Hindawi
Journal of Mathematics
Volume 2023, Article ID 8369711, 7 pages
https://doi.org/10.1155/2023/8369711

Research Article

One-Sided Version of Law of the Iterated Logarithm for Summations of Signum Functions

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Received 6 July 2023; Revised 20 August 2023; Accepted 5 September 2023; Published 19 September 2023

Academic Editor: Kenan Yildirim

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The law of the iterated logarithm (LIL), which describes the rate of convergence for a convergent lacunary series, was established by R. Salem and A. Zygmund. This rate is determined based on the variance-like term of the remainder after n terms of the series. In this article, we investigate a comparable one-sided LIL for sums of signum functions, which also relies on the remainder after n terms.

1. Introduction

LIL is a fundamental theorem in probability theory that characterizes the properties displayed by the sums of independent random variables and can be viewed as a refinement of two famous theorems, namely, central limit theorem (CLT) and law of large numbers (LLN). The LIL provides a more accurate estimation as compared to the CLT and LLN in the cases where the fluctuations of the sample average are anticipated to be small, yet the occurrence of significant deviations remains possible. The LIL was first introduced by Russian mathematician Khintchine [1] in 1924 while describing the size of deviation from the expected mean for Bernoulli's random variables. In 1929, Kolmogorov [2] extended this result to apply to the sums of independent random variables. Since then, the LIL has gained significance as a fundamental theorem and has found applications in diverse fields of mathematics and statistics. After the introduction of Kolmogorov's LIL, various extensions and generalizations of the LIL have been introduced in the different fields such as harmonic functions [3], martingales [4, 5], q-lacunary series, random walks, stochastic processes, etc. Salem and Zygmund [6] introduced an LIL for the sums of q-lacunary trigonometric series. We recall that a sequence $\{n_i\}$ is called Hadamard gap condition if it satisfies $q = n_{i+1}/n_i > 1$. A trigonometric series of the form $S(x) = \sum_{i=1}^{\infty} (a_i \cos n_i x + b_i \sin n_i x)$ in which n_i satisfies the Hadamard gap condition is known as q-lacunary trigonometric series. The LIL formulated by Salem and Zygmund is stated below.

Theorem 1. *Suppose that $S(\theta)$ is q-lacunary series and n_k are positive integers. Set $B_m^2 = 1/2 \sum_{i=1}^{m} (|a_i|^2 + |b_i|^2)$ and $M_m = \max_{1 \le i \le m} (|a_i|^2 + |b_i|^2)^{1/2}$. Assume that $\lim_{m \longrightarrow \infty} B_m = \infty$ and M_m satisfies $M_m^2 \le K_m B_m^2 / \log\log (e^e + B_m^2)$ for $K_m \downarrow 0$. Then,*

$$\limsup_{m \longrightarrow \infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log\log B_m}} \le 1, \tag{1}$$

for almost every θ in a unit circle T.

We note that the abovementioned theorem is not entirely analogous to the result introduced by Kolmogorov. In this direction, Erdös and Gál [7] derived a similar result for a particular form of q-lacunary series. Eventually, Weiss [8] succeeded in deriving a similar LIL for a general q-lacunary series analogous to the LIL introduced by Kolmogorov.

Theorem 2 (M. Weiss). *Let $S_m(\theta) = \sum_{i=1}^{m} (a_i \cos n_i \theta + b_i \sin n_i \theta)$ be a q-lacunary series and n_i be the integers. Set $B_m^2 = 1/2 \sum_{k=1}^{m} (|a_k|^2 + |b_k|^2)$ and $M_m = \max_{1 \le k \le m} (|a_k|^2 + |b_k|^2)^{1/2}$.*

Suppose also that $\lim_{m\longrightarrow\infty} B_m = \infty$ *and* M_m *satisfies* $M_m^2 \le K_m B_m^2/\log\log(e^e + B_m^2)$ *for some sequence of numbers* $K_m \downarrow 0$. *Then,*

$$\limsup_{m\longrightarrow\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log\log B_m}} = 1, \tag{2}$$

for almost every θ *in the unit circle.*

Salem and Zygmund [6] also introduced another LIL for q-lacunary series, stated as follows:

