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*Research Article*

# Existence of a Generalized Solution for the Fractional Contact Problem

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In this paper, we take into consideration the mathematical analysis of time-dependent quasistatic processes involving the contact between a solid body and an extremely rigid structure, referred to as a foundation. It is assumed that the constitutive law is fractional long-memory viscoelastic. The contact is considered to be bilateral and is modeled around Tresca's law. We establish the existence of the generalized solution's result. The proof is supported by the surjectivity of the multivalued maximum monotone operator, Rothe's semidiscretization method, and arguments for evolutionary variational inequality.

## 1. Introduction

It is well known that the empirical models containing the fractional derivative fit the experimental data more precisely than the model containing the integer derivative. The early application of the fractional derivative in viscoelasticity dates back to the beginning of this century. Germent and Baglet proposed the fractional Kelvin–Voigt model, and at the same time, Koeller obtained the fractional Maxwell model [1, 2]. The theoretical and numerical investigation has been done for fractional differential equations and inequalities in finite-dimensional spaces. On this subject, we cite, for instance [3–6]. The application of the variational framework, more precisely the variational inequalities, proves to be very useful in the study of engineering and mechanical problems. It was initiated by Panagiotopoulos; see monographs [7, 8], and several results have emerged; see [9–13]. In this paper, we are interested in studying the contact problem for a fractional viscoelastic law with long-term memory. The corresponding constitutive law is

$$\sigma = \mathfrak{A}\varepsilon\left({}_0^C D_\delta^\beta \mathfrak{w}\right) + \mathfrak{B}\left(\varepsilon(\mathfrak{w})\right) + \int_0^\delta \mathfrak{C}(t-\mu)\varepsilon(\mathfrak{w}(\mu))\mathrm{d}\mu, \quad (1)$$

where $\mathfrak{A}, \mathfrak{B}$, and $\mathfrak{C}$ represent the fractional viscoelasticity, elasticity operator, and relaxation tensor, $\sigma$ and $\mathfrak{w}$ stand for the stress and the displacement fields, respectively, and ${}_0^C D_\delta^\beta$ is related to Caputo fractional derivative of $\beta$ order and $0 < \beta < 1$.

The long-memory viscoelastic constitutive law has been the subject of numerous papers; see [14–16]. The distinction of this study is the selection of contact with Tresca's friction type, governed by a fractional constitutive law, and the use of Rothe's approach and the maximum monotone operators' surjectivity results to demonstrate the existence of a solution to a variational inequality. Actually, there are only a few contributions to the literature on the resolution of variational and hemivariational inequalities using Rothe's approach; see, for example, [18–21].

This article is structured according to the following outlines: in Section 2, we state the equilibrium equations modeling the process of viscoelastic material coming into contact with a foundation. We give some of the notations and the assumptions on data. Then, we obtain the variational formulation of the problem and present the main results concerning the existence of a weak solution. In Section 3, we first prove the existence of the discrete generalized solution to the discrete problem, in the second, we establish the a priori estimates, and in the last, we pass to the limit to obtain the existence of solutions to the variational problem.

## 2. Setting of the Problem

Before setting up the physical model, we give some definitions and preliminary results.

*Definition 1.* For any positive integer $n$ *and* $n-1\leq s<n$, the Caputo fractional derivative and fractional integral of orders of a given function $\mathfrak{w}\in L^1[0,b]$, are respectively defined as

$$\begin{aligned}{}_0D_\delta^s\mathfrak{w}(\delta)&=\frac{1}{\Gamma(n-s)}\int_0^\delta\frac{\mathfrak{w}^{(n)}(\tau)\mathrm{d}\tau}{(\delta-\tau)^{s-n+1}},\quad\forall\delta\in[0,b],\\ {}_0D_\delta^{-s}\mathfrak{w}(\delta)&=\frac{1}{\Gamma(s)}\int_0^\delta(\delta-\mu)^{s-1}\mathfrak{w}(\mu)\mathrm{d}\mu,\quad\forall\delta\in[0,b],\end{aligned}\tag{2}$$

where $\Gamma(\cdot)$ is the standard Gamma function.

*Definition 2.* Let $\mathscr{W}$ be a real Banach space. An operator $\mathscr{A}\colon\mathscr{W}\longrightarrow\mathscr{W}'$ is said to be

(a) $\mathscr{A}$est hemicontinuous if the real function $\delta\mapsto\langle\mathscr{A}(\mathfrak{u}+\delta\vartheta),\mathfrak{w}\rangle$ is continuous from [0,1] in $\mathbb{R}$, for all $\mathfrak{u},\vartheta,\mathfrak{w}\in\mathscr{W}$.

(b) bounded, if $\mathscr{A}$ maps bounded sets of $\mathscr{W}$ into bounded sets of $\mathscr{W}'$.

(c) $\mathscr{A}\colon\mathscr{W}\to2^{\mathscr{W}'}$ is monotone if, $\langle\mathfrak{y}-\mathfrak{y}_0,\mathfrak{x}-\mathfrak{x}_0\rangle\geq0$, $\forall\mathfrak{y}_0\in\mathscr{A}\mathfrak{x}_0$, $\mathfrak{y}\in\mathscr{A}\mathfrak{x}$.

(d) $\mathscr{A}$ is maximal monotone if, $\mathscr{A}$ is monotone, and for all $\mathfrak{x}\in D(\mathscr{A})$,$\mathfrak{y}\in2^{\mathscr{W}'}$, such that $\langle\mathfrak{y}-\mathfrak{z},\mathfrak{x}-\mathfrak{x}_0\rangle\geq0$, $\forall\mathfrak{x}_0\in\mathscr{W}$, $\mathfrak{z}\in\mathscr{A}\mathfrak{x}_0$, then $\mathfrak{y}\in\mathscr{A}\mathfrak{x}$.

(e) The generalized gradient (subdifferential) of a convex function $\mathfrak{I}$ *at* $\mathfrak{u}$ is a subset of the dual space $\mathscr{W}'$ *given by*

$$\partial\mathfrak{I}(\mathfrak{w})=\{\varrho\in\mathscr{W}'\colon\ \mathfrak{I}(\mathfrak{w})-\mathfrak{I}(\vartheta)\leq\langle\varrho,\mathfrak{w}-\vartheta\rangle,\forall\vartheta\in\mathscr{W}\}.\tag{3}$$

(f) A sequence $\{\mathfrak{w}_n\}\subset\mathscr{W}$ is weakly convergent to $\mathfrak{w}\in\mathscr{W}$, noted $\mathfrak{w}_n\rightharpoonup\mathfrak{w}$, if for all $\mathfrak{m}'\in\mathscr{W}'$, we have

$$\lim_{n\longrightarrow+\infty}\langle\mathfrak{w}_n\mathfrak{w}'\rangle_{\mathscr{W},\mathscr{W}'}=\langle\mathfrak{w},\mathfrak{w}'\rangle_{\mathscr{W},\mathscr{W}'},\tag{4}$$

where $\langle\cdot,\cdot\rangle_{\mathscr{W},\mathscr{W}'}$ represents the duality between $\mathscr{W}$ and $\mathscr{W}'$.

**Theorem 1** (see [22]). *Let* $F\colon\mathscr{W}\longrightarrow]-\infty,+\infty[$ *is convex, lower semicontinuous, and* $F\neq+\infty$. *Then,* $\partial F$ *is maximal monotone.*

**Theorem 2** (see [22]). *Let the following hypotheses hold*

($\mathscr{H}_1$): *the operator* $\mathscr{A}\colon\mathscr{W}\to2^{\mathscr{W}'}$ *is maximal monotone.*

($\mathscr{H}_2$): *the operator* $\mathscr{B}\colon\mathscr{W}\to\mathscr{W}'$ *is monotone hemicontinuous and bounded.*

($\mathscr{H}_3$): $\mathscr{B}$ *is* $\mathscr{A}$*-coercive, meaning that a point* $\mathfrak{u}_0\in D(\mathscr{A})$ *and a number* $r>0$ *exist, that way*

$$\frac{\langle\mathscr{B}\mathfrak{u},\mathfrak{u}-\mathfrak{u}_0\rangle}{\|\mathfrak{u}\|}\longrightarrow+\infty\text{ when }\|\mathfrak{u}\|\longrightarrow+\infty.\tag{5}$$

*Hence,* $\mathscr{A}+\mathscr{B}$ *is surjective.*

*2.1. Physical Setting.* Suppose that in the reference configuration, the domain $\mathfrak{D}\subset\mathbb{R}^d\,(d=2,3)$ which is considered bounded with a smooth boundary $\partial\mathfrak{D}=\Sigma$. Three open, disconnected sections make up this boundary: $\Sigma^D$, $\Sigma^{\aleph}$, and $\Sigma^C$ such that meas $(\Sigma^D)>0$. Let $[0,b]$ time interval of interest, where $b>0$.

The body is submitted to the action of body forces of density $f_0$. It also submitted to mechanical constraints on the boundary. The body is supposed squeezed into $\Sigma^D$, thus the displacement field vanishes over $\Sigma^D$. On $\Sigma^{\aleph}$, there are the density of traction forces denoted $f_\alpha$. At some point, the body comes into contact over $\Sigma^C$.

