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Research Article

On the Existence of a Normal Trimagic Square of Order $16n$

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The study of magic squares has a long history, and magic squares have been applied to many mathematical fields. In this paper, we give a complete solution to the existence of normal trimagic squares of all orders $16n$. In particular, we obtain a unified solution for the normal trimagic square of order $16n$ for $n > 3$ by means of set partitions, semibimagic squares, Latin squares, and new product construction. Since there exist normal trimagic squares of orders 16, 32, and 48, we prove that there exists a normal trimagic square of order $16n$ for every positive integer n.

1. Introduction

Magic squares are among the oldest known combinatorial designs and have been applied to many fields of mathematics [1–3]. Their origin can be found in the first-century book Da-Dai Liji in China [4]. A magic square of order n, denoted by MS (n), is an $n \times n$ matrix consisting of integers such that the sum of the entries in each row, each column, and each of two diagonals is the same number called the magic sum. A magic square of order n is normal if its entries are n^2 consecutive integers, say $0, 1, \ldots, n^2 - 1$.

During this long period of time, many subclasses have been proposed, such as bimagic squares and trimagic squares. Let t be a positive integer. A magic square of order n is a t-multimagic square, denoted by MS (n, t), if it remains magic when all its entries are replaced by its d-th powers for $d \in \{1, \ldots, t\}$. Usually, a 2-multimagic square is called a bimagic square and a 3-multimagic square is called a trimagic square. A lot of work has been done on normal magic squares and normal multimagic squares; for more details, the interested reader may refer to [5–11] and the references therein.

Next, let us review the history of the study of trimagic squares. In 1905, Tarry [12] constructed the first known normal trimagic square. Its order was 128. Later, smaller normal trimagic squares were found: order 64 by Cazalas [13] in 1933, 32 by Benson and Jacoby [14] in 1976, 16 by Chen and Chen ([15], see also [16]) in 2006, and 48 by Chen (see [16]) in 2007. Now, let us switch to the systematic investigation for the normal trimagic squares of orders $16n$. Let $\Omega_1 = \{n\colon 8 \le n \le 64\}$, $\Omega_2 = \{n\colon n = n_1 n_2, n_1 \equiv n_2 \pmod 2, n_1, n_2 \notin \{2, 3, 6\}\}$, $\Omega_3 = \{n\colon n \equiv 0 \pmod 4, n \ge 8\}$, and $\Omega_4 = \{\mathrm{mn}\colon \mathrm{mn} > 64, 8 \le m \le 64, m \equiv 2 \pmod 4, n \ge 5, n \equiv 1 \pmod 2\}$. In 2017, Li et al. [17] proved that there exists a normal MS $(\mathrm{mn}, 3)$ for $m \in \{2^l\colon l \ge 4\}$ and $n \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$. Now, a question of interest is this: Can we construct a normal trimagic square for every order divisible by 16?

The main purpose of this paper is to give an affirmative answer to the preceding question, that is, prove that there exists a normal trimagic square of order $16n$ for all positive integers n. Our construction tools include quasi-normal trimagic squares, bimagic subset pairs, semibimagic squares, Latin squares (see Section 2 for these definitions), and new product construction.

In this paper, we first present some related preliminaries (Section 2), then prove our main results and provide a brief discussion (Section 3), and finally give conclusions about the work (Section 4).

2. Preliminaries

This section presents some related preliminaries which are used in this paper.

Let J_n be an $n \times n$ matrix with the entries being all 1's and let $I_n = \{0, 1, \ldots, n-1\}$. For integers a and b with $a \le b$ and $a \equiv b \pmod 2$, let $[a, b]_2$ denote the set $\{a, a+2, \ldots, b-2, b\}$. For an $n \times n$ array A, the rows and columns of A are indexed by I_n. For any $n \times n$ array A, we denote by $\mathfrak{S}_A$ the set of the entries of A. Usually, the (i, j) entry of an array A is denoted by $a_{i,j}$. An MS $(n,t)\,A$ is quasi-normal if $\mathfrak{S}_A = [1-n^2, n^2-1]_2$.

For an integer set S, let $\overline{S} = \{-x\colon x \in S\}$. Let n be an integer greater than 3 and $S_n = (1/2)\sum_{k=0}^{4n-1}(2k+1)^2$. The pair of P and Q is called a bimagic subset pair, denoted by BSP (n), if the following three conditions are satisfied:

$(\mathscr{B}_1)$ $\{P, Q, \overline{P}, \overline{Q}\} = [1-8n, 8n-1]_2$ and $|P| = |Q| = 2n$

$(\mathscr{B}_2)$ $\sum_{x\in P} x = \sum_{x\in Q} x = 0$

$(\mathscr{B}_3)$ $\sum_{x\in P} x^2 = \sum_{x\in Q} x^2 = S_n$

Using a computer search, we obtain the following bimagic subset pairs.