Theorem 3 (Salem and Zygmund). *Let* $\widetilde{S}_N(\theta) = \sum_{i=N}^{\infty}(a_i \cos n_i\theta + b_i \sin n_i\theta)$ *with* $n_{i+1}/n_i > q > 1$ *and* $c_i^2 = a_i^2 + b_i^2$ *satisfy* $\sum_{i=1}^{\infty} c_i^2 < \infty$. *Define* $\widetilde{B}_M = (1/2\sum_{i=M}^{\infty} c_i^2)^{1/2}$ *and* $\widetilde{C}_M = \max_{i\ge M}|c_i|$. *Assume that* $\widetilde{B}_1 < \infty$ *with* $\widetilde{C}_M^2 \le K_M (\widetilde{B}_M^2/\ln\ln 1/\widetilde{B}_M)$ *for* K_M *approaches to 0 as M approaches to infinity. Then,*

$$\limsup_{M\longrightarrow\infty} \frac{\widetilde{S}_M(\theta)}{\sqrt{2\widetilde{B}_M^2 \ln\ln 1/\widetilde{B}_M}} \le 1, \tag{3}$$

for almost every θ *in the unit circle.*

Hence, it is evident that the LIL introduced by Salem and Zygmund is applicable to both divergent and convergent q-lacunary series. For the divergent series, they analyzed the partial sums and established the extent of their deviation, which is influenced by the term resembling variance, denoted as B_m^2. Meanwhile, in the convergent series, they analyzed the tail sums and derived another LIL that estimates the convergence rate of q-lacunary series, which also depends on the tail sums of variance-like term B_m^2. This LIL is commonly referred to as the "tail" LIL because of the tail sum component. The abovementioned result represents the one-sided version of the LIL. The other direction of the above theorem was estimated by Ghimire and Moore [9], who obtained the following outcome under the similar assumptions to those in the previous theorem.

Theorem 4. *Assuming the same notation and hypotheses as stated in the preceding theorem, we have*

$$\limsup_{N\longrightarrow\infty} \frac{\widetilde{S}_N(\theta)}{\sqrt{2\widetilde{B}_N^2 \log\log 1/\widetilde{B}_N}} \ge 1, \tag{4}$$

for a.e. $\theta \in [0, 2\pi]$.

In this article, we derive a similar LIL for the sums of signum functions. We recall that a general signum function $\operatorname{sgn}(t)$ is defined as follows:

$$\operatorname{sgn}(t) = \begin{cases} 1, & \text{if } t \ge 0; \\ -1, & \text{if } t < 0. \end{cases} \tag{5}$$

In order to form a sequence, we consider signum function as $u_i(t) = \operatorname{sgn}(\sin 2^i \pi t)$ on $[0, 1)$. Note that for $i = 1$, $u_1(t) = \operatorname{sgn}(\sin 2\pi t)$ will give value 1 on $[0, 1/2]$ and -1 on $(1/2, 1)$ and similarly the rest of the signum functions in the sequence will fluctuate between -1 and 1. Consider a sequence of real number $\{b_i\}_{i=1}^{\infty}$ and define $g_n(t) = \sum_{i=1}^{n} b_i u_i(t)$. Burkholder and Gundy in [10] proved that

$$\left\{ t : \sqrt{\sum_{i=1}^{\infty} [g_i(t) - g_{i-1}(t)]^2} < \infty \right\} = \{t : g_n(t) \text{converges}\}, \tag{6}$$