Let denote by $\mathbb{T}^d$ the space of the second order of symmetric tensors and by $(\cdot)$ and $|\cdot|$, respectively, the scalar product and the Euclidean norm in $\mathbb{T}^d$ (resp in $\mathbb{R}^d$). We adopt the summation convention over repeated indices. All indices $\imath$, $\mathfrak{j}$ take values from 1 to $d$.

$$\begin{aligned}&\sigma.\mathfrak{h}=\sigma_{\imath\mathfrak{j}}\mathfrak{h}_{\imath\mathfrak{j}},|\mathfrak{h}|=(\mathfrak{h}.\mathfrak{h})^{1/2},\quad\forall\sigma,\ \mathfrak{h}\in\mathbb{T}^d,\\ &\mathfrak{w}.\vartheta=\mathfrak{w}_\imath\vartheta_\imath,|\mathfrak{w}|=(\mathfrak{w}.\mathfrak{w})^{1/2},\quad\forall\mathfrak{w},\ \vartheta\in\mathbb{R}^d.\end{aligned}\tag{6}$$

Let $\mathfrak{D}\subset\mathbb{R}^d$, we shall use the notation

$$\begin{aligned}H&=\{\mathfrak{w}=(\mathfrak{w}_\imath)|\mathfrak{w}_\imath\in L^2(\mathfrak{D})\}=\left(L^2(\mathfrak{D})\right)^d,\\ \mathfrak{H}&=\{\sigma=(\sigma)_{\imath\mathfrak{j}}|\sigma_{\imath\mathfrak{j}}=\sigma_{\mathfrak{j}\imath}\in L^2(\mathfrak{D})\},\\ H_1&=\{\mathfrak{w}=(\mathfrak{w}_\imath)|\varepsilon(\mathfrak{w})\in\mathfrak{H}\},\\ \mathfrak{H}_1&=\{\sigma\in\mathfrak{H}|\mathrm{Div}\sigma\in H\},\end{aligned}\tag{7}$$

with $\epsilon\colon H\longrightarrow\mathfrak{H}$ and Div: $\mathfrak{H}\longrightarrow H$ are, respectively, operators of deformation and divergence defined by

$$\varepsilon(\mathfrak{w})=\left(\varepsilon_{\imath\mathfrak{j}}(\mathfrak{w})\right),\varepsilon_{\imath\mathfrak{j}}(\mathfrak{w})=\frac{1}{2}\left(\mathfrak{w}_{\imath\mathfrak{j}}+\mathfrak{w}_{\mathfrak{j}\imath}\right)\text{ and }\mathrm{Div}\sigma=\left(\sigma_{\imath\mathfrak{j},\mathfrak{j}}\right).\tag{8}$$

The space $H$, $\mathfrak{H}$, $H^1$, and $\mathfrak{H}^1$ are Hilbert spaces endowed with the inner products given by

$$\begin{aligned}(\mathfrak{w},\vartheta)_H&=\int_{\mathfrak{D}}\mathfrak{w}_\imath\vartheta_\imath\mathrm{d}\mathfrak{x},\\ (\sigma,\mathfrak{h})_{\mathfrak{H}}&=\int_{\mathfrak{D}}\sigma_{\imath\mathfrak{j}}\mathfrak{h}_{\imath\mathfrak{j}}\mathrm{d}\mathfrak{x},\\ (\mathfrak{w},\vartheta)_{H_1}&=(\mathfrak{w},\vartheta)_H+(\varepsilon(\mathfrak{w}),\varepsilon(\vartheta))_{\mathfrak{H}},\\ (\sigma,\mathfrak{h})_{\mathfrak{H}_1}&=(\sigma,\mathfrak{h})_{\mathfrak{H}}+(\mathrm{Div}\sigma,\mathrm{Div}\mathfrak{h})_H.\end{aligned}\tag{9}$$

We use the notation $\vartheta$ for the trace of $\vartheta$ on $\Sigma$. The normal and tangential components of $\vartheta$ on $\Sigma$ are represented by the symbols $\vartheta_\nu$ and $\vartheta_\tau$, respectively, and are determined by the formulas $\vartheta_\nu = \vartheta \cdot \nu$, $\vartheta_\tau = \vartheta - \vartheta_\nu \nu$, $\sigma_\nu = (\sigma\nu)\cdot\nu$, and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$. The trace of the regular stress field $\sigma$ applied to $\nu$ is noted $\sigma\nu$.

The physical model for the process is as follows:

*Problem 1.* Find a displacement field $\mathfrak{w}\colon \mathfrak{D}\times[0,b]\longrightarrow\mathbb{R}^d$

$$\sigma = \mathfrak{A}\varepsilon\left({}_0^C D_\delta^\beta \mathfrak{w}\right) + \mathfrak{B}(\varepsilon(\mathfrak{w})) + \int_0^\delta \mathfrak{C}(\delta-\mu)\varepsilon(\mathfrak{w}(\mu))\mathrm{d}\mu \text{ in } \mathfrak{D}\times(0,b), \tag{10}$$

$$\mathrm{Div}\sigma + f_0 = 0 \text{ in} \mathfrak{D}\times(0,b). \tag{11}$$

The boundary and initial conditions

$$\mathfrak{w} = 0, \text{ on } \Sigma^D \times (0,b), \tag{12}$$

$$\sigma_\nu = f_{\aleph}, \text{ on } \Sigma^{\aleph}\times(0,b), \tag{13}$$

$$\sigma_\nu = 0, \text{ on } \Sigma^C \times (0,b), \tag{14}$$

$$\begin{aligned} &\|\sigma_\tau\| \le g,\\ &\mathfrak{w}_\tau \ne 0 \Rightarrow \sigma_\tau = -g\frac{\mathfrak{w}_\tau}{\|\mathfrak{w}_\tau\|}, \text{ on } \Sigma^C\times(0,b), \end{aligned} \tag{15}$$

$$\mathfrak{w}(0) = \mathfrak{w}_0, \text{ in } \mathfrak{D}. \tag{16}$$

For the sake of simplifying the notations in equations (10)–(16) and below, we do not show explicitly the dependence of all functions on, respectively, space and time variables $\mathfrak{x}\in\mathfrak{D}\cup\Sigma$ and $\delta\in[0,b]$. The fractional viscoelastic constituency with memory law is given by (10), for $0<\beta<1$, where $\mathfrak{A}$ is linear viscosity function, $\mathfrak{B}$ is linear elasticity, and $\mathfrak{C}$ the linear relaxation function. The expression (11) is the equilibrium equation, we suppose the process is quasistatic. Here, the boundary conditions on displacement and traction are represented by (12) and (13). Conditions (14) and (15) are the bilateral contact and the friction conditions, respectively, it is modeled by Tresca's law with an imposed friction bound $g$ (hence $g\ge 0$ and the nonseparation condition $\mathfrak{u}_\nu = 0$). The equation given in (16) is the initial condition on displacement. To be able to give a variational formulation of the above problem, we give the following additional notations. We set

$$\begin{aligned} \mathscr{W} &= \left\{\mathfrak{w}\in H_1 \mid w = 0, \text{ on } \Sigma^D\right\},\\ \widetilde{\mathscr{W}} &= \left\{w\in\mathscr{W} \mid {}_0^C D_\delta^\beta w \in \mathscr{W}\right\}, \end{aligned} \tag{17}$$

equipped with the inner product $(\mathfrak{w},\vartheta)_{\mathscr{W}} = (\epsilon(\mathfrak{w}),\epsilon(\vartheta))_{\mathfrak{H}_1}$ for $\mathfrak{w}$, $\nu\in\mathscr{W}$. In view of meas $\Sigma^D>0$, there exist, respectively, $C_K>0$ and $c_F>0$, called Korn's and Poincaré constants, such that

$$|\varepsilon(\vartheta)|_{\mathfrak{H}_1} \ge c_K|\nu|_{H_1}, |\nabla\varrho|_H \ge c_F|\varrho|_{H^1}, \varrho\in\mathscr{W}. \tag{18}$$

Let denote by $\mathscr{V} = L^2(0,b;\mathscr{W})$ the Bochner–Lebesgue function space. Since the space $\mathscr{W}$ is reflexive, it remains that $\mathscr{V}$ and its dual space $\mathscr{V}' = L^2(0,b;\mathscr{W}')$ are reflexive too. To study the problem and present our main result, we make the following assumptions: the viscosity operator $\mathfrak{A}$, the elasticity one $\mathfrak{B}$, and the relaxation tensor $\mathfrak{C}$ satisfy the conditions

$$\begin{cases} (a)\ \mathfrak{A}\colon \mathfrak{D}\times\mathbb{T}^d \longrightarrow \mathbb{T}^d, \mathfrak{A} = \left(a_{\imath\mathfrak{j}\mathfrak{k}\ell}\right)\\ (b)\ \mathfrak{A}(\mathfrak{x},\mathfrak{h}) = \left(a_{\imath\mathfrak{j}\mathfrak{k}\ell}(\mathfrak{x})\mathfrak{h}_{\mathfrak{j}\mathfrak{k}}\right), \forall\mathfrak{h} = \left(\mathfrak{h}_{\imath\mathfrak{j}}\right)\in\mathbb{T}^d,\ a.e.\mathfrak{x}\in\mathfrak{D}.\\ (c)\ a_{\imath\mathfrak{j}\mathfrak{k}\ell}\in L^\infty(\mathfrak{D}), \forall\imath,\mathfrak{j},\mathfrak{k},\ell\\ (d)\ \text{there exist } m_{\mathfrak{A}}>0, \text{ such that}\\ \mathfrak{A}(\mathfrak{x},\varrho)\cdot\varrho \ge m_{\mathfrak{A}}\|\varrho\|^2, \text{ for all } \varrho\in\mathbb{T}^d. \end{cases} \tag{19}$$

$$\begin{cases} (a)\mathfrak{B}\colon \mathfrak{D}\times\mathbb{T}^d\to\mathbb{T}^d, \mathfrak{B} = \left(b_{\imath\mathfrak{j}\mathfrak{k}\ell}\right); b_{\imath\mathfrak{j}\mathfrak{k}\ell}\in L^\infty(\mathfrak{D}), \forall\imath,\mathfrak{j},\mathfrak{k},\ell.\\ (b)\mathfrak{B}(\mathfrak{x},\mathfrak{h}) = \left(b_{\imath\mathfrak{j}\mathfrak{k}\ell}(\mathfrak{x})\mathfrak{h}_{\mathfrak{j}\mathfrak{k}}\right), \forall\mathfrak{h} = \left(\mathfrak{h}_{\imath\mathfrak{j}}\right)\in\mathbb{T}^d,\ a.e.\mathfrak{x}\in\mathfrak{D}. \end{cases} \tag{20}$$

$$\begin{cases} (a)\mathfrak{C}\colon \mathfrak{D}\times[0,b]\times\mathbb{T}^d\to\mathbb{T}^d, \quad \mathfrak{C} = \left(\widetilde{e}_{\imath\mathfrak{j}\mathfrak{k}\ell}\right); \widetilde{e}_{\imath\mathfrak{j}\mathfrak{k}\ell}\in L^\infty(\mathfrak{D}),\\ (b)\mathfrak{C}(\mathfrak{x},\delta,\mathfrak{h}) = \left(\widetilde{e}_{\imath\mathfrak{j}\mathfrak{k}\ell}(\mathfrak{x},\delta)\mathfrak{h}_{\mathfrak{j}\mathfrak{k}}\right), \forall\mathfrak{h} = \left(\mathfrak{h}_{\imath\mathfrak{j}}\right)\in\mathbb{T}^{\mathrm{d}},\\ (\mathrm{c})\mathfrak{C} \text{ is Lipschitz continuous with constant } L_{\mathfrak{C}}>0. \end{cases} \tag{21}$$