Example 1. Set

$$
\begin{aligned}
&P_4 = \{1,-3,-13,15,21,-23,-25,27\}, \qquad Q_4 = \{5,-7,-9,11,17,-19,-29,31\},\\
&P_5 = \{1,-7,9,13,-21,23,25,29,-35,-37\}, \quad Q_5 = \{3,5,11,-15,17,-19,-27,31,33,-39\},\\
&P_6 = \{1,3,-5,-19,-21,23,-31,-33,-35,37,39,41\}, \; P_7 = \{1,-3,-7,11,21,-23,25,-27,-37,39,45,-47,-49,51\},\\
&Q_6 = \{7,-9,11,13,15,17,25,27,29,-43,-45,-47\}, \quad Q_7 = \{5,-9,-13,15,-17,19,29,-31,-33,35,41,-43,-53,55\},
\end{aligned}
\tag{1}
$$

then it is easy to check that the pair of P_n and Q_n is a BSP (n) for $n \in \{4, 5, 6, 7\}$.

Semimagic squares are a generalization of magic squares. A semimagic square of order n, denoted by SMS (n), is an $n \times n$ matrix consisting of integers such that the sum of the entries in each row and each column is the same number called the magic sum. Thus every magic square is a semimagic square, but not the converse. Let t be a positive integer. A semimagic square of order n is a semi t-multimagic square, denoted by SMS (n,t), if it remains semimagic when all its entries are replaced by its d-th powers for $d \in \{1, \ldots, t\}$. For results of semimultimagic squares, we refer the reader to [18] and the references therein. Usually, a semi 2-multimagic square is called a semi-bimagic square. An SMS (n,t) is also called a t-multimagic rectangle in [18]. A semibimagic square especially is called a bimagic square rectangle. Therefore, semimultimagic squares defined in this paper are special subclasses of multimagic rectangles defined in [18].

Based on the literature [6], we write $A^{*d} = (a_{i,j}^d)$ for any positive integer d. We call the matrix A consisting of distinct integers self-complementary if $\mathfrak{S}_{-A} = \mathfrak{S}_A$.

Let A, B, C, and D be integer matrices of order $4n$. We call the tuple (A, B, C, D) an extendable tuple of order $4n$, denoted by ET $(4n)$, if the following conditions are satisfied:

$(\mathscr{E}_1)$ $\mathfrak{S}_A \cup \mathfrak{S}_B \cup \mathfrak{S}_C \cup \mathfrak{S}_D = [1-(8n)^2, (8n)^2-1]_2$;

$(\mathscr{E}_2)$ A, B, C, and D are self-complementary and each of their diagonals consists of opposite numbers;

$(\mathscr{E}_3)$ A, B, C, and D are MS $(4n)$ s with magic sum 0, B^{*2} and C^{*2} are SMS $(4n)$ s with magic sum $2(64n^2+1)S_n$, and A^{*2} and D^{*2} are MS $(4n)$ s with magic sum $2(64n^2+1)S_n$.

The following definition is from [11]. Let T be an n-set. A diagonal Latin square of order n over T, denoted by DLS (n), is an $n \times n$ array such that the set of the entries in each row, each column, and each of two diagonals is T. Two DLS (n) s are called orthogonal if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. In this paper, we need the following.

Lemma 2 (see Abel et al. [19]). *Two orthogonal diagonal Latin squares of order n exist if and only if $n \notin \{2, 3, 6\}$.*

Using the literature [20], we give the following definition. An $n \times n$ array W with the entries in an n-set T is a balanced square if each element of T appears n times in W. Two balanced squares are called orthogonal if each symbol in the first square meets each symbol in the second square exactly once when they are superposed.

For an integer set T and an integer x, denote $\{x+t\colon t \in T\}$ by $x+T$.

3. Results and Discussion

In this section, first we shall show that a normal MS (n,t) can be obtained by constructing a quasi-normal MS (n,t), then construct bimagic subset pairs and semibimagic squares, next use new product construction to get a quasi-normal MS $(16n, 3)$ for $n > 3$ and prove our main theorem, and finally give a generalization.

3.1. A Transformation. For a normal MS $(n,t)\,A$, if the smallest entry integer is s, then it is readily verified that $A - sJ_n$ is a normal MS (n,t) over I_{n^2}. Furthermore, it is also readily verified that $2A - (n^2 - 1 + 2s)J_n$ is a quasi-normal MS (n,t) over $[1-n^2, n^2-1]_2$. Obviously, if B is a quasi-normal MS (n,t), then $(1/2)(B + (n^2-1)J_n)$ is a normal MS (n,t) over I_{n^2}. Based on the above observation, we have the following.

Lemma 3. *Let n and t be two positive integers. If there is a quasi-normal MS (n,t) over $[1-n^2, n^2-1]_2$, then there is a normal MS (n,t) over I_{n^2}.*

Therefore, we reduce the problem for the construction of a normal MS (n,t) to the problem for the construction of a quasi-normal MS (n,t).

3.2. Construction of Bimagic Subset Pairs. In this section, we shall prove the following core result.

Lemma 4. *There exists a BSP (n) for $n \geq 4$.*

Proof. Let $v \in I_4$ and u be a positive integer. Now, we prove the statement $\mathscr{P}_v(u)$. There exists a BSP $(4u+v)$ for $u \geq 1$ by induction. The base case $\mathscr{P}_v(1)$ is obviously true by Example 1.