where the sets are almost everywhere equal. Note that $\sum_{i=1}^{\infty}[g_i(t) - g_{i-1}(t)]^2 = \sum_{i=1}^{\infty}(b_i u_i(t))^2 = \sum_{i=1}^{\infty} b_i^2$. So the condition $\sum_{i=1}^{\infty} b_i^2 < \infty$ guarantees the existence of limit of $\{g_n = \sum_{i=1}^{n} b_i u_i\}_{n=1}^{\infty}$. Moreover, the sequence $\{g_n = \sum_{i=1}^{n} b_i u_i\}_{n=1}^{\infty}$ converges to $g = \sum_{i=1}^{\infty} b_i u_i$. Here, we derive a LIL for $\{g_n = \sum_{i=1}^{n} b_i u_i\}_{n=1}^{\infty}$ which is similar to the LIL introduced by Salem and Zygmund. Our result estimates the rate at which g_n converges to g and the rate at which it converges is governed by the tail sums of the square function $\sum_{i=1}^{\infty} b_i^2$. We only obtain the one-sided version of LIL and our main result is as follows:

Theorem 5. *Suppose* $\{u_i\}$ *is a sequence of signum functions defined by* $u_i(t) = sgn(\sin 2^i \pi t)$ *and* $\{b_i\}_{i=1}^{\infty}$ *with* $b_i \in \mathbb{R}$ *satisfies* $\sum_{i=1}^{\infty} b_i^2 < \infty$. *Then,*

$$\limsup_{n\longrightarrow\infty} \frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n+1}^{\infty} b_i^2 \ln\ln\left(1/\sum_{i=n+1}^{\infty} b_i^2\right)}} \le 1, \tag{7}$$

for almost every $t \in [0, 1)$.

2. Preliminaries

During the course of proving our main result, we will require certain estimates that will be utilized in the proof of the final outcome. We now prove these estimates.

Lemma 6. *Consider* $\{b_i\}_{i=1}^{\infty}$ *with* $b_i \in \mathbb{R}$ *and* $u_i(t) = sgn(\sin 2^i \pi t)$ *where*

$$\operatorname{sgn}(t) = \begin{cases} 1, & \text{if } t \ge 0; \\ -1, & \text{if } t < 0. \end{cases} \tag{8}$$

Then, we have the following estimate:

$$\int_0^1 \exp\left(\gamma \sum_{i=1}^{n} b_i u_i(t) - \frac{\gamma^2}{2}\sum_{i=1}^{n} b_i^2\right) dt \le 1, \tag{9}$$

where $\gamma \in \mathbb{R}$.

Proof. Let $g_n(t) = \sum_{i=1}^{n} b_i u_i(t), C_i(t) = g_i(t) - g_{i-1}(t)$. We have $g_n(t) = \sum_{i=1}^{n} C_i(t)$ with $g_0 = 0$. Let

$$h(n) = \int_0^1 e^{g_n(t) - 1/2\sum_{i=1}^{n} C_i^2(t)} dt = \int_0^1 e^{\sum_{i=1}^{n} C_i(t) - 1/2\sum_{i=1}^{n} C_i^2(t)} dt. \tag{10}$$

As n increases, we show that $h(n)$ decreases. For this,

$$\begin{aligned} h(n+1) &= \int_0^1 e^{\sum_{i=1}^{n+1} C_i(t) - 1/2\sum_{i=1}^{n+1} C_i^2(t)} \mathrm{d}t \\ &= \sum_{j=0}^{2^n} \int_{Q_{nj}} e^{\sum_{i=1}^{n} C_i(t) - 1/2\sum_{i=1}^{n} C_i^2(t)} e^{C_{n+1}(t) - 1/2C_{n+1}^2(t)} \mathrm{d}t \\ &= \sum_{j=0}^{2^n} \left[e^{\sum_{i=1}^{n} C_i(t) - 1/2\sum_{i=1}^{n} C_i^2(t)} \right]_{Q_{nj}} \int_{Q_{nj}} e^{C_{n+1}(t) - 1/2C_{n+1}^2(t)} \mathrm{d}t, \end{aligned} \tag{11}$$

where we used the fact of function g_n is constant on n th generation interval $Q_{nj} = [j/2^n, j + 1/2^n)$. Here, $C_{i+1}(t) = g_{i+1}(t) - g_i(t) = b_{i+1}u_{i+1}(t)$. Let Q_1 and Q_2 be the subinterval of n th generation interval $[j/2^n, j + 1/2^n)$ and $C_{n+1}(t) = b_{n+1}u_{n+1}(t)$ has value b_{n+1} on Q_1 and $-b_{n+1}$ on Q_2. Applying $\cosh t \le e^{t^2/2}$, we obtain