The initial data and the density forces satisfy

$$\mathfrak{w}_0 \in \mathscr{W}, \quad f_0 \in L^\infty\left(0, b; L^2(\mathfrak{D})^d\right), \; f_\aleph \in L^\infty\left(0, b; L^2\left(\Sigma^\aleph\right)^d\right). \tag{22}$$

## 3. Variational Formulation of the Problem

In this section, we establish the existence of a weak solution to Problem 1. First, we multiply the equilibrium equation by the test function, taking into account the initial and boundary conditions, and we obtain Problem 2 in the form of a variational inequality. Then, we reformulate the problem in operational form (Problem 3), which allows us to pass from a heavy scripture relating to tensorial products to a more convenient one. In order to apply the classical results of functional inclusions, we rewrite the last problem in the form of a nonlinear inclusion, thus giving Problem 4. Note that Problem 2, 3, and 4 are equivalent.

To obtain the variational formulation of Problem 1, we assume that the pair of functions $\mathfrak{u}$ and $\sigma$ are sufficiently smooth. We recall that Green's formula holds:

$$(\sigma, \varepsilon(\vartheta))_{\mathfrak{H}} + (\mathrm{Div}\sigma, \vartheta)_H = \int_\Sigma \sigma\nu \cdot \vartheta, \quad \forall \vartheta \in H_1. \tag{23}$$

From the (11), the boundary conditions (12)–(15) and equality (23), we obtain for all $\vartheta \in \mathscr{W}$

$$\begin{aligned}(\sigma, \varepsilon(\vartheta))_{\mathfrak{H}} &= \left[\int_{\mathfrak{D}} f_0\vartheta + \int_{\Sigma^\aleph} f_\aleph\vartheta\right] + \int_{\Sigma^C} \sigma_\tau\vartheta,\\ &= \langle f, \vartheta\rangle + \int_{\Sigma^C} \sigma_\tau\vartheta_\tau.\end{aligned} \tag{24}$$

Further,

$$(\sigma, \varepsilon(\vartheta))_{\mathfrak{H}} + \int_{\Sigma^C} g\|\vartheta_\tau\| \geq \langle f, \vartheta\rangle, \quad \forall \vartheta \in \mathscr{W}. \tag{25}$$

Let define $\mathscr{J}(\vartheta) = \int_{\Sigma^C} g\|\vartheta_\tau\|$, for all $\vartheta \in \mathscr{W}$, expression (25) leads to the following variational formulation:

*Problem 2.* Find $\mathfrak{w}$: $[0, b] \longrightarrow \widetilde{\mathscr{W}}$, such that

$$\begin{aligned}&\left(\mathfrak{A}\varepsilon\left({}_0^C D_\delta^\beta \mathfrak{w}(\delta)\right), \varepsilon(\vartheta) - \varepsilon(\mathfrak{w}(\delta))\right)_{\mathfrak{H}} + (\mathfrak{B}\varepsilon(\mathfrak{w}(\delta)), \varepsilon(\vartheta) - \varepsilon(\mathfrak{w}(\delta)))_{\mathfrak{H}}\\ &\quad + \left(\int_0^\delta \mathfrak{C}(\delta - \mu)\varepsilon(\mathfrak{w}(\mu))\mathrm{d}\mu, \varepsilon(\vartheta) - \varepsilon(\mathfrak{w}(\delta))\right)_{\mathfrak{H}} + \mathscr{J}(\vartheta) - \mathscr{J}(\mathfrak{w}(\delta))\\ &\geq (f(\delta), \varepsilon(\vartheta) - \varepsilon(\mathfrak{w}(\delta)))_{\mathscr{W}', \mathscr{W}}, \quad \forall \vartheta \in \mathscr{W}, a.e.\delta \in [0, b].\end{aligned} \tag{26}$$

Let define the operators $\mathscr{A}$, $\mathscr{B}$, and $C$ by the expressions

$$\begin{aligned}&(\mathscr{A}\mathfrak{w}(\delta), \vartheta)_{\mathscr{W}', \mathscr{W}} = (\mathfrak{A}\varepsilon(\mathfrak{w}(\delta)), \varepsilon(\vartheta))_{\mathfrak{H}}, \; (\mathscr{B}\mathfrak{w}(\delta), \vartheta)_{\mathscr{W}', \mathscr{W}} = (\mathfrak{B}\varepsilon(\mathfrak{w}(\delta)), \varepsilon(\vartheta))_{\mathfrak{H}},\\ &(C\mathfrak{w}(\delta), \vartheta)_{\mathscr{W}', \mathscr{W}} = \langle \int_0^\delta c(\delta, \mu)\mathfrak{w}(\mu)\mathrm{d}\mu, \vartheta\rangle_{\mathscr{W}', \mathscr{W}} = \left(\int_0^t \mathfrak{C}(t - s)\varepsilon(w(s))\mathrm{d}s, \varepsilon(\nu)\right)_{\mathfrak{H}},\end{aligned} \tag{27}$$

where $c$ is defined below. Note that since $\mathscr{J}$ is convex, we have

$$\mathscr{J}(\mathfrak{w}) - \mathscr{J}(\vartheta) \leq \langle \varrho, \mathfrak{w} - \vartheta\rangle, \; \forall \varrho \in \partial\mathscr{J}(\mathfrak{w}), \; \forall \vartheta \in \mathscr{W}. \tag{28}$$

The following problem can be established in light of all those considerations.

*Problem 3.* Find $\mathfrak{w} \in L^1(0, T; \widetilde{\mathscr{W}})$, such thatx

$$\begin{cases}\langle \mathscr{A}\left({}_0^C D_\delta^\beta \mathfrak{w}(\delta)\right), \vartheta - \mathfrak{w}(\delta)\rangle + \langle \mathscr{B}\mathfrak{w}(\delta), \vartheta - \mathfrak{w}(\delta)\rangle\\ +\langle \mathscr{C}\mathfrak{w}(\delta), \vartheta - \mathfrak{w}(\delta)\rangle + \mathscr{J}(\vartheta) - \mathscr{J}(\mathfrak{w}(\delta)) \geq \langle f(\delta), \vartheta - \mathfrak{w}(\delta)\rangle_{\mathscr{W}'},\\ \text{for all } \vartheta \in \mathscr{W} \text{ and } a.e.\delta \in (0, b).\end{cases} \tag{29}$$

Let now suppose that $\mathfrak{w} \in AC(0, b; \mathscr{W})$ be a solution to Problem 3 and put $\mathfrak{u} = {}_0^C D_\delta^\beta\, \mathfrak{w}$, for $0 < \beta < 1$ where ${}_0^C D_\delta^\beta$ is the Caputo fractional derivative. Note that when $\mathfrak{w} \in AC(0, b; \mathscr{W})$, then $\mathfrak{w}(\delta) = {}_0 D_\delta^{-\beta} \mathfrak{u}(\delta) + \mathfrak{w}_0$ have a sense.

So, we may rewrite our problem as a nonlinear inclusion, driven by a fractional integral operator, and it is given by the problem below.

*Problem 4.* Find $\mathfrak{u} \in L^1(0, b; \mathscr{W})$, such that

$$\begin{cases} \mathscr{A}(\mathfrak{u}(\delta)) + \mathscr{B}\left({}_0 D_\delta^{-\beta} \mathfrak{u}(\delta) + \mathfrak{w}_0\right) + C\left({}_0 D_\delta^{-\beta} \mathfrak{u}(\delta) + \mathfrak{w}_0\right) + \partial \mathscr{J}\left({}_0 D_\delta^{-\beta} \mathfrak{u}(\delta) + \mathfrak{w}_0\right) \ni f(\delta), \\ \text{for } a.e. \delta \in (0, b). \end{cases} \tag{30}$$

Using data from Problems 3 and 4, we reformulate the assumptions. $\mathscr{H}(\mathscr{A}) : \mathscr{A} \in \mathscr{L}(\mathscr{W}, \mathscr{W}')$ is coercive, i.e., there exists a constant $\widetilde{m}_\mathscr{A} > 0$, such that

$$\langle \mathscr{A}\vartheta, \vartheta \rangle \geq \widetilde{m}_\mathscr{A} \|\vartheta\|^2, \text{ for all } \vartheta \in \mathscr{W}. \tag{31}$$

$\mathscr{H}(\mathscr{B}) : \mathscr{B} \in \mathscr{L}(\mathscr{W}, \mathscr{W}')$. $\mathscr{H}(c)$: $c \in \mathscr{C}([0, b]^2, \mathscr{L}(\mathscr{W}; \mathscr{W}'))$; there exists $L_c > 0$, such that

$$\left\| c(\mu, \delta_1) - c(\mu, \delta_2) \right\|_{\mathscr{W}'} \leq L_c \left| \delta_2 - \delta_1 \right|, \tag{32}$$

$\mu, \delta_1, \delta_2 \in [0, b]$, and $\widetilde{m}_c = \max_{(\delta,\mu)\in[0,b]^2} \|c(\delta, \mu)\|$. $\mathscr{H}(f)$; $f \in L^\infty(0, b; \mathscr{W}')$.

