Hindawi
Journal of Mathematics
Volume 2023, Article ID 8390554, 7 pages
https://doi.org/10.1155/2023/8390554

Research Article

On Generalized Caristi Type Satisfying Admissibility Mappings

Mohammed M. A. Almazah,[1,2] **Basel Hardan**,[3] **Ahmed A. Hamoud**,[4] **and Faisal A. M. Ali**[5]

[1]Department of Mathematics, College of Sciences and Arts (Muhyil), King Khalid University, Muhyil 61421, Saudi Arabia
[2]Department of Mathematics and Computer, College of Sciences, Ibb University, Ibb 70270, Yemen
[3]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004, India
[4]Department of Mathematics, Taiz University, Taiz P.O. Box 6803, Taiz, Yemen
[5]Department of Data Science and Information Technology, College of Administrative Sciences, Taiz University, Taiz, Yemen

Correspondence should be addressed to Faisal A. M. Ali; faalmh@gmail.com

Received 2 November 2022; Revised 3 January 2023; Accepted 10 January 2023; Published 23 January 2023

Academic Editor: Tuncer Acar

In this article, the existence and uniqueness of a fixed point were investigated using the concept of (σ, γ)-contractive in the context of Hausdorff metric space. A well-known Caristi type is primarily generalized by the new results. The result is improved by building up an example.

1. Introduction

Fixed point theorems (FPTs) are essential in nonlinear functional analysis. The Banach contraction principle (BCP) [1] is one of the main findings of the FPTs that have been developed and implemented in various fields of study. BCP asserts that for any $\mathfrak{u}, \mathfrak{v} \in X$, where X is a complete metric space (MS), and τ is in the range of (0, 1), f has a FP.

BPC has been improved and expanded, as those made by Kannan, Chatterjee, and Reich in [2–4], respectively.

Newly, Isik et al. [5, 6] presented a new generalization of the BCP with an application, likewise, Biahdillah and Surjanto [7] introduced an application of the BCP in complex-valued Branciari b-MS, also Jleli et al. [8] researched further generalizations of the BCP, and on the other hand, Patil et al. [9–12] utilized the contractive, generalized contractive, Hardy-Rogers contractive, and generalized nonexpansive mappings on different spaces to get some new FPTs with applications.

Caristi employed a lower semicontinuous mapping (LSC) in 1976 to obtain a discernible expansion of BCP (see [13, 14]), and LSC maps are used by Abdeljawad and Karapinar [15] to generalize Cristi FPT on cone MS. In the literature, this theorem is known as the Caristi fixed point theorem (CFPT). New publications on the CFPT type are available: Aamri et al. [16] obtained CFPTs using the Száz principle in quasi-MS, Altun et al. [17] got CFPTs and some generalizations on M-MS, Aslantas et al. [18] gave some CFPTs, and a study on CFPTs in MS with a graph was given by Chuensupantharat and Gopal [19]. Direct proof of CFP was shown by Du [20], Kuhlmann et al. [21] showed the CFPTs from the point of view of ball spaces, and Hardan et al. [22] surveyed CFPTs of contractive mapping with application. A new type of Caristi's mapping on partial MSs shows that a partial MS is complete if and only if every Caristi mapping has an FP, given by Acar et al. [23], and on the same partial MS, Acar and Altun [24] also gave Bae and Suzuki-type generalizations of Caristi's FPT. Karapinar [25] showed this kind of FPT in compact partial MS before generalizing it to complete partial MS.

One of the helpful useful for our results is the Hausdorff space (HS), and it plays an important part in our theorem and its corollaries. The concept (α, ψ)-generalized contractions in a Hausdorff partial MS and its significance in obtaining some common FPTs for a pair of self-mappings which was discussed by Nazam and Acar [26].

An interesting notion is a different idea of applying the BCP, it contains the case of discontinuous functions, and this idea depends on using σ-admissible mappings (σ-AMs) to get FPT, which was introduced by Samet et al. in [27]. For these reasons, there is more study in the literature taking this case (see [28–31]). Two various separate developments of σ-AMs were presented. Budhia et al. in [32] applied a rectangular MS, while that Ansari in [33] applied the notion of $\mathscr{C}$-class maps. When we apply the researchers' idea on the Caristi type function, we obtain new FPTs on generalized MS, and we have used these results to find the existence and uniqueness (E & U) of a solution for problems for a lot of mathematical sections. On the other hand, of the most pleasure, discussed topics in FPTs, is the investigation of the E & U coincidence fixed point (COI-FP) of different maps in the given spaces. Agarwal and Karapinar [34] mentioned some comments on coupled FPTs in G-MS. Discussion on multidimensional coincidence points introduced by Al-Mezel et al. [35] and Cho et al. [36, 37] examined multidimensional FPTs in partially ordered complete MS and partially ordered fuzzy MS. Generalization of this study on coupled and triple FPTs was obtained by Karapinar et al. [38], and also Khojasteh and Rakoćević [39] used multivalued nonself-mappings to get some new common FPTs for generalized contractive. The generalization of contractive mappings satisfying the kind of an admissibility condition with the aid of $\mathscr{C}$-functions allows us to get novel common FPTs of the CFPT type in this study on HMS.