For the induction step, let w be a positive integer and $k = 4w+v$ and assume that $\mathscr{P}_v(w)$ is true; that is, there exists a BSP (k), P_k, and Q_k, satisfying the following conditions:

$$\begin{gathered}(B_1)\{P_k, Q_k, \overline{P_k}, \overline{Q_k}\} = [1-8k, 8k-1]_2,\\ |P_k| = |Q_k| = 2k,\\ (B_2)\sum_{x\in P_k} x = \sum_{x\in Q_k} x = 0,\\ (B_3)\sum_{x\in P_k} x^2 = \sum_{x\in Q_k} x^2 = S_k.\end{gathered} \tag{2}$$

Let $m = k+4$. We want to show that then $\mathscr{P}_v(w+1)$ is true; that is, there exists a BSP (m), P_m, and Q_m, satisfying the following conditions:

$$\begin{gathered}(B_4)\{P_m, Q_m, \overline{P_m}, \overline{Q_m}\} = [1-8m, 8m-1]_2,\\ |P_m| = |Q_m| = 2m,\\ (B_5)\sum_{x\in P_m} x = \sum_{x\in Q_m} x = 0,\\ (B_6)\sum_{x\in P_m} x^2 = \sum_{x\in Q_m} x^2 = S_m.\end{gathered} \tag{3}$$

Let

$$\begin{aligned}P^* &= \{8k+1, -8k-3, -8k-13, 8k+15, 8k+21, -8k-23, -8k-25, 8k+27\},\\ Q^* &= \{8k+5, -8k-7, -8k-9, 8k+11, 8k+17, -8k-19, -8k-29, 8k+31\}.\end{aligned} \tag{4}$$

Then, it is easy to check that

$$\begin{aligned}P^*\cup Q^*\cup\overline{P^*}\cup\overline{Q^*} &= [1-8m, -1-8k]_2 \cup [8k+1, 8m-1]_2,\\ \sum_{l\in P^*} l &= \sum_{l\in Q^*} l = 0,\\ \sum_{l\in P^*} l^2 &= \sum_{l\in Q^*} l^2\\ &= 512k^2 + 2048k + 2728\\ &= \frac{1}{2}\sum_{l=4k}^{4m-1}(2l+1)^2.\end{aligned} \tag{5}$$

Clearly, we have

$$\begin{aligned}[1-8m,8m-1]_2 &= [1-8m,-1-8k]_2 \cup [1-8k,8k-1]_2 \cup [8k+1,8m-1]_2\\ &= [1-8k,8k-1]_2 \cup \left([1-8m,-1-8k]_2 \cup [8k+1,8m-1]_2\right)\\ &= \left(P_k \cup Q_k \cup \overline{P_k} \cup \overline{Q_k}\right) \cup \left(P^* \cup Q^* \cup \overline{P^*} \cup \overline{Q^*}\right)\\ &= \left(P_k \cup P^*\right) \cup \left(Q_k \cup Q^*\right) \cup \left(\overline{P_k} \cup \overline{P^*}\right) \cup \left(\overline{Q_k} \cup \overline{Q^*}\right).\end{aligned} \tag{6}$$

By our induction hypothesis (B_2), we get

$$\begin{aligned}\sum_{l\in P_k\cup P^*} l &= \sum_{l\in P_k} l + \sum_{l\in P^*} l\\ &= 0+0=0,\\ \sum_{l\in Q_k\cup Q^*} l &= \sum_{l\in Q_k} l + \sum_{l\in Q^*} l\\ &= 0+0=0.\end{aligned} \tag{7}$$

By our induction hypothesis (B_3), we obtain

$$\begin{aligned}\sum_{l\in P_k\cup P^*} l^2 &= \sum_{l\in P_k} l^2 + \sum_{l\in P^*} l^2\\ &= \frac{1}{2}\sum_{l=0}^{4k-1}(2l+1)^2 + \frac{1}{2}\sum_{l=4k}^{4m-1}(2l+1)^2\\ &= \frac{1}{2}\sum_{l=0}^{4m-1}(2l+1)^2 = S_m,\\ \sum_{l\in Q_k\cup Q^*} l^2 &= \sum_{l\in Q_k} l^2 + \sum_{l\in Q^*} l^2\\ &= \frac{1}{2}\sum_{l=0}^{4k-1}(2l+1)^2 + \frac{1}{2}\sum_{l=4k}^{4m-1}(2l+1)^2 = S_m.\end{aligned} \tag{8}$$

Taking $P_m = P_k \cup P^*$ and $Q_m = Q_k \cup Q^*$, from the induction hypotheses (B_1), (B_2), and (B_3), we see that the conditions (B_4), (B_5), and (B_6) hold; in other words, the statement $\mathscr{P}_v(w+1)$ is true. Thus, assuming $\mathscr{P}_v(w)$ is true, it follows that $\mathscr{P}_v(w+1)$ is true. For $v \in I_4$, by induction, $\mathscr{P}_v(u)$ is true for $u \geq 1$. In summary, there exists a BSP (n) for $n \geq 4$. □

3.3. Construction of Extendable Tuples (Semibimagic Squares). In this section, we shall construct the building blocks forming quasi-normal trimagic squares by taking advantage of extendable tuples consisting of semibimagic squares. We shall take advantage of construction methods for quasi-normal MS $(4n, 2)$ s introduced by Pan and Huang [21], that is, combine the existence of a pair of orthogonal diagonal Latin squares of order $2n$ such that $n \geq 4$ with a BSP (n), to construct such an ET $(4n)$. Now, we state the following.