Let $\aleph \in \mathbb{N}_+$ be fixed, $\mathfrak{h} = b/\aleph$, $\delta_\mathfrak{k} = \mathfrak{k}\mathfrak{h}$, and for all $f \in L^1(0, b; \mathscr{W})$, $g_\mathfrak{h}^\mathfrak{k}$ be defined by $g_\mathfrak{h}^\mathfrak{k} = 1/\mathfrak{h} \int_{\delta_{\mathfrak{k}-1}}^{\delta_\mathfrak{k}} g(\mu) \mathrm{d}\mu$, for $\mathfrak{k} = 1, \ldots, \aleph$. Consider the so-called Rothe problem, i.e., the discretized problem corresponding to Problem 4, given by as follows:

*Problem 5.* Find $\left\{\mathfrak{u}_\mathfrak{h}^\mathfrak{k}\right\}_{\mathfrak{k}=1}^{\aleph} \subset \mathscr{W}$, such that $\mathfrak{u}_\mathfrak{h}^0 = 0$, and

$$\mathscr{A}\mathfrak{u}_\mathfrak{h}^\mathfrak{k} + \mathscr{B}\left(\mathfrak{w}_\mathfrak{h}^\mathfrak{k}\right) + C\left(\mathfrak{w}_\mathfrak{h}^\mathfrak{k}\right) + \partial \mathscr{J}\left(\mathfrak{w}_\mathfrak{h}^\mathfrak{k}\right) \ni f_\mathfrak{h}^\mathfrak{k}, \tag{33}$$

where $\mathfrak{w}_\mathfrak{h}^\mathfrak{k}$ and $C(\mathfrak{w}_\mathfrak{h}^\mathfrak{k})$ for $\delta \in (0, \delta_\mathfrak{k})$, and $\mathfrak{k} = 1, 2, \ldots, \aleph$, are defined by

$$\mathfrak{w}_\mathfrak{h}^\mathfrak{k} = \mathfrak{w}_0 + \frac{\mathfrak{h}^\beta}{\Gamma(\beta + 1)} \sum_{\mathfrak{j}=1}^{\mathfrak{k}} \mathfrak{u}_\mathfrak{h}^\mathfrak{j} \left[ (\mathfrak{k} - \mathfrak{j} + 1)^\beta - (\mathfrak{k} - \mathfrak{j})^\beta \right], \tag{34}$$

$$C\left(\mathfrak{w}_\mathfrak{h}^\mathfrak{k}\right) = \sum_{\mathfrak{j}=1}^{\mathfrak{k}} \int_{\delta_{\mathfrak{j}-1}}^{\delta_\mathfrak{j}} c(\delta_\mathfrak{k}, \mu) \mathfrak{w}_\mathfrak{h}^\mathfrak{j} \mathrm{d}\mu. \tag{35}$$

First, we shall show the existence of solution for a Problem 5.

**Theorem 3.** *Suppose that conditions $\mathscr{H}(\mathscr{A})$, $\mathscr{H}(\mathscr{B})$, $\mathscr{H}(c)$, and $\mathscr{H}(f)$ are verified. Hence, there exists $\mathfrak{h}_0 > 0$, such that for all $\mathfrak{h} \in (0, \mathfrak{h}_0)$, there is at least one solution to Problem 5.*

*Proof.* Given $\mathfrak{u}_\mathfrak{h}^0, \mathfrak{u}_\mathfrak{h}^1, \ldots, \mathfrak{u}_\mathfrak{h}^{\mathfrak{n}-1}$, we claim that there exist $\mathfrak{u}_\mathfrak{h}^\mathfrak{n} \in \mathscr{W}$, such that (33) hold. From equality (34), we obtain elements $\mathfrak{w}_\mathfrak{h}^0, \mathfrak{w}_\mathfrak{h}^1, \ldots, \mathfrak{w}_\mathfrak{h}^{\mathfrak{n}-1}$. We have to show that there exists elements $\mathfrak{u}_\mathfrak{h}^\mathfrak{n} \in \mathscr{W}$ and such that inclusion (33) holds. Let denote by

$$\vartheta_{\mathfrak{n}-1} = \mathfrak{w}_0 + \frac{\mathfrak{h}^\beta}{\Gamma(\beta + 1)} \sum_{\mathfrak{j}=1}^{\mathfrak{n}-1} \mathfrak{u}_\mathfrak{h}^\mathfrak{j} \left[ (\mathfrak{n} - \mathfrak{j} + 1)^\beta - (\mathfrak{n} - \mathfrak{j})^\beta \right], \widehat{c} = \frac{\mathfrak{h}^\beta}{\Gamma(\beta + 1)}. \tag{36}$$

To accomplish this, we will demonstrate that the multivalued operator

$$\mathscr{W} \ni \vartheta \mapsto \mathscr{A}\vartheta + (\mathscr{B} + C)\left(\vartheta_{\mathfrak{n}-1} + \widehat{c}\vartheta\right) + \partial \mathscr{J}\left(\left(\vartheta_{\mathfrak{n}-1} + \widehat{c}\vartheta\right)\right) \text{ is surjective.} \tag{37}$$

Suppositions $\mathscr{H}(\mathscr{A})$, $\mathscr{H}(\mathscr{B})$, and $\mathscr{H}(c)$ imply the linearity and boundedness of the operator

$$\mathscr{W} \ni \vartheta \mapsto \mathscr{A}\vartheta + (\mathscr{B} + C)\left(\vartheta_{\mathfrak{n}-1} + \widehat{c}\vartheta\right) \in \mathscr{W}^*, \tag{38}$$

so, it is hemicontinuous, and satisfies the condition

$$\begin{aligned} &\langle \mathscr{A}\vartheta + \mathscr{B}\left(\vartheta_{\mathfrak{n}-1} + \widehat{c}\vartheta\right) + C\left(\vartheta_{\mathfrak{n}-1} + \widehat{c}\vartheta\right), \vartheta \rangle \\ &\quad \geq \left(\widetilde{m}_\mathscr{A} - \widehat{c}\left(\|\mathscr{B}\| + b\widetilde{m}_c\right)\right)\|\vartheta\|^2 - \left(\|\mathscr{B}\| + b\widetilde{m}_c\right)\left\|\vartheta_{\mathfrak{n}-1}\right\|\|\vartheta\|. \end{aligned} \tag{39}$$

Thus, under the condition $(\widetilde{m}_\mathscr{A} - \widehat{c}[\|\mathscr{B}\| + b\widetilde{m}_c]) > 0$, the operator

$$\mathscr{A}\vartheta + (\mathscr{B} + C)(\vartheta_{\mathfrak{n}-1} + \hat{c}\vartheta), \quad \text{is coercive and monotone for all } \vartheta \in \mathscr{W}. \tag{40}$$

For the mapping $\mathfrak{x} \mapsto \mathscr{J}(\mathfrak{x})$, which it is, continuous and convex for all $\mathfrak{x} \in \mathscr{W}$, then by Theorem 1, we obtain that $\partial\mathscr{J}(\vartheta_{\mathfrak{n}-1} + \hat{c}\vartheta) \subset 2^{\mathscr{W}'}$ is maximal monotone for all $\vartheta \in \mathscr{W}$. The coercivity is so fulfilled if $(\Gamma(\beta+1)\widetilde{m}_{\mathscr{A}}/\|\mathscr{B}\| + b\widetilde{m}_c)^{1/\beta} > \mathfrak{h}$. Therefore, for all $\mathfrak{h} \in (0, \mathfrak{h}_0)$, where $\mathfrak{h}_0$ is given by

$$\mathfrak{h}_0 = \left(\frac{\widetilde{m}_{\mathscr{A}}\Gamma(1+\beta)}{2(\|\mathscr{B}\| + b\widetilde{m}_c)}\right)^{1/\beta}. \tag{41}$$

We may now use Theorem 2 to guarantee the surjectivity of the operator $\vartheta \mapsto \mathscr{A}\vartheta + (\mathscr{B} + C)(\vartheta_{\mathfrak{n}-1} + \hat{c}\vartheta) + \partial\mathscr{J}((\vartheta_{\mathfrak{n}-1} + \hat{c}\vartheta))$, for all $0 < \mathfrak{h} < \mathfrak{h}_0$. Thus, we conclude that there exists elements $\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}} \in \mathscr{W}$, satisfying the (33). This completes the proof of the theorem.