2. Preliminaries

In this section, we put the base for our major results.

Definition 1 (see [27]). Let f be a self-mapping on a MS (X, δ) and suppose $\sigma\colon X \times X \longrightarrow [0, \infty)$ be a function. f is called a σ-admissible function if $\sigma(f\mathfrak{u}, f\mathfrak{v}) \geq 1$ whenever $\sigma(\mathfrak{u}, \mathfrak{v}) \geq 1$, $\forall \mathfrak{u}, \mathfrak{v} \in X$.

Definition 2 (see [40]). Let (X, δ) be a MS, and let σ, γ as in Definition 1. X is said to be σ-orderly with respect to γ if for a sequence $\{\mathfrak{u}_i\}$ in X with $\sigma(\mathfrak{u}_i, \mathfrak{u}_{i+1}) \geq \gamma(\mathfrak{u}_i, \mathfrak{u}_{i+1})$ for all $i \geq N$ and $\mathfrak{u}_i \longrightarrow \mathfrak{u}$ as $i \longrightarrow \infty$; therefore, $\sigma(\mathfrak{u}_i, \mathfrak{u}) \geq \gamma(\mathfrak{u}_i, \mathfrak{u})$, for all $i \geq N$.

Definition 3 (see [27]). Let f be a self-mapping on a MS (X, δ). A map f is called a (σ, η)-contractive mapping if there exist two functions $\sigma\colon X \times X \longrightarrow [0, \infty)$ and $\eta\colon [0, +\infty) \longrightarrow [0, +\infty)$ such that

$$\sigma(\mathfrak{u}, \mathfrak{v})\delta(f\mathfrak{u}, f\mathfrak{v}) \leq \eta(\mathfrak{u}, \mathfrak{v}) \quad \forall \mathfrak{u}, \mathfrak{v} \in X, \tag{1}$$

where η is a nondecreasing function.

For extra ideas of σ-AM and (σ, η)-contractive mappings, see [27, 28, 30].

Definition 4 (see [31]). Suppose f be a self-mapping on a MS (X, δ) and let $\sigma, \gamma\colon X \times X \longrightarrow [0, \infty)$ are two mappings. A map f is called σ-AM with regard to γ if $\sigma(f\mathfrak{u}, f\mathfrak{v}) \geq \gamma(f\mathfrak{u}, f\mathfrak{v})$ where $\sigma(\mathfrak{u}, \mathfrak{v}) \geq \gamma(\mathfrak{u}, \mathfrak{v}), \forall \mathfrak{u}, \mathfrak{v} \in X$. Note that, if $\gamma(\mathfrak{u}, \mathfrak{v}) = 1$ for all $\mathfrak{u}, \mathfrak{v} \in X$, this definition leads us to Definition 1. Also, if we select $\sigma(\mathfrak{u}, \mathfrak{v}) = 1$, so we say that f is a γ-subadmissible functions.

Definition 5 (see [33]). A function $\psi\colon \mathscr{R}^+ \times \mathscr{R}^+ \longrightarrow \mathscr{R}$ is called $\mathscr{C}$-function, if

(i) $\psi(\mathfrak{u}_1, \mathfrak{u}_2) \leq \mathfrak{u}_1$
(ii) $\psi(\mathfrak{u}_1, \mathfrak{u}_2) = \mathfrak{u}_1$ implicit that either $\mathfrak{u}_1 = 0$ or $\mathfrak{u}_2 = 0$ for all $\mathfrak{u}_1, \mathfrak{u}_2 \in [0, \infty)$.

Example 1 (see [33]). The following functions are $\mathscr{C}$-functions.

(i) $\psi(\mathfrak{u}_1, \mathfrak{u}_2) = \eta(\mathfrak{u}_1)$, where $\eta\colon \mathscr{R}^+ \longrightarrow \mathscr{R}^+$ is a continuous map such that for $\mathfrak{u} > 0$, $\eta(\mathfrak{u}) < \mathfrak{u}$ and $\eta(0) = 0$
(ii) $\psi(\mathfrak{u}_1, \mathfrak{u}_2) = \tau\mathfrak{u}_1$, such that $\tau \in (0, 1)$
(iii) $\psi(\mathfrak{u}_1, \mathfrak{u}_2) = \mathfrak{u}_1 - \mathfrak{u}_2$

Definition 6 (see [41]). A nondecreasing continuous function $\eta\colon \mathscr{R}^+ \longrightarrow \mathscr{R}^+$ is called an altering distance mapping if $\eta(\mathfrak{u}) = 0$ if and only if $\mathfrak{u} = 0$