$$\begin{aligned} h(n+1) &= \sum_{j=0}^{2^n} \left[e^{\sum_{i=1}^{n} C_i(t) - 1/2\sum_{i=1}^{n} C_i^2(t)} \right]_{Q_{nj}} \left[\int_{Q_1} e^{b_{n+1} - 1/2b_{n+1}^2} \mathrm{d}t + \int_{Q_2} e^{-b_{n+1} - 1/2b_{n+1}^2} \mathrm{d}t \right] \\ &= \sum_{j=0}^{2^n} 2 \left[e^{\sum_{i=1}^{n} C_i(t) - 1/2\sum_{i=1}^{n} C_i^2(t)} \right]_{Q_{nj}} \cosh(b_{n+1}) e^{-1/2b_{n+1}^2} \frac{1}{2^{n+1}} \\ &\le \sum_{j=0}^{2^n} \left[e^{\sum_{i=1}^{n} C_i(t) - 1/2\sum_{i=1}^{n} C_i^2(t)} \right]_{Q_{nj}} e^{1/2b_{n+1}^2} e^{-1/2b_{n+1}^2} \frac{1}{2^n} = h(n). \end{aligned} \tag{12}$$

Thus we have $h(n+1) \le h(n)$. We next show $h(1) \le 1$. For this,

$$\begin{aligned} h(1) &= \int_0^1 e^{C_1(t) - 1/2C_1^2(t)} \mathrm{d}t \\ &= \int_{[0,1/2]} e^{C_1(t) - 1/2C_1^2(t)} \mathrm{d}t + \int_{[1/2,1]} e^{C_1(t) - 1/2C_1^2(t)} \mathrm{d}t \\ &\le e^{1/2b_1^2} e^{-1/2b_1^2} = 1. \end{aligned} \tag{13}$$

Hence, $h(1) \le 1$. This with $h(n+1) \le h(n)$ gives $h(n) \le 1$. Thus, we have

$$\begin{aligned} &\int_0^1 e\left(\sum_{i=1}^{n} C_i(t) - \frac{1}{2} \sum_{i=1}^{n} C_i^2(t) \right) \mathrm{d}t \le 1 \\ &\int_0^1 \exp\left(\sum_{i=1}^{n} b_i u_i(t) - \frac{1}{2} \sum_{i=1}^{n} b_i^2 \right) \mathrm{d}t \le 1. \end{aligned} \tag{14}$$

Then, γ scaling gives

$$\int_0^1 \exp\left(\gamma \sum_{i=1}^{n} b_i u_i(t) - \frac{\gamma^2}{2} \sum_{i=1}^{n} b_i^2 \right) \mathrm{d}t \le 1. \tag{15}$$

□

Lemma 7. *Let $\{b_i\}_{i=1}^{\infty}$ where $b_i \in \mathbb{R}$ and $\{u_i\}_{i=1}^{\infty}$ be a sequence of signum function defined by $u_i(t) = sgn(\sin 2^i \pi t)$ where*

$$\operatorname{sgn}(t) = \begin{cases} 1, & \text{if } t \ge 0; \\ -1, & \text{if } t < 0. \end{cases} \tag{16}$$

Then, for all $\eta > 0$ and for a fixed n, we have

$$\left| \left\{ t \in I : \sup_{l \ge n} \left| \sum_{i=n+1}^{l} b_i u_i(t) \right| > \eta \right\} \right| \le 6e\left(\frac{-\eta^2}{\sum_{i=n+1}^{\infty} b_i^2} \right), \tag{17}$$

where $I = [0, 1)$.