Note that $\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}} \in \mathscr{W}$ is solution of (33), also means that there exists $\varsigma_{\mathfrak{h}}^{\mathfrak{n}} \in \partial\mathscr{J}(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}})$, such that

$$\mathscr{A}\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{k}} + \mathscr{B}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}}\right) + C\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}}\right) + \varsigma_{\mathfrak{h}}^{\mathfrak{n}} = f_{\mathfrak{h}}^{\mathfrak{k}}. \tag{42}$$

□

It is noteworthy that hereafter, we use *Cte* to denote a generic positive constant independent of $\mathfrak{h}$. The sequence of solutions to Rothe Problem 5 is estimated in the finding that follows:

**Lemma 1.** *Let's say that all four hypotheses $\mathscr{H}(\mathscr{A})$, $\mathscr{H}(\mathscr{B})$, $\mathscr{H}(c)$, and $\mathscr{H}(f)$, are fulfilled, then there exists $\mathfrak{h}_1 > 0$ independent of h, such that for all $\mathfrak{h} \in (0, \mathfrak{h}_1)$, the solutions of Problem 5$(\mathfrak{k} = \mathfrak{n})$ satisfy*

$$\max_{\mathfrak{n}=1,2,\ldots,\aleph}\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\| \le Cte, \tag{43}$$

$$\max_{\mathfrak{n}=1,2,\ldots,\aleph}\left\|\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}\right\| \le Cte, \tag{44}$$

*where the constant Cte here is independent of $\mathfrak{h}$ and $\mathfrak{n}$.*

*Proof.* We multiply equation (33) by taking $\mathfrak{k} = \mathfrak{n}$ and $\vartheta = \mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}} - \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}} \in \mathscr{W}$ to get

$$\langle\mathscr{A}\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}, \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle + \langle\mathscr{B}\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}, \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle + \langle C\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}, \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle + \mathscr{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}\right) - \mathscr{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}} - \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right) \le \langle f_{\mathfrak{h}}^{\mathfrak{n}}, \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle. \tag{45}$$

By the expression (34) of $\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}$ and hypotheses $\mathscr{H}(\mathscr{B})$ and $\mathscr{H}(c)$, we have

$$\begin{aligned}
\langle\mathscr{B}\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}, \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle &= \langle\mathscr{B}\left(\mathfrak{w}_0 + \hat{c}\sum_{\mathfrak{j}=1}^{\mathfrak{n}}\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{j}}\left((\mathfrak{n}-\mathfrak{j}+1)^\beta - (\mathfrak{n}-\mathfrak{j})^\beta\right)\right), \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle \\
&\ge -\left\|\mathscr{B}\mathfrak{u}_0\right\|_{\mathscr{W}'}\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\| - \|\mathscr{B}\|\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{j}}\right\|\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\|\hat{c}\sum_{\mathfrak{j}=1}^{\mathfrak{n}}\left[\left((\mathfrak{n}-\mathfrak{j}+1)^\beta - (\mathfrak{n}-\mathfrak{j})^\beta\right)\right] - \hat{c}\|\mathscr{B}\|\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\|^2,
\end{aligned} \tag{46}$$

$$\begin{aligned}
\langle C\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}, \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle &= \langle\sum_{\mathfrak{j}=1}^{\mathfrak{n}}\int_{\delta_{\mathfrak{j}-1}}^{\delta_{\mathfrak{j}}} c(\delta_{\mathfrak{n}}, \mu)\left[\mathfrak{u}_0 + \hat{c}\sum_{\ell=1}^{\mathfrak{j}}\mathfrak{u}_{\mathfrak{h}}^{\ell}\left((\mathfrak{n}-\ell+1)^\beta - (\mathfrak{n}-\ell)^\beta\right)\right]d\mu, \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle \\
&= \langle\varrho_{\mathfrak{h}}^{\mathfrak{n}}, \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle + \langle\int_{\delta_{\mathfrak{n}-1}}^{\delta_{\mathfrak{n}}} c(\delta_{\mathfrak{n}}, \mu)\left[\mathfrak{w}_0 + \frac{\mathfrak{h}^\beta}{\Gamma(\beta+1)}\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right]d\mu, \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle \\
&\ge -2b\widetilde{m}_c\left[\left\|\mathfrak{w}_0\right\| + \hat{c}\sum_{\ell=1}^{\mathfrak{n}-1}\left\|\mathfrak{u}_{\mathfrak{h}}^{\ell}\right\|\left((\mathfrak{n}-\ell+1)^\beta - (\mathfrak{n}-\ell)^\beta\right)\right]\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\| \\
&\quad - \frac{b\widetilde{m}_c\mathfrak{h}^\beta}{\Gamma(\beta+1)}\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\|^2,
\end{aligned} \tag{47}$$

where

$$\begin{aligned}\varrho_{\mathfrak{h}}^{\mathfrak{n}} = \sum_{\mathfrak{j}=1}^{\mathfrak{n}-1}\int_{\delta_{\mathfrak{j}-1}}^{\delta_{\mathfrak{j}}} c(\delta_{\mathfrak{n}},\mu)\left[\mathfrak{w}_0 + \frac{\mathfrak{h}^{\beta}}{\Gamma(\beta+1)}\sum_{\ell=1}^{\mathfrak{j}}\mathfrak{u}_{\mathfrak{h}}^{\ell}\left((\mathfrak{n}-\ell+1)^{\beta}-(\mathfrak{n}-\ell)^{\beta}\right)\right]\mathrm{d}\mu \\ + \int_{\delta_{\mathfrak{n}-1}}^{\delta_{\mathfrak{n}}} c(\delta_{\mathfrak{n}},\mu)\left[\mathfrak{u}_0 + \frac{\mathfrak{h}^{\beta}}{\Gamma(\beta+1)}\sum_{\ell=1}^{\mathfrak{n}-1}\mathfrak{u}_{\mathfrak{h}}^{\ell}\left((\mathfrak{n}-\ell+1)^{\beta}-(\mathfrak{n}-\ell)^{\beta}\right)\right]\mathrm{d}\mu.\end{aligned} \tag{48}$$

According to the definition of $\mathcal{J}$, we have

$$\left|\mathcal{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}}\right) - \mathcal{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}} - \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right)\right| \le g\int_{\Sigma^C}\left|\left\|\mathfrak{w}_{\tau\mathfrak{h}}^{\mathfrak{k}}\right\| - \left\|\mathfrak{w}_{\tau\mathfrak{h}}^{\mathfrak{k}} - \mathfrak{u}_{\tau\mathfrak{h}}^{\mathfrak{k}}\right\|\right|\mathrm{d}a \le gc_F\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{j}}\right\|, \tag{49}$$

where $c_F$ is Poincaré's constant. This implies that

$$\langle\varsigma_{\mathfrak{h}}^{\mathfrak{n}},\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle \ge \mathcal{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}}\right) - \mathcal{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}} - \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right) \ge -gc_F\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\|. \tag{50}$$

Now, we conclude that because of the coercivity of the operator $\mathcal{A}$ and expressions (46), (47), and (50),

$$\begin{aligned}\langle f_{\mathfrak{h}}^{\mathfrak{n}},\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle &\ge \langle\mathcal{A}\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}},\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle + \langle\mathcal{B}\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}},\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle + \langle C\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}},\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\rangle + \mathcal{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}}\right) - \mathcal{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}} - \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right) \\ &\ge m_{\mathcal{A}}\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\|^2 - \widehat{c}\|\mathcal{B}\|\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\|^2 - \left(\|\mathcal{B}\mathfrak{w}_0\|_{\mathcal{W}'} + gc_p\right)\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\| \\ &\quad - \widehat{c}\|\mathcal{B}\|\sum_{\mathfrak{j}=1}^{\mathfrak{n}-1}\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{j}}\right\|\left((\mathfrak{n}-\mathfrak{j}+1)^{\beta}-(\mathfrak{n}-\mathfrak{j})^{\beta}\right)\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\| \\ &\quad - 2b\widetilde{m}_c\left[\|\mathfrak{w}_0\| + \widehat{c}\sum_{\ell=1}^{\mathfrak{n}-1}\left\|\mathfrak{u}_{\mathfrak{h}}^{\ell}\right\|\left((\mathfrak{n}-\ell+1)^{\beta}-(\mathfrak{n}-\ell)^{\beta}\right)\right]\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\| \\ &\quad - b\widetilde{m}_c\widehat{c}\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\|^2.\end{aligned} \tag{51}$$

Thus, gives

$$\begin{aligned}&\left\|f_{\mathfrak{z}}^{\mathfrak{n}}\right\|_{\mathcal{W}'} + \widehat{c}\left(\|\mathcal{B}\| + 2b\widetilde{m}_c\right) + \|\mathfrak{w}_0\|\Big)\sum_{\mathfrak{j}=1}^{\mathfrak{n}-1}\left\|\mathfrak{u}_{\mathfrak{z}}^{\mathfrak{j}}\right\|\left((\mathfrak{n}-\mathfrak{j}+1)^{\beta}-(\mathfrak{n}-\mathfrak{j})^{\beta}\right) \\ &+\|\mathcal{B}\mathfrak{w}_0\|_{\mathcal{W}'} + gc_p \ge \left(\widetilde{m}_{\text{Å}} - \widehat{c}\left(\|\mathcal{B}\| + b\widetilde{m}_c\right)\right)\left\|\mathfrak{u}_{\mathfrak{z}}^{\mathfrak{n}}\right\|.\end{aligned} \tag{52}$$

If we choose $\mathfrak{h}_1 = (\widetilde{m}_{\mathcal{A}}\Gamma(1+\beta)/2(\|\mathcal{B}\| + b\widetilde{m}_c))^{1/\beta}$, we deduce that

$$\widetilde{m}_{\mathcal{A}} - \widehat{c}\left(\|\mathcal{B}\| + b\widetilde{m}_c\right) \ge \frac{\widetilde{m}_{\mathcal{A}}}{2} \text{ for all } \mathfrak{h} \in (0,\mathfrak{h}_1), \tag{53}$$

(note that we have $\mathfrak{h}_0 > \mathfrak{h}_1$). Subsequently,

$$\begin{aligned}&\frac{2\left\|f_{\mathfrak{h}}^{\mathfrak{n}}\right\|_{\mathcal{W}'}}{\widetilde{m}_{\mathcal{A}}} + \frac{2\left(\|\mathcal{B}\mathfrak{w}_0\|_{\mathcal{W}'} + gc_p\right)}{\widetilde{m}_{\mathcal{A}}} \\ &+ \frac{2\widehat{c}\left(\|\mathcal{B}\| + 2b\widetilde{m}_c\right) + \|\mathfrak{u}_0\|\Big)}{\widetilde{m}_{\mathcal{A}}}\sum_{\mathfrak{j}=1}^{\mathfrak{n}-1}\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{j}}\right\|\left((\mathfrak{n}-\mathfrak{j}+1)^{\beta}-(\mathfrak{n}-\mathfrak{j})^{\beta}\right) \ge \left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\|.\end{aligned} \tag{54}$$

From hypothesis $\mathcal{H}(f)$, $\|f_{\mathfrak{h}}^{\mathfrak{n}}\|_{\mathcal{W}'} \leq \sup_{\delta \in [0,b]} \|f(\delta)\|_{\mathcal{W}'}$, for all $\mathfrak{h} > 0$; $\mathfrak{n} = 1, \overline{\aleph}$. Next, thanks to the generalized discrete Gronwall inequality, see that for

$$c_0 = \frac{2\left\|f_{\mathfrak{h}}^{\mathfrak{n}}\right\|_{\mathcal{W}'}}{\widetilde{m}_{\mathcal{A}}} + \frac{2\left(\left\|\mathcal{B}\mathfrak{w}_0\right\|_{\mathcal{W}'} + gc_p\right)}{\widetilde{m}_{\mathcal{A}}}$$
$$\left\|\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right\| \leq c_0 \exp \frac{2}{\widetilde{m}_{\mathcal{A}}}\left(\widehat{c}\left(\|\mathcal{B}\| + 2b\widetilde{m}_c\right) + \left\|\mathfrak{w}_0\right\|\right) \sum_{\mathfrak{j}=1}^{\mathfrak{n}-1}\left((\mathfrak{n} - \mathfrak{j} + 1)^{\beta} - (\mathfrak{n} - \mathfrak{j})^{\beta}\right) \leq Cte. \tag{55}$$

So, the estimate (43) is verified.