Remark 1. We refer to the class of altering distance functions by v

Definition 7 (see [42]). Suppose f, g are self-mappings on X, then

(i) A point $\mathfrak{z} \in X$ is said to be a common FP of f, g if $\mathfrak{z} = f\mathfrak{z} = g\mathfrak{z}$.
(ii) A point $\mathfrak{u} \in X$ is called a COI-P of f and g if $f\mathfrak{u} = g\mathfrak{u}$. In case $\mathfrak{z} = f\mathfrak{u} = g\mathfrak{u}$, then $\mathfrak{z}$ is said to be a COI-P of f, g.
(iii) The self-mappings f, g are said to be weakly compatible if they commute at their COI-FP that is, $fg\mathfrak{u} = gf\mathfrak{u}$ whenever $g\mathfrak{u} = f\mathfrak{u}$.

In the next section, with the help of type $\mathscr{C}$-functions, we will give a new coincidence and common from the type of CFPT by generalized of (σ, η)-contractive mappings satisfy σ-admissibility on HMS.

3. Main Results

Theorem 1. *Let (X, δ) be an HMS and let $f, g\colon X \longrightarrow X$ are weakly compatible fulfilling σ-AMs with respect to γ. Let $\psi \in \mathscr{C}$-functions and $\eta, \xi \in v$ such that for all $\mathfrak{u}, \mathfrak{v} \in X$.*

$$\sigma(\mathfrak{u}, \mathfrak{v}) \geq \gamma(\mathfrak{u}, \mathfrak{v}) \Rightarrow \eta(\delta(f\mathfrak{u}, f\mathfrak{v})) \leq \psi[\eta(\zeta(\mathfrak{u}, \mathfrak{v})), \xi(\zeta(\mathfrak{u}, \mathfrak{v}))], \tag{2}$$

where

$$\zeta(\mathfrak{u}, \mathfrak{v}) = \max\{|\vartheta f\mathfrak{v} - \vartheta g\mathfrak{v}|, |\vartheta f\mathfrak{u} - \vartheta g\mathfrak{u}|\}. \tag{3}$$

Such that for all $\mathfrak{u} \in X$, we have

$$\vartheta \mathfrak{u} = \| f\mathfrak{u} - g\mathfrak{u}\|. \tag{4}$$

Suppose that

(a) There exists $\mathfrak{u}_0 \in X$ such that $\sigma(g\mathfrak{u}_0, f\mathfrak{u}_0) \geq \gamma((g\mathfrak{u}_0, f\mathfrak{u}_0)$
(b) $\sigma(\mathfrak{u}_{i_{m-1}}, \mathfrak{u}_{j_{m-1}}) \geq \gamma(\mathfrak{u}_{i_{m-1}}, \mathfrak{u}_{j_{m-1}})$, for all $m \longrightarrow \infty$
(c) Either f, g are continuous or $\sigma(\mathfrak{u}_i, \mathfrak{z}) \geq \gamma(\mathfrak{u}_i, \mathfrak{z})$, for all $\mathfrak{u}_i \subset X$, $i \in \mathcal{N}$, and for some $\mathfrak{z} \in X$

Then, there exists $\mathfrak{u} \in X$ such that $f^n x = g\mathfrak{u} = \mathfrak{u}$, for some $n \in \mathcal{N}$, that is, $\mathfrak{u}$ is a periodic point, but if for each periodic point $\mathfrak{u}$ satisfying $\sigma(f\mathfrak{u}, g\mathfrak{u}) \geq \gamma(f\mathfrak{u}, g\mathfrak{u})$, then f has a fixed point. Moreover, the FP is an unique if for all $\mathfrak{u}, \mathfrak{v} \in \psi(f) = \{\mathfrak{u} \in X: f\mathfrak{u} = g\mathfrak{u} = \mathfrak{u}\}$, such that $\sigma(\mathfrak{u}, \mathfrak{v}) \geq \gamma(\mathfrak{u}, \mathfrak{v})$.

Proof. As $\mathfrak{u}_0 \in X$, given

$$\sigma(g\mathfrak{u}_0, f\mathfrak{u}_0) \geq \gamma(g\mathfrak{u}_0, f\mathfrak{u}_0). \tag{5}$$

Consider the iteration

$$f^i\mathfrak{u}_0 = f\mathfrak{u}_{i-1} = g\mathfrak{u}_i = \vartheta\mathfrak{u}_i. \tag{6}$$

Such that $\mathfrak{u}_i \neq \mathfrak{u}_{i+1}$, for all $i \in \mathcal{N}$. So, by (6) and since f satisfied Definition 4 and use (5), we have

$$\sigma(\mathfrak{u}, \mathfrak{v}) = \sigma\left(f\mathfrak{u}_0, f^2\mathfrak{u}_0\right) \geq \gamma\left(f\mathfrak{u}_0, f^2\mathfrak{u}_0\right) = \gamma(\mathfrak{u}, \mathfrak{v}). \tag{7}$$

By induction, we get

$$\sigma(\mathfrak{u}_i, \mathfrak{u}_{i+1}) \geq \gamma(\mathfrak{u}_i, \mathfrak{u}_{i+1}), \quad \forall i \in \mathcal{N}. \tag{8}$$

We will follow an incremental approach to build and accomplish our proof of E & U.