Lemma 5. *There exists an extendable tuple of order* $4n$ *for* $n \geq 4$.

Proof. Let k^* denote $4n-1-k$ for $k \in I_{4n}$ and let $\delta_k = (-1)^{\lfloor k/n \rfloor}$ for $k \in I_{2n}$. Let the pair of P and Q be a BSP (n) and $(L,R) \in \{(P,P),(P,Q),(Q,P),(Q,Q)\}$. Suppose that E and F are orthogonal diagonal Latin squares over L and R, respectively. Write $E = (e_{i,j})$ and $F = (f_{i,j})$. If $L = R$, we suppose that $e_{k,k} = f_{2n-1-k,2n-1-k}$ for $k \in I_{2n}$. Define $4n\times 4n$ matrices G and H by

$$\begin{aligned}\begin{pmatrix} g_{i,j} & g_{i,j^*}\\ g_{i^*,j} & g_{i^*,j^*}\end{pmatrix} &= \begin{pmatrix} e_{i,j} & e_{i,j}\\ -e_{i,j} & -e_{i,j}\end{pmatrix},\\ \begin{pmatrix} h_{i,j} & h_{i,j^*}\\ h_{i^*,j} & h_{i^*,j^*}\end{pmatrix} &= \begin{pmatrix} \delta_j f_{i,j} & -\delta_j f_{i,j}\\ \delta_j f_{i,j} & -\delta_j f_{i,j}\end{pmatrix}, \quad i,j \in I_{2n}.\end{aligned} \tag{9}$$

When $(L,R) = (P,P)$, denote $G + 8nH$ by A; when $(L,R) = (P,Q)$, denote $G+8nH$ by B; when $(L,R) = (Q,P)$, denote $G+8nH$ by C; when $(L,R) = (Q,Q)$, denote $G+8nH$ by D. Let $W \in \{A,B,C,D\}$. Next, we shall prove that the square tuple (A,B,C,D) is an ET $(4n)$.

First, we prove that A, B, C, and D satisfy $(\mathscr{E}_1)$. Since E and F are orthogonal, αE and βF are orthogonal, where $(\alpha,\beta) \in \{(1,-1),(-1,1),(-1,-1)\}$. We have

$$\begin{aligned}\left\{\left(g_{u,v},h_{u,v}\right): u,v \in I_{4n}\right\} &= \bigcup_{i,j\in I_{2n}} \left\{\left(g_{i,j},h_{i,j}\right),\left(g_{i,j^*},h_{i,j^*}\right),\left(g_{i^*,j},h_{i^*,j}\right),\left(g_{i^*,j^*},h_{i^*,j^*}\right)\right\}\\ &= \bigcup_{i,j\in I_{2n}} \left\{\left(e_{i,j},\delta_j f_{i,j}\right),\left(e_{i,j},-\delta_j f_{i,j}\right),\left(-e_{i,j},\delta_j f_{i,j}\right),\left(-e_{i,j},-\delta_j f_{i,j}\right)\right\}\\ &= \bigcup_{i,j\in I_{2n}} \left\{\left(e_{i,j},f_{i,j}\right),\left(e_{i,j},-f_{i,j}\right),\left(-e_{i,j},f_{i,j}\right),\left(-e_{i,j},-f_{i,j}\right)\right\}\\ &= (L\times R)\cup(L\times\overline{R})\cup(\overline{L}\times R)\cup(\overline{L}\times\overline{R}) = (L\cup\overline{L})\times(R\cup\overline{R}).\end{aligned} \tag{10}$$

Therefore, we obtain

$$\begin{aligned}
&\mathfrak{S}_A \cup \mathfrak{S}_B \cup \mathfrak{S}_C \cup \mathfrak{S}_D \\
&\quad = \{x + 8ny\colon\ (x, y) \in (P \cup \overline{P}) \times (P \cup \overline{P})\} \cup \{x + 8ny\colon\ (x, y) \in (P \cup \overline{P}) \times (Q \cup \overline{Q})\} \\
&\quad \cup \{x + 8ny\colon\ (x, y) \in (Q \cup \overline{Q}) \times (P \cup \overline{P})\} \cup \{x + 8ny\colon\ (x, y) \in (Q \cup \overline{Q}) \times (Q \cup \overline{Q})\} \\
&\quad = \{x + 8ny\colon\ (x, y) \in (P \cup \overline{P} \cup Q \cup \overline{Q}) \times (P \cup \overline{P} \cup Q \cup \overline{Q})\} \\
&\quad = \{x + 8ny\colon\ (x, y) \in [1 - 8n, 8n - 1]_2 \times [1 - 8n, 8n - 1]_2\} \\
&\quad = \left[1 - (8n)^2, (8n)^2 - 1\right]_2.
\end{aligned} \tag{11}$$