Next, by the definition of the $\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}$, which is written in terms of $\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}$, we find the following estimate:

$$\begin{aligned}\left\|\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}\right\| &\leq \left\|\mathfrak{w}_0\right\| + \frac{1}{\Gamma(\beta+1)} \sum_{\ell=1}^{n}\left\|\mathfrak{u}_{\mathfrak{h}}^{\ell}\right\|\left(\left(\delta_{\mathfrak{n}-\ell+1}^{\beta} - \delta_{\mathfrak{n}-\ell}^{\beta}\right)\right. \\ &\leq \left\|\mathfrak{w}_0\right\| + \frac{\delta_{\mathfrak{n}}^{\beta} Cte}{\Gamma(\beta+1)} \leq \left\|\mathfrak{w}_0\right\| + \frac{b^{\beta} Cte}{\Gamma(\beta+1)} = Cte.\end{aligned} \tag{56}$$

Thus concludes the proof of Lemma 1. □

It is now necessary to demonstrate our major result, that a solution to Problem 3 exists. We define the interpolate functions $\overline{\mathfrak{u}}_{\mathfrak{h}}, \overline{\mathfrak{w}}_{\mathfrak{h}}$: $[0,b] \longrightarrow \mathcal{W}$ by

$$\begin{aligned}\overline{\mathfrak{u}}_{\mathfrak{h}}(\delta) &= \mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}, \overline{\mathfrak{w}}_{\mathfrak{h}}(\delta) \\ &= \mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}, \text{ and } \varsigma(\delta) \\ &= \varsigma_{\mathfrak{h}}^{\mathfrak{n}}, \quad \delta \in [\delta_{\mathfrak{n}-1}, \delta_{\mathfrak{n}}], \ \mathfrak{n} = 1, \ldots, \aleph,\end{aligned} \tag{57}$$

as $\mathfrak{h} \longrightarrow 0$, where $\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}} \in L^{1/\kappa}(0,b;\mathcal{W})$ is a solution to Problem 4.

**Lemma 2.** *Assume that $\mathcal{H}(\mathcal{A})$, $\mathcal{H}(\mathcal{B})$, $\mathcal{H}(c)$, and $\mathcal{H}(f)$, hold. Let $\kappa \in (0,\eta)$, where $\eta = max(1/2,\beta)$, and $\{\mathfrak{h}_{\mathfrak{n}}\}$be a sequence such that $\mathfrak{h}_{\mathfrak{n}} \longrightarrow 0$, as $\mathfrak{n} \longrightarrow \infty$, to make simple, let it denoted by $\mathfrak{h}$. Then, we have*

$$\overline{\mathfrak{u}}_{\mathfrak{h}} \rightharpoonup \mathfrak{u} \text{ weakly in } L^{1/\kappa}(0,b;\mathcal{W}), \tag{58}$$

$$\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta) \rightharpoonup {}_0D_{\delta}^{-\beta}\mathfrak{u}(\delta) + \mathfrak{w}_0 \text{ weakly in } \mathcal{W}, \text{ as } \mathfrak{h} \longrightarrow 0. \tag{59}$$

*Proof.* Let $0 < \kappa < \eta = \max(1/2,\beta)$, then by applying estimate (43), we have

$$\left\|\overline{\mathfrak{u}}_{\mathfrak{h}}\right\|_{L^{1/\kappa}(0,b;\mathcal{W})}^{1/\kappa} = \int_0^b \left\|\overline{\mathfrak{u}}_{\mathfrak{h}}(\mu)\right\|^{1/\kappa} d\mu = \sum_{\iota=1}^{\aleph} \int_{\delta_{\iota-1}}^{\delta_{\iota}} \left\|\mathfrak{u}_{\mathfrak{h}}^{\iota}\right\|^{1/\kappa} d\mu = \ell_1^{1/\kappa} \sum_{\iota=1}^{\aleph} \int_{\delta_{\iota-1}}^{\delta_{\iota}} d\mu \leq Cte. \tag{60}$$

Hence, it follows that $\{\overline{\mathfrak{u}}_{\mathfrak{h}}\}$ is bounded in $L^{1/\kappa}(0,b;\mathcal{W})$ and consequently, we deduce that there exists $\mathfrak{u} \in L^{1/\kappa}(0,b;\mathcal{W})$, such that

$$\overline{\mathfrak{u}}_{\mathfrak{h}} \longrightarrow \mathfrak{u} \text{ weakly in } L^{1/\kappa}(0,b;\mathcal{W}), \text{ as } \mathfrak{h} \longrightarrow 0. \tag{61}$$

For any $g \in \mathcal{W}'$ and $\delta \in [0,b]$, note that function $(\delta-\mu)^{\beta-1}\chi_{[0,t]}(\mu)g$ for $\mu \in (0,b)$, is in $L^{1/\kappa'}(0,b;\mathcal{W}')$, where $\kappa' = 1-\kappa$, indeed

$$\begin{aligned}\left\|(\delta - \cdot)^{\beta-1}\chi_{[0,\delta]}g\right\|_{L^{1/\kappa}(0,b;\mathcal{W})}^{\acute{\kappa}} &= \int_0^{\delta} \|g\|_{\mathcal{W}}^{1/\acute{\kappa}}(\delta-\mu)^{\beta-1/\kappa'} d\mu \\ &\leq \frac{1-\kappa}{\beta-\kappa}\|g\|_{\mathcal{W}}^{1/\acute{\kappa}}|\delta-\mu|^{\beta-\kappa/1-\kappa}|_0^{\delta} < \infty.\end{aligned} \tag{62}$$

Hence, this leads

$$\begin{aligned}&\left|\langle g, \frac{1}{\Gamma(\beta)}\int_0^{\delta}(\delta-\mu)^{\beta-1}\overline{\mathfrak{u}}_{\mathfrak{h}}(\mu)\mathrm{d}\mu - \frac{1}{\Gamma(\beta)}\int_0^{\delta}(\delta-\mu)^{\beta-1}\mathfrak{u}(\mu)\mathrm{d}\mu\rangle\right|\\ &\leq \frac{1}{\Gamma(\beta)}\int_0^{\delta}\left|\langle(\delta-\mu)^{\beta-1}g, \overline{\mathfrak{u}}_{\mathfrak{h}}(\mu)-\mathfrak{u}(\mu)\rangle\right|\mathrm{d}\mu\\ &\leq \frac{1}{\Gamma(\beta)}\left\|(\delta-\cdot)^{\beta-1}\chi_{[0,\delta]}g\right\|_{L^{1/\kappa'}(0,b;\mathscr{W}')}\left\|\overline{\mathfrak{u}}_{\mathfrak{h}}-\mathfrak{u}\right\|_{L^{1/\kappa}(0,b;\mathscr{W})}\longrightarrow 0, \text{ as } \mathfrak{h}\longrightarrow 0.\end{aligned} \tag{63}$$

Thus, we deduce that

$${}_0D_{\delta}^{-\beta}\overline{\mathfrak{u}}_{\mathfrak{h}}(\delta)\longrightarrow {}_0D_{\delta}^{-\beta}\mathfrak{u}(\delta) \text{ weakly in } \mathscr{W}, \text{ as } \mathfrak{h}\longrightarrow 0, \tag{64}$$

for all $\delta \in [0,b]$. Moreover, remark that we have

$$\begin{aligned}{}_0D_{\delta}^{-\beta}\overline{\mathfrak{u}}_{\mathfrak{h}}(\delta) &= \widehat{c}\sum_{\mathfrak{j}=1}^{\mathfrak{n}}\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{j}}\left((\mathfrak{n}-\mathfrak{j}+1)^{\beta}-(\mathfrak{n}-\mathfrak{j})^{\beta}\right)\\ &= \frac{1}{\Gamma(\beta)}\int_0^{\delta_{\mathfrak{n}}}(\delta_{\mathfrak{n}}-\mu)^{\beta-1}\overline{\mathfrak{u}}_{\mathfrak{h}}(\mu)\mathrm{d}\mu\\ &\cdot\frac{1}{\Gamma(\beta)}\int_0^{\delta}(\delta_{\mathfrak{n}}-\mu)^{\beta-1}\overline{\mathfrak{u}}_{\mathfrak{h}}(\mu)\mathrm{d}\mu - \int_{\delta}^{\delta_{\mathfrak{n}}}(\delta_{\mathfrak{n}}-\mu)^{\beta-1}\overline{\mathfrak{u}}_{\mathfrak{h}}(\mu)\mathrm{d}\mu,\end{aligned} \tag{65}$$

so, it entails

$$\begin{aligned}(I) = \left\|{}_0D_{\delta}^{-\beta}\overline{\mathfrak{u}}_{\mathfrak{h}}(\delta)-\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta)+\mathfrak{w}_0\right\| &\leq \frac{1}{\Gamma(\beta)}\left\|\int_{\delta}^{\delta_{\mathfrak{n}}}(\delta_{\mathfrak{n}}-\mu)^{\beta-1}\overline{\mathfrak{u}}_{\mathfrak{h}}(\mu)\mathrm{d}\mu\right\|\\ &+\frac{1}{\Gamma(\beta)}\left\|\int_0^{\delta}\left((\delta_{\mathfrak{n}}-\mu)^{\beta-1}-(\delta-\mu)^{\beta-1}\right)\overline{\mathfrak{u}}_{\mathfrak{h}}(\mu)\mathrm{d}\mu\right\|.\end{aligned} \tag{66}$$