Proof. Define $g_n(t) = \sum_{i=1}^{n} b_i u_i(t)$. Let n be fixed and Q_l be the interval in $[0, 1)$ of length $1/2^l$. Then

$$g_l(t) = \frac{1}{|Q_l|} \int_{Q_l} g_n(x) \mathrm{d}x, \tag{18}$$

for all $t \in Q_l$ with $l \ge n$. Fix t.

$$\begin{aligned} e^{\eta |g_l(t)|} &= \exp\left(\alpha \left| \int_{Q_l} g_n(x) d\left(\frac{x}{|Q_l|} \right) \right| \right) \\ &\le \frac{1}{|Q_l|} \int_{Q_l} e^{(\alpha |g_n(x)|)} \mathrm{d}x \\ &\le M\left(e^{\alpha |g_l|} \right)(t). \end{aligned} \tag{19}$$

where $M(e^{\alpha |g_l|})$ denote the Hardy–Littlewood maximal function of the function $e^{\alpha |g_l|}$. Employing the Hardy–Littlewood maximal theorem, we have

$$\begin{aligned}\left|\left\{t\in I:\sup_{1\le l\le n}\left|g_l(t)\right|>\eta\right\}\right| &= \left|\left\{t\in I:\sup_{1\le l\le n}e^{\eta\left|g_l(t)\right|}>e^{\alpha\eta}\right\}\right|\\ &\le \left|\left\{t\in I: M\left(e^{\alpha\left|g_l\right|}\right)(t)>e^{\alpha\eta}\right\}\right|\\ &\le \frac{3}{e^{\alpha\eta}}\int_0^1 e^{\alpha\left|g_n(t)\right|}\mathrm{d}t\\ &= \frac{3}{e^{\alpha\eta}}\exp\left(\frac{\alpha^2}{2}\sum_{i=1}^n b_i^2\right)\int_0^1\exp\left(\alpha\left|g_n(t)\right|-\frac{\alpha^2}{2}\sum_{i=1}^n b_i^2\right)\mathrm{d}y.\end{aligned}\tag{20}$$

Using Lemma 6, we obtain

$$\int_0^1 e^{\left(\alpha\left|g_n(t)\right|-\alpha^2/2\sum_{i=1}^n b_i^2\right)}\mathrm{d}y\le 2.\tag{21}$$

This gives,

$$\left|\left\{t\in I:\sup_{1\le l\le n}\left|g_l(t)\right|>\eta\right\}\right|\le\frac{6}{e^{\alpha\eta}}\exp\left(\frac{\alpha^2}{2}\sum_{i=1}^n b_i^2\right).\tag{22}$$

Choose $\alpha=\eta/\sum_{i=1}^n b_i^2$ and using $\sum_{i=1}^n b_i^2\nearrow\sum_{i=1}^\infty b_i^2$, we obtain

$$\left|\left\{t\in I:\sup_{1\le l\le n}\left|g_l(t)\right|>\eta\right\}\right|\le 6\exp\left(\frac{-\eta^2}{2\sum_{i=1}^\infty b_i^2}\right).\tag{23}$$

Continuity property gives,

$$\left|\left\{t\in I:\sup_{l\ge 1}\left|g_l(t)\right|>\eta\right\}\right|\le 6\exp\left(\frac{-\eta^2}{2\sum_{i=1}^\infty b_i^2}\right),\tag{24}$$

i.e.

$$\left|\left\{t\in I:\sup_{l\ge 1}\left|\sum_{i=1}^l b_iu_i(t)\right|>\eta\right\}\right|\le 6\exp\left(\frac{-\eta^2}{2\sum_{i=1}^\infty b_i^2}\right).\tag{25}$$

We note that $\sum_{i=1}^\infty b_i^2=\sum_{i=1}^\infty[(t)b_iu_i]^2=\sum_{i=1}^\infty[g_i(t)-g_{i-1}(t)]^2$.