Next, we prove that A, B, C, and D satisfy ($\mathscr{E}_2$). Since $\mathfrak{S}_A \cup \mathfrak{S}_B \cup \mathfrak{S}_C \cup \mathfrak{S}_D = [1 - (8n)^2, (8n)^2 - 1]_2$, we see at once that W consists of $(4n)^2$ distinct integers. From (9), for $i, j \in I_{2n}$, we have

$$\begin{aligned}
w_{i,j} &= g_{i,j} + 8nh_{i,j} \\
&= e_{i,j} + 8n\delta_j f_{i,j}, \\
w_{i^*,j^*} &= g_{i^*,j^*} + 8nh_{i^*,j^*} \\
&= -e_{i,j} - 8n\delta_j f_{i,j}, \\
w_{i,j^*} &= g_{i,j^*} + 8nh_{i,j^*} \\
&= e_{i,j} - 8n\delta_j f_{i,j}, \\
w_{i^*,j} &= g_{i^*,j} + 8nh_{i^*,j} \\
&= -e_{i,j} + 8n\delta_j f_{i,j}.
\end{aligned} \tag{12}$$

Hence, the square W is self-complementary and each of two diagonals consists of opposite numbers. Since $W \in \{A, B, C, D\}$, we prove that A, B, C, and D are self-complementary and each of their diagonals consists of opposite numbers.

Now, we prove that A, B, C, and D satisfy ($\mathscr{E}_3$). From (9), for $i \in I_{2n}$, we have

$$\begin{aligned}
\sum_{v=0}^{4n-1} w_{i,v} &= \sum_{j\in I_{2n}} w_{i,j} + \sum_{j\in I_{2n}} w_{i,j^*} \\
&= \sum_{j\in I_{2n}} \left(g_{i,j} + 8nh_{i,j}\right) + \sum_{j\in I_{2n}} \left(g_{i,j^*} + 8nh_{i,j^*}\right) \\
&= \sum_{j\in I_{2n}} \left(e_{i,j} + 8n\delta_j f_{i,j}\right) + \sum_{j\in I_{2n}} \left(e_{i,j} - 8n\delta_j f_{i,j}\right) \\
&= 2\sum_{j\in I_{2n}} e_{i,j} = 2\sum_{x\in L} x = 0, \\
\sum_{v=0}^{4n-1} w_{i^*,v} &= \sum_{j\in I_{2n}} w_{i^*,j} + \sum_{j\in I_{2n}} w_{i^*,j^*} \\
&= \sum_{j\in I_{2n}} \left(g_{i^*,j} + 8nh_{i^*,j}\right) + \sum_{j\in I_{2n}} \left(g_{i^*,j^*} + 8nh_{i^*,j^*}\right) \\
&= \sum_{j\in I_{2n}} \left(-e_{i,j} + 8n\delta_j f_{i,j}\right) + \sum_{j\in I_{2n}} \left(-e_{i,j} - 8n\delta_j f_{i,j}\right) \\
&= -2\sum_{j\in I_{2n}} e_{i,j} \\
&= -2\sum_{x\in L} x = 0.
\end{aligned} \tag{13}$$

Similarly, for $j \in I_{2n}$, we have

$$\begin{aligned}
\sum_{u=0}^{4n-1} w_{u,j} &= 0, \\
\sum_{u=0}^{4n-1} w_{u,j^*} &= 0.
\end{aligned} \tag{14}$$

Noting that each of two diagonals of W consists of opposite numbers, we know that W is an MS $(4n)$ with magic sum 0. Therefore, we prove that A, B, C, and D are four MS $(4n)$ s with magic sum 0.

Again, from (9), for $i \in I_{2n}$, we have

$$\begin{aligned}
\sum_{v=0}^{4n-1} w_{i,v}^2 &= \sum_{j\in I_{2n}} w_{i,j}^2 + \sum_{j\in I_{2n}} w_{i,j^*}^2 \\
&= \sum_{j\in I_{2n}} \left(g_{i,j} + 8nh_{i,j}\right)^2 + \sum_{j\in I_{2n}} \left(g_{i,j^*} + 8nh_{i,j^*}\right)^2 \\
&= \sum_{j\in I_{2n}} \left(e_{i,j} + 8n\delta_j f_{i,j}\right)^2 + \sum_{j\in I_{2n}} \left(e_{i,j} - 8n\delta_j f_{i,j}\right)^2 \\
&= \sum_{j\in I_{2n}} \left(e_{i,j}^2 + 16n\delta_j e_{i,j} f_{i,j} + 64n^2 f_{i,j}^2\right) \\
&\quad + \sum_{j\in I_{2n}} \left(e_{i,j}^2 - 16n\delta_j e_{i,j} f_{i,j} + 64n^2 f_{i,j}^2\right) \\
&= 2\sum_{j\in I_{2n}} e_{i,j}^2 + 128n^2 \sum_{j\in I_{2n}} f_{i,j}^2 \\
&= 2\sum_{x\in L} x^2 + 128n^2 \sum_{y\in R} y^2 = 2\left(64n^2 + 1\right)S_n, \\
\sum_{v=0}^{4n-1} w_{i^*,v}^2 &= 2\left(64n^2 + 1\right)S_n.
\end{aligned} \tag{15}$$

Similarly, for $j \in I_{2n}$, we have

$$\begin{aligned}
\sum_{u=0}^{4n-1} w_{u,j}^2 &= 2\left(64n^2 + 1\right)S_n, \\
\sum_{u=0}^{4n-1} w_{u,j^*}^2 &= 2\left(64n^2 + 1\right)S_n.
\end{aligned} \tag{16}$$

The above results show that W^{*2} is an SMS $(4n)$ with magic sum $2(64n^2 + 1)S_n$. Therefore, we prove that A^{*2}, B^{*2}, C^{*2}, and D^{*2} are four SMS $(4n)$ s with magic sum $2(64n^2 + 1)S_n$.