Using estimate (43) again, one has

$$\begin{aligned}(I) &\leq \frac{\ell_1}{\Gamma(\beta)}\left(\int_{\delta}^{\delta_{\mathfrak{n}}}(\delta_{\mathfrak{n}}-\mu)^{\beta-1}\mathrm{d}\mu + \int_0^{\delta}\left|(\delta-\mu)^{\beta-1}-(\delta_{\mathfrak{n}}-\mu)^{\beta-1}\right|\mathrm{d}\mu\right)\\ &\leq \frac{\ell_1}{\Gamma(\beta+1)}\left((\delta_{\mathfrak{n}}-\delta)^{\beta}-\delta_{\mathfrak{n}}^{\beta}+(\delta_{\mathfrak{n}}-\delta)^{\beta}+\delta_{\mathfrak{n}}^{\beta}\right),\end{aligned} \tag{67}$$

for $\delta \in (\delta_{\mathfrak{n}-1}, \delta_{\mathfrak{n}}]$. Therefore, estimate (67), ensure that, for all $\delta \in [0,b]$

$$\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta)-\mathfrak{w}_0-{}_0D_{\delta}^{-\beta}\overline{\mathfrak{u}}_{\mathfrak{h}}(\delta)\longrightarrow 0 \text{ strongly in } \mathscr{W}, \text{ as } \mathfrak{h}\longrightarrow 0. \tag{68}$$

Employing the latter and taking convergence (64) into account, we get

$$\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta)\rightharpoonup {}_0D_{\delta}^{-\beta}\mathfrak{u}(\delta)+\mathfrak{w}_0 \text{ weakly in } \mathscr{W}, \text{ as } \mathfrak{h}\longrightarrow 0. \tag{69}$$

□

**Lemma 3.** *Let* $\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}} \in \mathscr{W}$ *and* $\varsigma_{\mathfrak{h}}^{\mathfrak{n}} \in \partial\mathscr{J}(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}})$ *are solutions of equation (42). Then, we have*

$$\varsigma_{\mathfrak{h}}^{\mathfrak{n}} \longrightarrow \varsigma \text{ weakly in } \mathscr{W}', \tag{70}$$

*where* $\varsigma(\delta) \in \partial\mathscr{J}(\mathfrak{w}(\delta))$ *and* $\mathfrak{w}(\delta)$ *is given by the convergence (59).*

*Proof.* Let's remind that $\mathscr{J}(\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta)) = \int_{\Sigma^C} g\|\gamma\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta)\|d\mu$, where $\gamma(\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta)) = \overline{\mathfrak{w}}_{\tau\mathfrak{h}}(\delta)$; furthermore, using the compactness of trace map $\gamma: \mathscr{W} \longrightarrow L^2(\Sigma^C)$, it follows from the weak convergence $\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta) \rightharpoonup \mathfrak{w}(\delta) = {}_0D_\delta^{-\beta}\mathfrak{u}(\delta) + \mathfrak{w}_0$, in $\mathscr{W}$ that $\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta) \longrightarrow \mathfrak{w}(\delta)$, strongly in $L^2(\Sigma^C)$. Subsequently, we deduce that

$$\mathscr{J}\left(\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta)\right) \longrightarrow \mathscr{J}(\overline{\mathfrak{w}}(\delta)). \tag{71}$$

For all $\delta \in [\delta_{\mathfrak{n}-1}, \delta_{\mathfrak{n}}]$, we have first

$$\left|\langle\varsigma_{\mathfrak{h}}^{\mathfrak{n}}, \vartheta\rangle\right| \le \left|\mathscr{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}} - \vartheta\right) - \mathscr{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}}\right)\right|, \tag{72}$$

and second

$$\begin{aligned}\left|\langle\varsigma_{\mathfrak{h}}^{\mathfrak{n}}, \vartheta\rangle\right| &\le \left|\mathscr{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}} - \vartheta\right) - \mathscr{J}\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{k}}\right)\right| \le g\int_{\Sigma^C} \left\|\gamma\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta) - \gamma\vartheta\right\| - \left\|\gamma\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta)\right\|da, \\ &\le g\int_{\Sigma^C} \|\gamma\vartheta\| da \le gc_F\|\vartheta\|.\end{aligned} \tag{73}$$

This leads to

$$\left\|\varsigma_{\mathfrak{h}}^{\mathfrak{n}}\right\|_{\mathscr{W}'} \le gc_F. \tag{74}$$

Then, there exists a subsequence noted $\varsigma_{\mathfrak{h}}^{\mathfrak{n}}$, such that $\varsigma_{\mathfrak{h}}^{\mathfrak{n}} \longrightarrow \varsigma$ weakly in $\mathscr{W}'$ ($\mathscr{W}'$ is reflexive space). We claim that $\varsigma(\delta) \in \partial\mathscr{J}(\mathfrak{w}(\delta))$. Indeed, we have that the subdifferential $\partial\mathscr{J}(\mathfrak{w}(\delta))$ is identical to $\gamma^*\partial\mathfrak{J}(\gamma\mathfrak{w}(\delta))$, where $\mathfrak{J}: L_2(\Sigma^C) \longrightarrow \mathbb{R}$. It is sufficient to prove then that

$$\mathfrak{J}(\gamma\mathfrak{w}(\delta)) - \mathfrak{J}(\gamma\vartheta) \le \langle\tilde{\varsigma}, \gamma\mathfrak{w}(\delta) - \gamma\vartheta\rangle_{L_2(\Sigma^C)} = \langle\gamma^*\tilde{\varsigma}, \mathfrak{w}(\delta) - \vartheta\rangle_{\mathscr{W}'}, \text{ for all } \vartheta \in \mathscr{W}. \tag{75}$$

Otherwise, let us show that $\gamma^*\tilde{\varsigma}(\delta) = \varsigma(\delta)$, with $\tilde{\varsigma}(\delta) \in \partial\mathfrak{J}(\gamma\mathfrak{w}(\delta))$. Combining the strong convergence

$$\gamma\mathfrak{w}_{\mathfrak{h}}(\delta) \longrightarrow \gamma\mathfrak{w}(\delta) \text{ in } L_2\left(\Sigma^C\right). \tag{76}$$

with the weak convergence of a subsequence of $\tilde{\varsigma}_{\mathfrak{h}}(\delta)$ to $\tilde{\varsigma}(\delta)$, we pass to the limit as $\mathfrak{h} \longrightarrow 0$, we have

$$\begin{aligned}\lim\mathfrak{J}\left(\gamma\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta)\right) - \mathfrak{J}(\gamma\vartheta) &\le \lim\langle\tilde{\varsigma}_{\mathfrak{h}}(\delta), \gamma\overline{\mathfrak{w}}_{\mathfrak{h}}(\delta) - \gamma\vartheta\rangle_{L_2(\Sigma^C)} \\ &= \langle\tilde{\varsigma}(\delta), \gamma(\mathfrak{w}(\delta) - \vartheta)\rangle_{L_2(\Sigma^C)}, \text{ for all } \vartheta \in \mathscr{W}.\end{aligned} \tag{77}$$

Since $\gamma^*\tilde{\varsigma}_{\mathfrak{h}} = \varsigma_{\mathfrak{h}}$, and $\gamma^*$ is linear, it follows that

$$\varsigma_{\mathfrak{h}}(\delta) = \gamma^*\tilde{\varsigma}_{\mathfrak{h}}(\delta) \rightharpoonup \gamma^*\tilde{\varsigma}(\delta) \text{ weakly in } \mathscr{W}'. \tag{78}$$

The uniqueness of limit guarantees that $\gamma^*\tilde{\varsigma}(\delta) = \varsigma(\delta)$. Now by definition of the subdifferential of $\mathscr{J}$, we have

$$\mathscr{J}(\mathfrak{w}(\delta)) - \mathscr{J}(\vartheta) = \mathfrak{J}(\gamma\mathfrak{w}(\delta)) - \mathfrak{J}(\gamma\vartheta) \le \langle\gamma^*\tilde{\varsigma}(\delta), \mathfrak{w}(\delta) - \vartheta\rangle_{\mathscr{W}'}, \text{ for all } \vartheta \in \mathscr{W}. \tag{79}$$

This entails $\varsigma(\delta)$ is in $\partial\mathscr{J}(\mathfrak{w}(\delta))$. Thus, complete the proof of the lemma. □

**Theorem 4.** *Let* $\mathscr{H}(\mathscr{A})$, $\mathscr{H}(\mathscr{B})$, $\mathscr{H}(c)$, *and* $\mathscr{H}(f)$ *be satisfied. Assume that* $\kappa \in (0, \eta)$, *where* $\eta = max(1/2, \beta)$. *Then, there exists a solution* $\mathfrak{w} \in L^1(0, b; \mathscr{W})$ *and* ${}_0^C D_\delta\mathfrak{w} \in L^{1/\kappa}(0, b; \mathscr{W})$, *satisfying Problem 3.*