For a fixed n, define

$$f_l(t)=\begin{cases}0, & \text{if } l\le n;\\ g_l(t)-g_n(t), & \text{if } l>n.\end{cases}\tag{26}$$

For this function, the above estimate becomes

$$\left|\left\{t\in I:\sup_{l\ge 1}\left|f_l(t)\right|>\eta\right\}\right|\le 6\exp\left(\frac{-\eta^2}{2\sum_{i=1}^\infty[g_i(t)-g_{i-1}(t)]^2}\right).\tag{27}$$

Here,

$$\begin{aligned}\sum_{i=1}^\infty[f_i(t)-f_{i-1}(t)]^2 &= \sum_{i=n+1}^\infty[f_i(t)-f_{i-1}(t)]^2\\ &= \sum_{i=n+1}^\infty[g_i(t)-g_n(t)-g_{i-1}(t)+g_n(t)]^2\\ &= \sum_{i=n+1}^\infty[g_i(t)-g_{i-1}(t)]^2\\ &= \sum_{i=n+1}^\infty[b_iu_i(t)]^2=\sum_{i=n+1}^\infty b_i^2.\end{aligned}\tag{28}$$

But $f_l(t)=0$ for $l\le n$ and $f_l(t)=g_l(t)-g_n(t)=\sum_{i=n+1}^l b_iu_i(t)$. This gives

$$\left|\left\{t\in I:\sup_{l\ge n}\left|\sum_{i=n+1}^l b_iu_i(t)\right|>\eta\right\}\right|\le 6\exp\left(\frac{-\eta^2}{\sum_{i=n+1}^\infty b_i^2}\right).\tag{29}$$

□

Lemma 8. *Let $\{b_i\}_{i=1}^\infty$ where $b_i\in\mathbb{R}$ and $\{u_i\}_{i=1}^\infty$ be a sequence of signum function defined by $u_i(t)=sgn(\sin 2^i\pi t)$ where*

$$\operatorname{sgn}(t)=\begin{cases}1, & \text{if } t\ge 0;\\ -1, & \text{if } t<0.\end{cases}\tag{30}$$

Then for all $\alpha>0$ and a fixed number n, we have

$$\left|\left\{t\in I:\sup_{m\ge n}\left|\sum_{i=m+1}^\infty b_iu_i(t)\right|>\alpha\right\}\right|\le 12\exp\left(\frac{-\alpha^2}{2\sum_{i=n+1}^\infty b_i^2}\right)\tag{31}$$

where $I=[0,1)$.

Proof. Let $C_i(t) = \sum_{k=1}^{i} b_k u_k(t) - \sum_{k=1}^{i-1} b_k u_k(t) = b_i u_i(t)$. Here, each C_i is independent and symmetric with mean 0 and variance 1. Invoking Levy's inequality, we have

$$\left|\left\{t \in I : \left|\max_{0\le j\le n} \sum_{i=1}^{j} C_i(t)\right| > \alpha\right\}\right| \le 2\left|\left\{t \in I : \left|\sum_{i=1}^{n} C_i(t)\right| > \alpha\right\}\right|. \tag{32}$$

Let $M \gg n$ Then, we obtain

$$\left|\left\{t \in I : \left|\max_{0\le j\le M-n-1} \sum_{i=0}^{j} C_{M-i}(t)\right| > \alpha\right\}\right| \le 2\left|\left\{t \in I : \left|\sum_{i=0}^{n} C_{M-n-1} C_{M-i}(t)\right| > \alpha\right\}\right|. \tag{33}$$

Thus,

$$\begin{aligned} &\left|\left\{t \in I : \left|\max C_M(t), C_M(t) + C_{M-1}(t) \ldots, C_M(t) + C_{M-1}(t) + \cdots + C_{n+1}(t)\right| > \alpha\right\}\right| \\ &\le 2\left|\left\{t \in I : \left|C_M(t) + C_{M-1}(t) + \cdots + C_{n+1}(t)\right| > \alpha\right\}\right|. \end{aligned} \tag{34}$$

We have $C_i(t) = g_i(t) - g_{i-1}(t)$ and $g_k(t) = \sum_{i=1}^{k} b_i u_i(t)$; we obtain

$$\left|\left\{t \in I : \left|\max_{M-1\ge m\ge n} g_M(t) - g_m(t)\right| > \alpha\right\}\right| \le 2\left|\left\{t \in I : \left|g_M(t) - g_n(t)\right| > \alpha\right\}\right|. \tag{35}$$