Further, for $L = R$, we shall show that the sum of the entries in each of two diagonals of W^{*2} is $2(64n^2+1)S_n$. We have

$$\begin{aligned}\sum_{u=0}^{4n-1} w_{u,u}^2 &= \sum_{i=0}^{2n-1}\left(\left(e_{i,i}+8n\delta_i f_{i,i}\right)^2+\left(-e_{i,i}-8n\delta_i f_{i,i}\right)^2\right)\\
&= 2\sum_{i=0}^{2n-1}\left(e_{i,i}^2+16n\delta_i e_{i,i}f_{i,i}+64n^2 f_{i,i}^2\right)\\
&= 2\sum_{i=0}^{2n-1} e_{i,i}^2+128n^2\sum_{i=0}^{2n-1} f_{i,i}^2+32n\sum_{i=0}^{2n-1}\delta_i e_{i,i}f_{i,i}\\
&= 2\left(64n^2+1\right)S_n+32n\sum_{i=0}^{n-1}\left(\delta_i e_{i,i}f_{i,i}+\delta_{2n-1-i}e_{2n-1-i,2n-1-i}f_{2n-1-i,2n-1-i}\right)\\
&= 2\left(64n^2+1\right)S_n+32n\sum_{i=0}^{n-1}\left(e_{i,i}e_{2n-1-i,2n-1-i}-e_{2n-1-i,2n-1-i}e_{i,i}\right)\\
&= 2\left(64n^2+1\right)S_n,\\
\sum_{u=0}^{4n-1} w_{u,u^*}^2 &= \sum_{i=0}^{2n-1}\left(\left(e_{i,i}-8n\delta_i f_{i,i}\right)^2+\left(-e_{i,i}+8n\delta_i f_{i,i}\right)^2\right)\\
&= 2\sum_{i=0}^{2n-1} e_{i,i}^2+128n^2\sum_{i=0}^{2n-1} f_{i,i}^2-32n\sum_{i=0}^{2n-1}\delta_i e_{i,i}f_{i,i}\\
&= 2\left(64n^2+1\right)S_n.\end{aligned} \tag{17}$$

Hence, A^{*2} and D^{*2} are MS $(4n)$ s with magic sum $2(64n^2+1)S_n$. □

3.4. New Product Construction and Proof of Main Theorem. In this section, using an ET $(4n)$ and a new product construction, we shall give a unified construction of all quasi-normal MS $(16n,3)$ s for $n \geq 4$ and prove our main theorem.

Lemma 6. *If there exists an ET $(4n)$, then there exists a quasi-normal MS $(16n,3)$.*

Proof. Let (A,B,C,D) be an ET $(4n)$, and let

$$\begin{aligned}E &= \begin{pmatrix}-3 & 1 & 3 & -1\\ 3 & -1 & -3 & 1\\ -1 & 3 & 1 & -3\\ 1 & -3 & -1 & 3\end{pmatrix},\\
F &= \left(F_{u,v}\right)_{4\times 4}\\
&= \begin{pmatrix}A & B & -B & -A\\ C & D & -D & -C\\ -C & -D & D & C\\ -A & -B & B & A\end{pmatrix},\\
G &= (8n)^2 E\otimes J_{4n}+F,\end{aligned} \tag{18}$$

where $E\otimes J_{4n}$ is the Kronecker product of E and J_{4n}, that is, $E\otimes J_{4n} = (e_{i,j}J_{4n})$. Set $G = (g_{i,j})$ $(i,j \in I_{16n})$. Write $F_{u,v} = (f_{r,s}^{(u,v)})$, and then we have

$$\begin{aligned}g_{i,j} &= (8n)^2 e_{u,v}+f_{r,s}^{(u,v)},\\
i &= 4nu+r,\\
j &= 4nv+s,\quad u,v\in I_4, r,s\in I_{4n}.\end{aligned} \tag{19}$$

In the following, we prove that G is a quasi-normal MS $(16n,3)$.

First, we shall prove that G is quasi-normal. Noting that $\mathfrak{S}_{-A} = \mathfrak{S}_A$, $\mathfrak{S}_{-B} = \mathfrak{S}_B$, $\mathfrak{S}_{-C} = \mathfrak{S}_C$, and $\mathfrak{S}_{-D} = \mathfrak{S}_D$, we get a special array F^* as follows:

$$\begin{aligned}F^* &= \begin{pmatrix}\mathfrak{S}_A & \mathfrak{S}_B & \mathfrak{S}_{-B} & \mathfrak{S}_{-A}\\ \mathfrak{S}_C & \mathfrak{S}_D & \mathfrak{S}_{-D} & \mathfrak{S}_{-C}\\ \mathfrak{S}_{-C} & \mathfrak{S}_{-D} & \mathfrak{S}_D & \mathfrak{S}_C\\ \mathfrak{S}_{-A} & \mathfrak{S}_{-B} & \mathfrak{S}_B & \mathfrak{S}_A\end{pmatrix}\\
&= \begin{pmatrix}\mathfrak{S}_A & \mathfrak{S}_B & \mathfrak{S}_B & \mathfrak{S}_A\\ \mathfrak{S}_C & \mathfrak{S}_D & \mathfrak{S}_D & \mathfrak{S}_C\\ \mathfrak{S}_C & \mathfrak{S}_D & \mathfrak{S}_D & \mathfrak{S}_C\\ \mathfrak{S}_A & \mathfrak{S}_B & \mathfrak{S}_B & \mathfrak{S}_A\end{pmatrix}.\end{aligned} \tag{20}$$