We shall show that $\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1}) \longrightarrow 0$ as $i \longrightarrow \infty$, i.e., $\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1})$ is nonincreasing.

By inequality (2), we get

$$\begin{aligned}\eta(\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1})) &= \eta(\delta(fx_{i-1}, fx_i))\\ &\leq \psi[\eta(\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i)), \xi(\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i))],\end{aligned} \tag{9}$$

where

$$\begin{aligned}\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i) &= \max\left\{\left|\vartheta fx_i - \vartheta gx_i\right|, \left|\vartheta fx_{i-1} - \vartheta gx_{i-1}\right|\right\}\\ &= \max\left\{\left|\vartheta\mathfrak{u}_{i+1} - \vartheta\mathfrak{u}_i\right|, \left|\vartheta\mathfrak{u}_i - \vartheta\mathfrak{u}_{i-1}\right|\right\}\\ &= \max\{\delta(\vartheta\mathfrak{u}_{i+1}, \vartheta\mathfrak{u}_i), \delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i-1})\}.\end{aligned} \tag{10}$$

We have two proposals:

Proposal 1: If $\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i) = \delta(\vartheta\mathfrak{u}_{i+1}, \vartheta\mathfrak{u}_i)$, then

$$\begin{aligned}&\eta(\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1}))\\ &= \eta(\delta(fx_{i-1}, fx_i))\\ &\leq \psi[\eta(\delta(\vartheta\mathfrak{u}_{i+1}, \vartheta\mathfrak{u}_i)), \xi(\delta(\vartheta\mathfrak{u}_{i+1}, \vartheta\mathfrak{u}_i))].\end{aligned} \tag{11}$$

From Definition 5, we find either $\eta(\delta(\vartheta\mathfrak{u}_{i+1}, \vartheta\mathfrak{u}_i)) = 0$ or $\xi(\delta(\vartheta\mathfrak{u}_{i+1}, \vartheta\mathfrak{u}_i)) = 0$, i.e., $\delta(\vartheta\mathfrak{u}_{i+1}, \vartheta\mathfrak{u}_i) = 0$, but this is a contradiction, due to $f\mathfrak{u}_{i+1} \neq f\mathfrak{u}_i$

Proposal 2: If $\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i) = \delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i-1})$, then

$$\begin{aligned}&\eta(\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1}))\\ &= \eta(\delta(fx_{i-1}, fx_i))\\ &\leq \psi[\eta(\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i-1})), \xi(\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i-1}))]\\ &\leq \eta(\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i-1})).\end{aligned} \tag{12}$$

Since η is nondecreasing map and ϑ is a continuous SLF, then

$$\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1}) \leq \delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i-1}). \tag{13}$$

Thus, $\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1})$ is nonincreasing sequence. So,

$$\begin{aligned}&\lim_{i\longrightarrow\infty} \delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1}) = \chi,\\ &\lim_{i\longrightarrow\infty} \zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i) = \chi.\end{aligned} \tag{14}$$

By the continuity property for the functions η and ξ, we have

$$\begin{aligned}\lim_{i\longrightarrow\infty} \eta(\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1})) &\leq \lim_{i\longrightarrow\infty} \psi[\eta(\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i)), \xi(\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i))],\\ &= \psi\left[\lim_{i\longrightarrow\infty} \eta(\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i)), \lim_{i\longrightarrow\infty} \xi(\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i))\right],\\ &\leq \lim_{i\longrightarrow\infty} \eta(\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i-1})),\end{aligned} \tag{15}$$

or

$$\eta(\chi) \leq \psi[\eta(\chi), \xi(\chi)] \leq \eta(\chi). \tag{16}$$

Hence, by Definition 5, we obtain $\chi = 0$, i.e., $\lim_{i\longrightarrow\infty}\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+1}) = 0$.