*Proof.* Consider now $\tilde{\mathscr{A}}$, $\tilde{\mathscr{B}}$, and $\tilde{C}$, the Nemytskii operators that correspond to $\mathscr{A}$, $\mathscr{B}$, and $C$, respectively, which are defined by $(\tilde{\mathscr{A}}\mathfrak{u})(\delta) = \mathscr{A}\mathfrak{u}(\delta)$, $(\tilde{\mathscr{B}}\mathfrak{u})(\delta) = \mathscr{B}(\mathfrak{w}_0 + {}_0D_\delta^{-\beta}\mathfrak{u}(\delta))$, and $(\tilde{C}\mathfrak{u})(\delta) = C(\mathfrak{w}_0 + {}_0D_\delta^{-\beta}\mathfrak{u}(\delta))$. Since $\mathscr{A} \in \mathscr{L}(\mathscr{W}, \mathscr{W}')$ and $\overline{\mathfrak{u}}_{\mathfrak{h}} \longrightarrow \mathfrak{u}$ weakly in $L^{1/\kappa}(0, b; \mathscr{W})$, as $\mathfrak{h} \longrightarrow 0$, then we have

$$\tilde{\mathscr{A}}\overline{\mathfrak{u}}_{\mathfrak{h}} \rightharpoonup \tilde{\mathscr{A}}\mathfrak{u} \text{ weakly in } L^{1/\kappa}(0, b; \mathscr{W}'). \tag{80}$$

Note that $1/\kappa > 2$, then $L^{1/\kappa}(0,b;\mathscr{W}') \subset \mathscr{V}$. Now by hypothesis that $\mathscr{B} \in \mathscr{L}(\mathscr{W},\mathscr{W}')$, one has

$$\langle \mathscr{B}\left(\mathfrak{w}_0 + {}_0D_\delta^{-\beta}\overline{\mathfrak{u}}_{\mathfrak{h}}(\delta)\right), \vartheta(\delta)\rangle \leq \left[\|\mathscr{B}\mathfrak{w}_0\| + \frac{\ell_1 b^\beta}{\Gamma(\beta+1)}\|\mathscr{B}\|\right]\|\vartheta(\delta)\|. \tag{81}$$

Furthermore, we use the convergence

$$\langle \mathscr{B}\left(\mathfrak{w}_0 + {}_0D_\delta^{-\beta}\overline{\mathfrak{u}}_{\mathfrak{h}}(\delta)\right), \vartheta(\delta)\rangle \longrightarrow \langle \mathscr{B}\left(\mathfrak{w}_0 + {}_0D_\delta^{-\beta}\mathfrak{u}(\delta)\right), \vartheta(\delta)\rangle \text{ a.e. in } [0,b]. \tag{82}$$

Applying the Lebesgue dominated convergence theorem, we deduce that

$$\widetilde{\mathscr{B}}\overline{\mathfrak{u}}_{\mathfrak{h}} \rightharpoonup \widetilde{\mathscr{B}}\mathfrak{u} \text{ weakly in } \mathscr{V}'. \tag{83}$$

Now by hypothesis on $c$, $(C \in \mathscr{L}(\mathscr{W},\mathscr{W}'), c \in C(0,b;\mathscr{L}(\mathscr{W},\mathscr{W}')))$, for $\delta \in [\delta_{\mathfrak{n}-1},\delta_{\mathfrak{n}}]$

$$\begin{aligned}
\left\|C\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}\right) - \int_0^\delta c(\delta,\mu)\mathfrak{w}(\mu)\mathrm{d}\mu\right\| &= \left\|\sum_{\mathfrak{j}=1}^{\mathfrak{n}}\left[\int_{\delta_{\mathfrak{j}-1}}^{\delta_{\mathfrak{j}}} c(\delta_{\mathfrak{n}},\mu)\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{j}}\mathrm{d}\mu - \int_{\delta_{\mathfrak{j}-1}}^{\delta} c(\delta,\mu)\mathfrak{w}(\mu)\mathrm{d}\mu\right]\right\| \\
&\leq \int_0^\delta\left\|c(\delta_{\mathfrak{n}},\mu)\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{j}}\mathrm{d}\mu - c(\delta,\mu)\mathfrak{w}(\mu)\mathrm{d}\mu\right\| \\
&\quad + \sum_{\mathfrak{j}=1}^{\mathfrak{n}}\int_\delta^{\delta_{\mathfrak{j}}} c(\delta_{\mathfrak{n}},\mu)\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{j}}\mathrm{d}\mu.
\end{aligned} \tag{84}$$

Using the boundedness of $\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{j}}$, $\mathfrak{j} = 1,\ldots,\mathfrak{n}$ and the Lipschitz continuity of $c(\cdot,\mu)$, this implies

$$\left\|C\left(\mathfrak{u}_{\mathfrak{h}}^{\mathfrak{n}}\right) - \int_0^\delta c(\delta,\mu)\mathfrak{u}(\mu)\mathrm{d}\mu\right\| \leq \mathfrak{h}Cte. \tag{85}$$

That is to conclude

$$C\left(\mathfrak{w}_{\mathfrak{h}}^{\mathfrak{n}}\right) \longrightarrow \int_0^\delta c(\delta,\mu)\mathfrak{w}(\mu)\mathrm{d}\mu, \text{ strongly in } \mathscr{W}, \text{ as } \mathfrak{h} \longrightarrow 0. \tag{86}$$

Since $c \in C(0,b;\mathscr{L}(\mathscr{W},\mathscr{W}'))$, it is obvious that $\int_0^\delta c(\delta,\mu)\mathfrak{w}(\mu)\mathrm{d}\mu$ belongs to $\mathscr{V}$. That leads to

$$\int_0^b \langle \widetilde{C}\left(\overline{\mathfrak{w}}_{\mathfrak{h}}\right), \vartheta(\delta)\rangle \mathrm{d}\delta \longrightarrow \int_0^b \langle \int_0^\delta c(\delta,\mu)\mathfrak{w}(\mu)\mathrm{d}\mu, \vartheta(\delta)\rangle \mathrm{d}\delta, \ \forall \vartheta \in \mathscr{V}. \tag{87}$$

It is clear that $f_{\mathfrak{h}} \longrightarrow f$ strongly in $\mathscr{V}'$, as $\mathfrak{h} \longrightarrow 0$.

The above convergences entail for all $v \in \mathscr{V}$

$$\begin{aligned}
&\langle \widetilde{\mathscr{A}}\mathfrak{u}_{\mathfrak{h}}, \vartheta\rangle_{\mathscr{V}',\mathscr{V}} \longrightarrow \langle \widetilde{\mathscr{A}}\mathfrak{u}, \vartheta\rangle_{\mathscr{V}',\mathscr{V}}, \langle \widetilde{\mathscr{B}}\mathfrak{u}_{\mathfrak{h}}, \vartheta\rangle_{\mathscr{V}',\mathscr{V}} \longrightarrow \langle \widetilde{\mathscr{B}}\mathfrak{u}, \vartheta\rangle_{\mathscr{V}',\mathscr{V}}, \\
&\langle \widetilde{C}\mathfrak{w}_{\mathfrak{h}}, \vartheta\rangle_{\mathscr{V}',\mathscr{V}} \longrightarrow \langle \widetilde{C}\mathfrak{w}, \vartheta\rangle_{\mathscr{V}',\mathscr{V}}, \langle \varsigma_{\mathfrak{h}}, \vartheta\rangle_{\mathscr{V}',\mathscr{V}} \longrightarrow \langle \varsigma, \vartheta\rangle_{\mathscr{V}',\mathscr{V}},
\end{aligned} \tag{88}$$

for $\vartheta \in \mathscr{W}$ and $\mathfrak{h} \longrightarrow 0$. By the strong convergence of $f_{\mathfrak{h}}$ to $f$ in $\mathscr{W}'$ and the fact that $f_{\mathfrak{h}}$ and $f$ belong to $L^\infty(0,b;\mathscr{W}')$, we have

$$\langle f_{\mathfrak{h}}, \mathfrak{w}_{\mathfrak{h}}\rangle_{\mathscr{V}',\mathscr{V}} \longrightarrow \langle f, \mathfrak{w}\rangle_{\mathscr{V}',\mathscr{V}}, \text{ as } \mathfrak{h} \longrightarrow 0. \tag{89}$$

All above convergences imply that

$$0 \leq \lim_{\mathfrak{h} \longrightarrow 0} \sup \langle \widetilde{\mathscr{A}} \mathfrak{u}_{\mathfrak{h}} + \widetilde{\mathscr{B}} \mathfrak{u}_{\mathfrak{h}} + \widetilde{C} \mathfrak{w}_{\mathfrak{h}} + \varsigma_{\mathfrak{h}}, \vartheta \rangle_{\mathscr{V}', \mathscr{V}} - \lim_{\mathfrak{h} \longrightarrow 0} \inf \langle f_{\mathfrak{h}}, \vartheta \rangle_{\mathscr{V}, \mathscr{V}'}. \tag{90}$$

We obtain the following inequality, for all $\vartheta \in \mathscr{V}$

$$\langle \mathscr{A}\left( {}_{0}^{C}D_{\delta}^{\beta} \mathfrak{w} \right) + \mathscr{B} \mathfrak{w} + C \mathfrak{w} + \varsigma, \vartheta \rangle_{\mathscr{V}', \mathscr{V}} \geq \langle f, \vartheta \rangle_{\mathscr{V}', \mathscr{V}}. \tag{91}$$

Finally, taking into account the formulation (25), we deduce that $\mathfrak{w} \in L^1(0, b; \mathscr{W})$ and ${}_{0}^{C}D_{\delta} \mathfrak{w}$ in $L^{1/\kappa}(0, b; \mathscr{W})$ is a solution of Problem 3. This completes the proof of Theorem 4. □

## 4. Conclusion

In this paper, we have already considered the problem of the deformation of a body that comes into contact with a rigid foundation. The novelty here was that we took into account the combination of a fractional Kelvin–Voigt's constitutive law and a frictional Tresca's contact [17] law. Also, we have established the existence of weak solutions by going through a series of problems that allows us to find the first discrete solutions by means of Rothe's method and some results on nonlinear inclusions. Then, by passing to the limit, we have found a weak solution to the problem, satisfying a certain regularity.

## Data Availability

No data were used to support this study.

## Disclosure

This work is performed as part of the employment of the authors.

## Conflicts of Interest

No Conflicts of Interest are related to this work


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