Therefore, F^* is an array of order 4 over the symbol set $\{\mathfrak{S}_A, \mathfrak{S}_B, \mathfrak{S}_C, \mathfrak{S}_D\}$, that is, $\mathfrak{S}_{F^*}$. Obviously, E and F^* are both balanced squares, and E and F^* are orthogonal.

Therefore, noting that $\mathfrak{S}_F = \mathfrak{S}_A \cup \mathfrak{S}_B \cup \mathfrak{S}_C \cup \mathfrak{S}_D$, for $\alpha \in \{-3, -1, 1, 3\}$, we obtain

$$\begin{aligned}
&\left\{(8n)^2 e_{u,v} + f_{r,s}^{(u,v)} \colon e_{u,v} = \alpha, r, s \in I_{4n}, u, v \in I_4\right\} \\
&\quad = \left(\alpha(8n)^2 + \mathfrak{S}_A\right) \cup \left(\alpha(8n)^2 + \mathfrak{S}_B\right) \cup \left(\alpha(8n)^2 + \mathfrak{S}_C\right) \cup \left(\alpha(8n)^2 + \mathfrak{S}_D\right) \\
&\quad = \alpha(8n)^2 + \left(\mathfrak{S}_A \cup \mathfrak{S}_B \cup \mathfrak{S}_C \cup \mathfrak{S}_D\right) \\
&\quad = \alpha(8n)^2 + \mathfrak{S}_F = \alpha(8n)^2 + \left[1 - (8n)^2, (8n)^2 - 1\right]_2 \\
&\quad = \left[1 + (\alpha - 1)(8n)^2, (\alpha + 1)(8n)^2 - 1\right]_2.
\end{aligned} \tag{21}$$

It follows that

$$\begin{aligned}
\left\{g_{i,j} \colon i, j \in I_{16n}\right\} &= \bigcup_{\alpha \in \{-3,-1,1,3\}} \left[1 + (\alpha - 1)(8n)^2, (\alpha + 1)(8n)^2 - 1\right]_2 \\
&= \left[1 - (16n)^2, (16n)^2 - 1\right]_2,
\end{aligned} \tag{22}$$

which means that G is quasi-normal.

Now, we show that G is a magic square. Noting that E is a DLS(4) over $\{-3, -1, 1, 3\}$ and $\pm A$, $\pm B$, $\pm C$ and $\pm D$ are magic squares with magic sum 0, for $i \in I_{16n}$, we have

$$\begin{aligned}
\sum_{j=0}^{16n-1} g_{i,j} &= (8n)^2 \sum_{s=0}^{4n-1} \sum_{v=0}^{3} e_{u,v} + \sum_{v=0}^{3} \sum_{s=0}^{4n-1} f_{r,s}^{(u,v)} \\
&= (8n)^2 \times 4n \times 0 + 4 \times 0 = 0.
\end{aligned} \tag{23}$$

Similarly, one can prove that

$$\begin{aligned}
\sum_{i=0}^{16n-1} g_{i,j} &= 0, \quad j \in I_{16n}, \\
\sum_{i=0}^{16n-1} g_{i,i} &= 0, \\
\sum_{i=0}^{16n-1} g_{i,16n-1-i} &= 0.
\end{aligned} \tag{24}$$

Next, we show that G^{*2} is a magic square. Let $T_n = 2(64n^2+1)S_n$. Noting that E^{*2} is a magic square with magic sum 20, matrices $(\pm A)^{*2}$ and $(\pm D)^{*2}$ are magic squares with magic sum T_n, and $(\pm B)^{*2}$ and $(\pm C)^{*2}$ are semimagic squares with magic sum T_n, for $i \in I_{16n}$, we obtain

$$\begin{aligned}
\sum_{j=0}^{16n-1} h_{i,j}^2 &= (8n)^4 \sum_{s=0}^{4n-1} \sum_{v=0}^{3} e_{u,v}^2 + 2(8n)^2 \sum_{v=0}^{3} e_{u,v} \sum_{s=0}^{4n-1} f_{r,s}^{(u,v)} \\
&\quad + \sum_{v=0}^{3} \sum_{s=0}^{4n-1} \left(f_{r,s}^{(u,v)}\right)^2 \\
&= (8n)^4 \times (4n) \times 20 + 2(8n)^2 \times 0 \times 0 + 4 \times T_n \\
&= (8n)^4 \times 80n + 4T_n.
\end{aligned} \tag{25}$$

Write $T = (8n)^4 \times 80n + 4T_n$. Similarly, one can prove that

$$\begin{aligned}
\sum_{i=0}^{16n-1} h_{i,j}^2 &= T, \quad j \in I_{16n}, \\
\sum_{i=0}^{16n-1} h_{i,i}^2 &= T, \\
\sum_{i=0}^{16n-1} h_{i,16n-1-i}^2 &= T.
\end{aligned} \tag{26}$$

Therefore, G^{*2} is a magic square; that is, G is a quasi-normal bimagic square.

