) \Rightarrow (a)$ This is clear. □

Theorem 3. *Let M be a reduced multiplication R-module. Then, for each finitely generated submodule K of M, we have the following:*

(a) *$V(\mathrm{Ann}_R(K)M) = \mathrm{Min}^p(M)/V(K)$*

(b) *$V(K) = V((0\colon {}_M \mathrm{Ann}_R(K)))$*

Proof

(a) If $P \in V(K)$, then $\mathrm{Ann}_R(K) \subseteq (P\colon {}_R M)$ by Lemma 2. Thus, $\mathrm{Ann}_R(K)M \subseteq P$. Therefore, $V(\mathrm{Ann}_R(K)M) \cap V(K) = \varnothing$. On the other hand, if $P \in \mathrm{Min}^p(M)/V(K)$, then there exists $x \in K/P$. Hence, for any $b \in \mathrm{Ann}_R(K)$, we have $bx = 0 \subseteq P$. Since $x \in P$ and P is prime, $bM \subseteq P$. This implies that $P \in V(\mathrm{Ann}_R(K)M)$. Thus, $V(\mathrm{Ann}_R(K)M) = \mathrm{Min}^p(M)/V(K)$.

(b) Clearly, $V((0\colon {}_M \mathrm{Ann}_R(K))) \subseteq V(K)$. Now, let P be a minimal prime submodule of M containing K. Then, there exists $a \in \mathrm{Ann}_R(K)/(P\colon {}_R M)$ by Lemma 2. Thus, for any $y \in (0\colon {}_M \mathrm{Ann}_R(K))$, we have $ay = 0 \in P$. It follows that $y \in P$. Therefore, $(0\colon {}_M \mathrm{Ann}_R(K)) \subseteq P$. Thus, $V(K) \subseteq V((0\colon {}_M \mathrm{Ann}_R(x)))$. □

Corollary 2. *Let M be a reduced multiplication R-module. Then, for each finitely generated submodule K of M, $(0\colon {}_M \mathrm{Ann}_R(K)) = \mathfrak{P}_K$.*

Proof. This follows from Theorem 2 in [10] and Theorem 3(b). □

Theorem 4. *Let N be a submodule of a reduced multiplication R-module M. Then, for finitely generated submodules K and H of M, the following are equivalent:*

(a) *$\mathfrak{P}_K = \mathfrak{P}_H$ and $K \subseteq N$ imply that $H \subseteq N$*

(b) *$V(K) = V(H)$ and $K \subseteq N$ imply that $H \subseteq N$*

(c) *$\mathrm{Ann}_R(K) = \mathrm{Ann}_R(H)$ and $K \subseteq N$ imply that $H \subseteq N$*

Proof

$(a) \Rightarrow (b)$ Assume that $V(K) = V(H)$ and $K \subseteq N$. Then, $\mathfrak{P}_K = \mathfrak{P}_H$. Thus, by part (a), $H \subseteq N$.

$(b) \Rightarrow (c)$ Assume that $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(H)$ and $K \subseteq N$. Then, $(0\colon {}_M \mathrm{Ann}_R(K)) = (0\colon {}_M \mathrm{Ann}_R(H))$. So, by Theorem 3(b), $V(K) = V(H)$. Hence, $H \subseteq N$ by part (b).

$(c) \Rightarrow (a)$ Assume that $\mathfrak{P}_K = \mathfrak{P}_H$ and $K \subseteq N$. Then, $(0\colon {}_M \mathrm{Ann}_R(K)) = (0\colon {}_M \mathrm{Ann}_R(H)$ by Corollary 2. Thus,

$$\begin{aligned}\mathrm{Ann}_R(K) &= \mathrm{Ann}_R\big(0\colon {}_M \mathrm{Ann}_R(K)\big)\\ &= \mathrm{Ann}_R\big(0\colon {}_M \mathrm{Ann}_R(H)\big) = \mathrm{Ann}_R(H).\end{aligned} \tag{1}$$

So, $H \subseteq N$ by part (c). □

Proposition 4. *Let N be a strong z°-submodule of a faithful multiplication R-module M. Then, for each $r \in R$, $(N\colon {}_M r)$ is a strong z°-submodule of M.*

Proof. As M is a faithful multiplication R-module, one can see that $r\mathfrak{P}_K \subseteq \mathfrak{P}_{rK}$ for each submodule K of M. Suppose that $r \in R$ and $K \subseteq (N\colon {}_M r)$. Then, $rK \subseteq N$. So, by assumption, $\mathfrak{P}_{rK} \subseteq N$. Since $r\mathfrak{P}_K \subseteq \mathfrak{P}_{rK}$, we have $r\mathfrak{P}_K \subseteq N$. Thus, $\mathfrak{P}_K \subseteq (N\colon {}_M r)$. □

Theorem 5. *Let M be a Noetherian reduced multiplication R-module. Then, the following are equivalent:*

(a) *For submodules K and H of M, $V(K) = V(H)$ and $K \subseteq N$ imply that $H \subseteq N$*

(b) *For a submodule K of $M, K \subseteq N$ implies that $(0\colon {}_M \mathrm{Ann}_R(K)) \subseteq N$*

(c) *For submodules K and H of M, $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(H)$andK$\subseteq N$ imply that $H \subseteq N$*

Proof

$(a) \Rightarrow (b)$ Let K be a submodule of M. As M is Noetherian, K is finitely generated. Thus, $V(K) = V((0\colon {}_M \mathrm{Ann}_R(K)))$ by Theorem 3. Hence, by part (b), $(0\colon {}_M \mathrm{Ann}_R(K)) \subseteq N$.