Employing $\sup_j |b_j| > \alpha$ implies $|\sup_j b_j| > \alpha$ or $|\sup_j (-b_j)| > \alpha$, we have

$$\left|\left\{t \in I : \max_{M\ge m\ge n}\left|g_M(t) - g_m(t)\right| > \alpha\right\}\right| \le 2\left|\left\{t \in : \left|g_M(t) - g_n(t)\right| > \alpha\right\}\right|. \tag{36}$$

Using Lemma 6, we obtain

$$\left|\left\{t \in I : \sup_{m\ge n}\left|g_m(t) - g_n(t)\right| > \alpha\right\}\right| \le 6 \exp\left(\frac{-\alpha^2}{2\sum_{i=n+1}^{\infty} b_i^2}\right). \tag{37}$$

Using (36) and (37), we obtain

$$\left|\left\{t \in I : \sup_{M\ge m\ge n}\left|g_M(t) - g_n(t)\right| > \alpha\right\}\right| \le 12 \exp\left(\frac{-\alpha^2}{2\sum_{i=n+1}^{\infty} b_i^2}\right). \tag{38}$$

Let $A_M = \{x \in I : \sup_{M\ge m\ge n} |g_M(t) - g_m(t)| > \alpha\}$ and $A = \cup_{i=1}^{\infty} A_i$. Then $A_m \subset A_{m+1}$. Then, continuity property implies $|A| = \lim_{M\longrightarrow\infty} |A_M|$. One can check that

$$\left\{t \in I : \sup_{m\geq n}\left|g(t) - g_m(t)\right| > \alpha\right\} \subset A, \tag{39}$$

where we have $g(t) = \sum_{i=1}^{\infty} b_i u_i(t)$. Then

$$\begin{aligned}\left|\left\{t \in I : \sup_{m\geq n}\left|g(t) - g_m(t)\right| > \alpha\right\}\right| &\leq |A| \\ &= \lim_{M\longrightarrow\infty} \left|A_M\right| \\ &= \lim_{M\longrightarrow\infty} \left|\left\{t \in I : \sup_{M\geq m\geq n}\left|g_M(t) - g_m(t)\right| > \alpha\right\}\right| \\ &\leq 12 \exp\left(\frac{-\alpha^2}{2\sum_{i=n+1}^{\infty} b_i^2}\right).\end{aligned} \tag{40}$$

This gives

$$\left|\left\{t \in I : \sup_{m\geq n}\left|g(t) - g_m(t)\right| > \alpha\right\}\right| \leq 12 \exp\left(\frac{-\alpha^2}{2\sum_{i=n+1}^{\infty} b_i^2}\right). \tag{41}$$

□

3. Proof of Our Main Result

Consider $\theta > 1$ and define stopping times $n_1 \leq n_2 \leq \cdots \leq n_i \cdots \longrightarrow \infty$ as:

$$n_i = \min\left(n: \sum_{j=n+1}^{\infty} b_j^2 < \frac{1}{\theta^i}\right). \tag{42}$$

Using Lemma 8 for a fixed m, we obtain

$$\left|\left\{t \in I : \sup_{n\geq m}\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > \alpha\right\}\right| \leq 12e\left(\frac{-\alpha^2}{2\sum_{i=m+1}^{\infty} b_i^2}\right). \tag{43}$$

For stopping time n_i, this becomes

$$\left|\left\{t \in I : \sup_{n\geq n_i}\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > \alpha\right\}\right| \leq 12e\left(\frac{-\alpha^2}{2\sum_{i=n_i+1}^{\infty} b_i^2}\right). \tag{44}$$

Choose $\alpha = (1+\varepsilon)\sqrt{2\ln(\ln\theta^i)/\theta^i}$ where $\varepsilon > 0$.

$$\left|\left\{t \in I : \sup_{n\geq n_i}\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > (1+\varepsilon)\sqrt{\frac{2\ln\left(\ln\theta^i\right)}{\theta^k}}\right\}\right| \leq 12 \exp\left(\frac{-(1+\varepsilon)^2 \ln\left(\ln\theta^i\right)}{\theta^i \sum_{i=n_i+1}^{\infty} b_i^2}\right). \tag{45}$$