Using the abovementioned steps to prove $\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+2}) \longrightarrow 0$ and by inequality (2), we have

$$\begin{aligned}\eta(\delta(\vartheta\mathfrak{u}_i, \vartheta\mathfrak{u}_{i+2})) &= \eta(\delta(fx_{i-1}, fx_{i+1}))\\ &\leq \psi[\eta(\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_{i+1})), \xi(\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_{i+1}))]\\ &\leq \eta(\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_{i+1})),\end{aligned} \tag{17}$$

which could be

$$\delta(\vartheta \mathfrak{u}_i, \vartheta \mathfrak{u}_{i+2}) \leq \zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_{i+1}), \tag{18}$$

where η is changing the functions' space. We obtain

$$\begin{aligned} \zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_{i+1}) &= \max\{|\vartheta f x_{i+1} - \vartheta g x_{i+1}|, |\vartheta f x_{i-1} - \vartheta g x_{i-1}|\} \\ &= \max\{|\vartheta \mathfrak{u}_{i+2} - \vartheta \mathfrak{u}_{i+1}|, |\vartheta \mathfrak{u}_i - \vartheta \mathfrak{u}_{i-1}|\} \\ &= \max\{\delta(\vartheta \mathfrak{u}_{i+2}, \vartheta \mathfrak{u}_{i+1}), \delta(\vartheta \mathfrak{u}_i, \vartheta \mathfrak{u}_{i-1})\}. \end{aligned} \tag{19}$$

Again, we have two proposals:

Proposal 1: If $\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i) = \delta(\vartheta \mathfrak{u}_{i+2}, \vartheta \mathfrak{u}_{i+1})$, then

$$\begin{aligned} &\eta(\delta(\vartheta \mathfrak{u}_{i+2}, \vartheta \mathfrak{u}_{i+1})) \\ &= \eta(\delta(f x_{i+1}, f x_i)) \\ &\leq \psi[\eta(\delta(\vartheta \mathfrak{u}_{i+2}, \vartheta \mathfrak{u}_{i+1})), \xi(\delta(\vartheta \mathfrak{u}_{i+2}, \vartheta \mathfrak{u}_{i+1}))]. \end{aligned} \tag{20}$$

From the definition of $\mathscr{C}$-functions, we find either $\eta(\delta(\vartheta \mathfrak{u}_{i+2}, \vartheta \mathfrak{u}_{i+1})) = 0$ or $\xi(\delta(\vartheta \mathfrak{u}_{i+2}, \vartheta \mathfrak{u}_{i+1})) = 0$, i.e., $\delta(\vartheta \mathfrak{u}_{i+2}, \vartheta \mathfrak{u}_{i+1}) = 0$. But this is a contradiction, due to $f \mathfrak{u}_{i+1} \neq f \mathfrak{u}_i \, \forall i \in \mathscr{N}$.

Proposal 2: If $\zeta(\mathfrak{u}_{i-1}, \mathfrak{u}_i) = \delta(\vartheta \mathfrak{u}_i, \vartheta \mathfrak{u}_{i-1})$, this case has been discussed before.

Obviously, $\vartheta \mathfrak{u}_i$ is not necessity to be subsequent in arranging for all $i \in \mathscr{N}$ of the convergence in X.

By contradiction, we shall prove that $\{\vartheta \mathfrak{u}_i\}$ is a Cauchy sequence. For that, the following lemma is helpful for the remainder of the theorem's proof. Its proof is classic so we skip it. □

Lemma 1. *Let (X, δ) be a MS and let $\{\vartheta \mathfrak{u}_i\}$ be a sequence in X such that*

$$\lim_{i \longrightarrow \infty} (\vartheta \mathfrak{u}_i, \vartheta \mathfrak{u}_{i+1}) = \lim_{i \longrightarrow \infty} (\vartheta \mathfrak{u}_i, \vartheta \mathfrak{u}_{i+2}) = 0, \tag{21}$$

where $\mathfrak{u}_i \neq x_j$, for all $i \neq j$. If $\{\vartheta \mathfrak{u}_i\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two subsequences $\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{j_m} \subset \{\vartheta \mathfrak{u}_i\}$, where $m < j_m < i_m$, $m \in \mathscr{N}$. Furthermore,

$$\begin{aligned} \delta(\mathfrak{u}_{i_m}, \mathfrak{u}_{j_m}) &\geq \epsilon, \\ \delta(\mathfrak{u}_{i_m}, \mathfrak{u}_{j_m - 1}) &< \epsilon. \end{aligned} \tag{22}$$

Thus, for the following sequences,

$$\delta(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{i_{m-1}}), \delta(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{i_{m-1}}), \delta(\vartheta \mathfrak{u}_{j_m}, \vartheta \mathfrak{u}_{j_{m-1}}), \tag{23}$$

it satisfies

$$\begin{aligned} \lim_{i \longrightarrow \infty} \inf \delta(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{j_m}) &\leq \lim_{i \longrightarrow \infty} \max \delta(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{j_m}) \leq \epsilon, \\ \lim_{i \longrightarrow \infty} \inf \delta(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{i_{m-1}}) &\leq \lim_{i \longrightarrow \infty} \max \delta(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{i_{m-1}}) \leq \epsilon, \\ \lim_{i \longrightarrow \infty} \inf \delta(\vartheta \mathfrak{u}_{j_m}, \vartheta \mathfrak{u}_{j_{m-1}}) &\leq \lim_{i \longrightarrow \infty} \max \delta(\vartheta \mathfrak{u}_{j_m}, \vartheta \mathfrak{u}_{j_{m-1}}) \leq \epsilon. \end{aligned} \tag{24}$$