$(b) \Rightarrow (c)$ Let K, H be submodules of M such that $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(H)$ and $K \subseteq N$. Then, $(0\colon {}_M \mathrm{Ann}_R(K) = (0\colon {}_M \mathrm{Ann}_R(H))$. By part (c), $(0\colon {}_M \mathrm{Ann}_R(K)) \subseteq N$. Thus, $H \subseteq (0\colon {}_M \mathrm{Ann}_R(H)) \subseteq N$.

$(c) \Rightarrow (a)$ This follows from Theorem 4.

Now, we have the following corollary. □

Corollary 3. *Let M be a Noetherian reduced multiplication R-module. Then, for a proper submodule N of M, the following are equivalent:*

(a) *N is a strong z°-submodule of M*
(b) *N is an intersection of minimal prime submodules of M*
(c) *$(N:_{R}M)$ is a strong z°-ideal of R*
(d) *For a submodule K of M, $K\subseteq N$ implies that $(0:_{M}\mathrm{Ann}_R(K))\subseteq N$*
(e) *For submodules K and H of M, $V(K)=V(H)$ and $K\subseteq N$ imply that $H\subseteq N$*
(f) *For submodules K and H of M, $\mathrm{Ann}_R(K)=\mathrm{Ann}_R(H)$ and $K\subseteq N$ imply that $H\subseteq N$*
(g) *For submodules K, H of M, $\mathfrak{P}_K=\mathfrak{P}_H$ and $K\subseteq N$ imply that $H\subseteq N$*
(h) *If K is a submodule of N, H a submodule of M, and $V(K)\subseteq(H)$, then $H\subseteq N$;*
(i) *$N=\sum_{K\in\Lambda}\mathfrak{P}_K$, where Λ is the collection of all submodules of N.*

An R-module M is said to be a comultiplication module if for every submodule N of M there exists an ideal I of R such that $N=(0:_{M}I)$ equivalently for each submodule N of M, we have $N=(0:_{M}\mathrm{Ann}_R(N))$ [15].

Example 1. Let M be a Noetherian reduced multiplication and comultiplication R-module. Then, every proper submodule of M is a strong z°-submodule of M. In particular,

(a) If p is a prime number and $n>1$, then every proper submodule of the $\mathbb{Z}_{p^n}$-module $p^{n-1}\mathbb{Z}_{p^n}$ is a strong z°-submodule
(b) If n is square-free, every proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}_n$ is a strong z°-submodule

Theorem 6. *Let M be a Noetherian reduced multiplication R-module and N be a strong z°-submodule of M. Then, every minimal prime submodule over N is a prime strong z°-submodule of M. In particular, $\mathrm{rad}(N)$ is a strong z°-submodule of M.*

Proof. Let P be a minimal prime submodule over N. Assume that $\mathrm{Ann}(K)=\mathrm{Ann}(H)$, where K,H are submodules of M with $K\subseteq P$. Since P/N is a minimal prime submodule of M/N, by Lemma 2, there exists $c\in\mathrm{Ann}_R(K+N))/(P/N:_{R}M/N)$. Thus, $cK\subseteq N$ and $c\in(P:_{R}M)$. Clearly, $\mathrm{Ann}(cK)=\mathrm{Ann}(cH)$. As N is a strong z°-submodule of M, we have $cH\subseteq N\subseteq P$. As $c\in(P:_{R}M)$ and P is a prime submodule, $H\subseteq P$ as needed. The last assertion is clear. □

Corollary 4. *Let M be a Noetherian reduced multiplication R-module. If $f\colon M\longrightarrow M/N$ is the natural epimorphism, where N is a strong z°-submodule of M, then every strong z°-submodule of M/N contracts to a strong z°-submodule of M.*

Proposition 5. *Let M be a Noetherian reduced multiplication R-module. Let N_i for $1\le i\le n$ be a proper submodule of M such that for each $i\neq j$, $(N_i:_{R}M)$ and $(N_j:_{R}M)$ are co-prime ideals of R. Then, $\cap_{i=1}^{n}N_i$ is a strong z°-submodule of M if and only if each N_j for $1\le j\le n$ is a strong z°-submodule of M.*

Proof. Assume that $\cap_{i=1}^{n}N_i$ is a strong z°-submodule of M and $1\le j\le n$. We show that N_j is a strong z°-submodule of M. So, assume that $\mathrm{Ann}_R(K)=\mathrm{Ann}_R(H)$ for some submodules K,H of M with $K\subseteq N_j$. Since, for each $i\neq j$, $(N_i:_{R}M)$ and $(N_j:_{R}M)$ are co-prime ideals of R, $\cap_{i=1,i\neq j}^{n}(N_i:_{R}M)$ and $(N_j:_{R}M)$ are co-prime ideals of R. Thus, $1=a+b$ for some $a\in\cap_{i=1,i\neq j}^{n}(N_i:_{R}M)$ and $b\in(N_j:_{R}M)$. So, $H=aH+bH$ and $\mathrm{Ann}_R(aK)=\mathrm{Ann}_R(aH)$. Now, we have $aK\subseteq\cap_{i=1}^{n}N_i$. Thus, $\cap_{i=1}^{n}N_i$ is a strong z°-submodule of M which implies that $aH\cap_{i=1}^{n}N_i\subseteq\subseteq N_j$. Now, since $bH\subseteq N_j$, we have $H\subseteq N_j$ and we are done. The converse is clear. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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