Finally, we show that G^{*3} is a magic square. Noting that E^{*3} is a magic square with magic sum 0 and each column of F consists of opposite numbers, for $i \in I_{16n}$, we obtain

$$\begin{aligned}\sum_{j=0}^{16n-1} g_{i,j}^3 &= (8n)^6 \sum_{s=0}^{4n-1}\sum_{v=0}^{3} e_{u,v}^3 + 3(8n)^4 \sum_{v=0}^{3} e_{u,v}^2 \sum_{s=0}^{4n-1} f_{r,s}^{(u,v)} \\ &\quad + 3(8n)^2 \sum_{v=0}^{3} e_{u,v} \sum_{s=0}^{4n-1} \left(f_{r,s}^{(u,v)}\right)^2 + \sum_{v=0}^{3}\sum_{s=0}^{4n-1}\left(f_{r,s}^{(u,v)}\right)^3 \\ &= (8n)^6 \times (4n) \times 0 + 3(8n)^4 \times 20 \times 0 + 3(8n)^2 \times 0 \times T_n + \sum_{v=0}^{3}\sum_{s=0}^{4n-1}\left(f_{r,s}^{(u,v)}\right)^3 \\ &= \sum_{v=0}^{3}\sum_{s=0}^{4n-1}\left(f_{r,s}^{(u,v)}\right)^3 = 0.\end{aligned} \tag{27}$$

Similarly, one can prove that

$$\begin{aligned}\sum_{i=0}^{16n-1} g_{i,j}^3 &= 0, \quad j \in I_{16n}, \\ \sum_{i=0}^{16n-1} g_{i,i}^3 &= 0, \\ \sum_{i=0}^{16n-1} g_{i,16n-1-i}^3 &= 0.\end{aligned} \tag{28}$$

Therefore, G^{*3} is magic; that is, G is a quasi-normal trimagic square.

We are now in a position to prove the main result. □

Theorem 7. *There exists a normal MS $(16n, 3)$ for all positive integers n.*

Proof. For $n \in \{1, 2, 3\}$, by Benson and Jacoby [14], Chen and Chen (see [15]; see also [16]), and Chen (see [16]), there exists a normal MS $(16n, 3)$. For every positive integer n greater than 3, by Lemma 4, there exists a BSP $(2n)$; hence, from Lemma 5, there exists an ET $(4n)$, and finally using Lemma 6, there exists a quasi-normal MS $(16n, 3)$; therefore, by Lemma 3, there exists a normal MS $(16n, 3)$. □

3.5. Discussion. Our methods can be generalized. Let n, d, and t be three positive integers and A, B, C, and D be integer matrices of order $4n$. Set $T_{4n} = [1-(8n)^2, (8n)^2-1]_2$ and $S_{4n,d} = 1/8n\sum_{x\in T_{4n}} x^d$. We call the tuple (A, B, C, D) a $(2t+1)$-extendable tuple of order $4n$, denoted by ET $(4n, 2t+1)$, if the following conditions are satisfied:

$(\mathscr{E}_1')$ $\mathfrak{S}_A \cup \mathfrak{S}_B \cup \mathfrak{S}_C \cup \mathfrak{S}_D = T_{4n}$.

$(\mathscr{E}_2')$ A, B, C, and D are self-complementary.

$(\mathscr{E}_3')$ A^{*d}, B^{*d}, C^{*d}, and D^{*d} are SMS $(4n)$ s with magic sum $S_{4n,d}$ for $d = 1, \ldots, 2t$, and the sums of entries of the left main diagonal of A^{*d} and D^{*d} are $S_{4n,d}$ for $d = 1, \ldots, 2t+1$.

Similar to the proof of Lemma 6, we can prove that the following conclusion: if there exists an ET $(4n, 2t+1)$, then there exists a quasi-normal MS $(16n, 2t+1)$. Therefore, by Lemma 3, if there exists an ET $(4n, 2t+1)$, then there exists a normal MS $(16n, 2t+1)$.

4. Conclusion

In the paper, we reduce the problem for the construction of a normal t-multimagic square of order n to the problem for the construction of a quasi-normal t-multimagic square of order n and give a complete solution to the existence of normal trimagic squares of all orders $16n$ with the help of many other combinatorial configures, such as set partitions, bimagic subset pairs, semibimagic squares, Latin squares, and classical product construction. More precisely, we prove that there exists a normal trimagic square of order $16n$ for all positive integers n. Meanwhile, the concept of the extendable tuple and the conclusion of Lemma 6 are generalized to the case of $(2t+1)$-th power.

Data Availability

All the data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

M. Su, F. Pan, and C. Hu conceptualized the study. M. Su, F. Pan, and C. Hu proposed the methodology. F. Pan, J. Meng, and S. Xiong wrote the original draft. F. Pan and C. Hu reviewed and edited the manuscript. All authors have read and agreed to the published version of the manuscript.

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