Now,

$$\lim_{m \longrightarrow \infty} \delta(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{j_m}) = \lim_{m \longrightarrow \infty} \delta(f \mathfrak{u}_{i_{m-1}}, f \mathfrak{u}_{j_{m-1}}), \tag{25}$$

for all $i, j, m \in \mathscr{N}$, by inequality (2), we have

$$\begin{aligned} &\eta(\delta(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{j_m}) \\ &= \eta(\delta(f \mathfrak{u}_{i_{m-1}}, f \mathfrak{u}_{j_{m-1}})) \\ &\leq \psi[\eta(\zeta(\mathfrak{u}_{i_{m-1}}, \mathfrak{u}_{j_{m-1}})), \xi(\zeta(\mathfrak{u}_{i_{m-1}}, \mathfrak{u}_{j_{m-1}}))], \end{aligned} \tag{26}$$

where

$$\begin{aligned} \zeta(\mathfrak{u}_{i_{m-1}}, \mathfrak{u}_{j_{m-1}}) &= \max\{|\vartheta f x_{j_{m-1}} - \vartheta g x_{j_{m-1}}|, |\vartheta f x_{i_{m-1}} - \vartheta g x_{i_{m-1}}|\} \\ &= \max\{|\vartheta \mathfrak{u}_{j_m} - \vartheta \mathfrak{u}_{j_{m-1}}|, |\vartheta \mathfrak{u}_{i_m} - \vartheta \mathfrak{u}_{i_{m-1}}|\} \\ &= \max\{\delta(\vartheta \mathfrak{u}_{j_m}, \vartheta \mathfrak{u}_{j_{m-1}}), \delta(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{i_{m-1}})\}. \end{aligned} \tag{27}$$

Then, by Lemma 1 and since,

$$\sigma(\vartheta \mathfrak{u}_{j_m}, \vartheta \mathfrak{u}_{j_{m-1}}) \geq \gamma(\vartheta \mathfrak{u}_{j_m}, \vartheta \mathfrak{u}_{j_{m-1}}), \tag{28}$$

also,

$$\sigma(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{i_{m-1}}) \geq \gamma(\vartheta \mathfrak{u}_{i_m}, \vartheta \mathfrak{u}_{i_{m-1}}). \tag{29}$$

When $i, m \longrightarrow \infty$, the inequality (26) will come like this.

$$\eta(\epsilon) \leq \psi[\eta(\epsilon), \xi(\epsilon)] \leq \eta(\epsilon), \tag{30}$$

where η, ξ, ψ, and ϑ are continuous functions. Therefore, we have $\eta(\epsilon) = 0$ or $\xi(\epsilon) = 0$. Thus, $\epsilon = 0$, which is a contradiction. We conclude that $\{\vartheta \mathfrak{u}_i\}$ is a Cauchy sequence. Since X is a complete MS, then $\vartheta \mathfrak{u}_i \longrightarrow \mathfrak{u}_*$ as $i \longrightarrow \infty$, for some $\mathfrak{u}_* \in X$.

If f and g are continuous mappings and by iteration relation (6), we find

$$\vartheta \mathfrak{u}_{i+1} = g \mathfrak{u}_{i+1} = f \mathfrak{u}_i = f \mathfrak{u}_*, \quad \text{as } i \longrightarrow \infty. \tag{31}$$

Then, f and g have a periodical point, since X is a HS, i.e., $f \mathfrak{u}_* = g \mathfrak{u}_* = \mathfrak{u}_*$. Otherwise, if X is an σ-regular with regard to γ, then by condition **b**, we obtain

$$\eta(\delta(\vartheta f \mathfrak{u}_i, \vartheta f \mathfrak{u}_*)) \leq \psi[\eta(\zeta(\mathfrak{u}_i, \mathfrak{u}_*)), \xi(\zeta(\mathfrak{u}_i, \mathfrak{u}_*))], \tag{32}$$

where

$$\begin{aligned} \zeta(\mathfrak{u}_i, \mathfrak{u}_*) &= \max\{|\vartheta f \mathfrak{u}_* - g \vartheta \mathfrak{u}_*|, |\vartheta f \mathfrak{u}_i - g \vartheta \mathfrak{u}_i|\} \\ &= \max\{|\vartheta f \mathfrak{u}_* - \vartheta \mathfrak{u}_*|, |\vartheta \mathfrak{u}_{i+1} - \vartheta \mathfrak{u}_i|\} \\ &= \max\{\delta(\vartheta f \mathfrak{u}_*, \vartheta \mathfrak{u}_*), \delta(\vartheta \mathfrak{u}_{i+1}, \vartheta \mathfrak{u}_i)\}. \end{aligned} \tag{33}$$