Using $\sum_{i=n_i+1}^{\infty} b_i^2 < 1/\theta^i$, we have

$$\left|\left\{t \in I : \sup_{n\geq n_i}\left|\sum_{i=n+1}^{\infty} b u_i(t)_i\right| > (1+\varepsilon)\sqrt{\frac{2\ln\left(\ln\theta^i\right)}{\theta^i}}\right\}\right| \leq 12 \exp\left(\frac{-(1+\varepsilon)^2 \ln\left(\ln\theta^i\right)}{\theta^i 1/\theta^i}\right) = \frac{12}{(\ln\theta)^{(1+\varepsilon)^2}} \frac{1}{i^{(1+\varepsilon)^2}}. \tag{46}$$

Then

$$\sum_{i=1}^{\infty}\left|\left\{t \in I : \sup_{n\geq n_i}\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > (1+\varepsilon)\sqrt{\frac{2\ln\left(\ln\theta^i\right)}{\theta^i}}\right\}\right| < \frac{12}{(\ln\theta)^{(1+\varepsilon)^2}} \sum_{i=1}^{\infty} \frac{1}{i^{(1+\varepsilon)^2}} < \infty. \tag{47}$$

So, by Borel-Cantelli Lemma [11] for a.e. t, we have

$$\sup_{n\geq n_i}\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| \leq (1+\varepsilon)\sqrt{\frac{2\ln\left(\ln\theta^i\right)}{\theta^i}}, \tag{48}$$

for a large i, with $i \geq N$, N being fixed. Clearly, the value of N is based on t. We fix t and then consider $n \geq n_N$. We can find $i \geq N$ satisfying $n_i \leq n < n_{i+1}$. Clearly $\sum_{i=n_{i+1}+1}^{\infty} b_i^2 < 1/\theta^{i+1}$. But $n_{i+1} > n$. So $\sum_{i=n+1}^{\infty} b_i^2 \geq 1/\theta^{i+1}$. Moreover, we have $\sum_{i=n+1}^{\infty} b_i^2 < 1/\theta^i$. Therefore

$$\frac{1}{\theta^{i+1}} \leq \sum_{i=n+1}^{\infty} b_i^2 < \frac{1}{\theta^i}. \tag{49}$$

Using this, we have

$$\begin{aligned}\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| &\leq \sup_{m\geq n_i}\left|\sum_{i=m+1}^{\infty} b_i u_i(t)\right| \\ &\leq (1+\varepsilon)\sqrt{\frac{2\ln\left(\ln\theta^i\right)}{\theta^i}} \\ &= (1+\varepsilon)\sqrt{\frac{2\theta\ln\left(\ln\theta^i\right)}{\theta^{i+1}}} \\ &< (1+\varepsilon)\sqrt{2\theta\sum_{i=n+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sum_{i=n+1}^{\infty} b_i^2}\right)}.\end{aligned} \tag{50}$$

Hence, for a.e. $t \in I$, we obtain

$$\limsup_{n\longrightarrow\infty}\frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n+1}^{\infty} b_i^2 \ln\ln\left(1/\sum_{i=n+1}^{\infty} b_i^2\right)}} < (1+\varepsilon)\sqrt{\theta}. \tag{51}$$

Taking limit as $\theta \longrightarrow 1$, we have

$$\limsup_{n\longrightarrow\infty}\frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n+1}^{\infty} b_i^2 \ln\ln\left(1/\sum_{i=n+1}^{\infty} b_i^2\right)}} < (1+\varepsilon). \tag{52}$$

This can be done for all $\varepsilon > 0$, we have

$$\limsup_{n\longrightarrow\infty}\frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n+1}^{\infty} b_i^2 \ln\ln\left(1/\sum_{i=n+1}^{\infty} b_i^2\right)}} \leq 1, \tag{53}$$

for a.e. $t \in I$.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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