Since $\delta(\vartheta \mathfrak{u}_{i+1}, \vartheta \mathfrak{u}_i) = 0$ as $i \longrightarrow \infty$ (from the first step), then

$$\lim_{i \longrightarrow \infty} \zeta(\mathfrak{u}_i, \mathfrak{u}_*) = \delta(\vartheta f \mathfrak{u}_*, \vartheta \mathfrak{u}_*). \tag{34}$$

By substituting equation (34) into inequality (32), we get

$$\eta(\vartheta f \mathfrak{u}_*, \vartheta \mathfrak{u}_*) \leq \psi[\eta(\delta(\vartheta f \mathfrak{u}_*, \vartheta \mathfrak{u}_*)), \xi(\delta(\vartheta f \mathfrak{u}_*, \vartheta \mathfrak{u}_*))], \tag{35}$$

where $i \longrightarrow \infty$. Thus, $\eta(\delta(\vartheta f \mathfrak{u}_*, \vartheta \mathfrak{u}_*)) = 0$ or $\xi(\delta(\vartheta f \mathfrak{u}_*, \vartheta \mathfrak{u}_*)) = 0$; therefore, $\delta = (\vartheta f \mathfrak{u}_*, \vartheta \mathfrak{u}_*) = 0$. Hence, $\vartheta f \mathfrak{u}_* = \vartheta \mathfrak{u}_*$; based on the definition of ϑ, we conclude that f and g have a periodical point $f\mathfrak{u}_* = g\mathfrak{u}_* = \mathfrak{u}_*$. Thus, we have shown the validity of the c part of our theory.

In the remainder of the proof, we will show that f and g have a COI-FP of the periodical point. This is what we will discuss in the next step.

Suppose $\vartheta f^k \mathfrak{z} = \vartheta g^k \mathfrak{z} = \mathfrak{z}$, $\mathfrak{z} \in X$. Obviously, $\mathfrak{z}$ is a FP of f where $k = 1$. We will prove $\vartheta f^{k-1}\mathfrak{z} = g^{k-1}\mathfrak{z} = \mathfrak{z}_*$, where $k > 1$, $\forall k > 1$, $k \in \mathcal{N}$. We have $\sigma(f\mathfrak{z}, g\mathfrak{z}) \geq \gamma(f\mathfrak{z}, g\mathfrak{z})$ for a periodical point $\mathfrak{z}$. If potential, let $\vartheta f^{k-1}\mathfrak{z} \neq \vartheta f^k \mathfrak{z}$ and $\vartheta g^{k-1}\mathfrak{z} \neq \vartheta g^k \mathfrak{z}$ for all $k > 1$, $k \in \mathcal{N}$. Therefore, by inequalities (2),

$$\begin{aligned} &\eta\left(\delta\left(f^{k-1}\mathfrak{z}, f^k\mathfrak{z}\right)\right) \\ &\leq \psi\left[\eta\left(\zeta\left(f^{k-2}\mathfrak{z}, f^{k-1}\mathfrak{z}\right)\right), \xi\left(\zeta\left(f^{k-2}\mathfrak{z}, f^{k-1}\mathfrak{z}\right)\right)\right], \end{aligned} \tag{36}$$

where

$$\begin{aligned} \zeta\left(f^{k-2}\mathfrak{z}, f^{k-1}\mathfrak{z}\right) &= \max\left\{\left|\vartheta f f^{k-1}\mathfrak{z} - \vartheta g f^{k-1}\mathfrak{z}\right|, \left|\vartheta f f^{k-2}\mathfrak{z} - \vartheta g f^{k-2}\mathfrak{z}\right|\right\} \\ &= \max\left\{\left|\vartheta f^k\mathfrak{z} - \vartheta f^{k-1}\mathfrak{z}\right|, \left|\vartheta f^{k-1}\mathfrak{z} - \vartheta f^{k-2}\mathfrak{z}\right|\right\} \\ &= \left\{\delta\left(\vartheta f^k\mathfrak{z}, \vartheta f^{k-1}\mathfrak{z}\right), \delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^{k-2}\mathfrak{z}\right)\right\}. \end{aligned} \tag{37}$$

We have two proposals again.

Proposal 1: If $\zeta(f^{k-2}\mathfrak{z}, f^{k-1}\mathfrak{z}) = \delta(\vartheta f^k\mathfrak{z}, \vartheta f^{k-1}\mathfrak{z})$ for some k, then by (36), we have

$$\begin{aligned} &\eta\left(\delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^k\mathfrak{z}\right)\right) \\ &\leq \psi\left[\eta\left(\delta\left(\vartheta f^k\mathfrak{z}, \vartheta f^{k-1}\mathfrak{z}\right)\right), \xi\left(\delta\left(\vartheta f^k\mathfrak{z}, \vartheta f^{k-1}\mathfrak{z}\right)\right)\right] \\ &\leq \eta\left(\delta\left(\vartheta f^k\mathfrak{z}, \vartheta f^{k-1}\mathfrak{z}\right)\right). \end{aligned} \tag{38}$$

Hence, $\delta(\vartheta f^k\mathfrak{z}, \vartheta f^{k-1}\mathfrak{z}) = 0$, by Definition 6. Thus, $\vartheta f^k\mathfrak{z} = \vartheta f^{k-1}\mathfrak{z}$, which is contradiction.

Proposal 2: If $\zeta(f^{k-2}\mathfrak{z}, f^{k-1}\mathfrak{z}) = \delta(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^{k-2}\mathfrak{z})$ for all k. Here, f is an σ-admissible with regard to γ and $\sigma(f\mathfrak{z}, g\mathfrak{z}) \geq \gamma(f\mathfrak{z}, g\mathfrak{z})$, and we obtain

$$\begin{aligned} &\eta\left(\delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^k\mathfrak{z}\right)\right) \\ &\leq \psi\left[\eta\left(\delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^{k-2}\mathfrak{z}\right)\right), \xi\left(\delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^{k-2}\mathfrak{z}\right)\right)\right] \\ &\leq \eta\left(\delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^{k-2}\mathfrak{z}\right)\right). \end{aligned} \tag{39}$$

Consequently, $\delta(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^{k-2}\mathfrak{z})$ is a nonincreasing sequence of $\mathcal{R}^+$, and we get

$$\begin{aligned} \eta(\delta(\mathfrak{z}, \vartheta f\mathfrak{z})) &= \eta(\delta(\vartheta g\mathfrak{z}, \vartheta f\mathfrak{z})) \\ &= \eta\left(\delta\left(\vartheta g f^k\mathfrak{z}, \vartheta f^{k+1}\mathfrak{z}\right)\right) \\ &\leq \eta\left(\delta\left(\vartheta g f^{k-1}\mathfrak{z}, \vartheta f^k\mathfrak{z}\right)\right) \\ &\leq \psi\left[\eta\left(\delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^{k-2}\mathfrak{z}\right)\right), \xi\left(\delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^{k-2}\mathfrak{z}\right)\right)\right] \\ &\leq \eta\left(\delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^{k-2}\mathfrak{z}\right)\right) \\ &\vdots \\ &\leq \psi[\eta(\delta(\mathfrak{z}, \vartheta f\mathfrak{z})), \xi(\delta(\vartheta\mathfrak{z}, \vartheta f\mathfrak{z}))] \\ &\leq \eta(\delta(\mathfrak{z}, \vartheta f\mathfrak{z})) \\ &\eta\left(\delta\left(\vartheta f^{k-1}\mathfrak{z}, \vartheta f^k\mathfrak{z}\right)\right) \\ &\leq \psi\left[\eta\left(\delta\left(\vartheta f^k\mathfrak{z}, \vartheta f^{k-1}\mathfrak{z}\right)\right), \xi\left(\delta\left(\vartheta f^k\mathfrak{z}, \vartheta f^{k-1}\mathfrak{z}\right)\right)\right]. \end{aligned} \tag{40}$$

Thus, $\vartheta f^k\mathfrak{z} = \vartheta f^{k-1}\mathfrak{z}$. And implicitly indicates that $\vartheta g^k\mathfrak{z} = \vartheta g^{k-1}\mathfrak{z}$ which is contradiction. Accordingly, the claim that $f^{k-1}\mathfrak{z} = g^{k-1}\mathfrak{z} = \mathfrak{z}_*$ is not true. Hence, f, g have a COI-FP $\mathfrak{z}$. Since f and g are weakly compatible, and by Definition 7 (iii), then, $\mathfrak{z}$ is a common FP of f and g on X.

The uniqueness of the fixed point increases its strength and renders the solution is not subject to other possibilities. Consequently, we finish our argument by explaining that the FP we have established is unique. This is what we will clarify in the final step of the proof.

Suppose that $\mathfrak{z}_1, \mathfrak{z}_2 \in X$ are two common FPs of f and g such that $\mathfrak{z}_1 \neq \mathfrak{z}_2$. Use (2) and since $\sigma(\mathfrak{z}_1, \mathfrak{z}_2) \geq \gamma(\mathfrak{z}_1, \mathfrak{z}_2)$, we obtain

$$\begin{aligned}\eta(\delta(\mathfrak{z}_1, \mathfrak{z}_2)) &= \eta(\delta(\vartheta f \mathfrak{z}_1, \vartheta f \mathfrak{z}_2)) \\ &\leq \psi[\eta(\zeta(\mathfrak{z}_1, \mathfrak{z}_2)), \xi(\zeta(\mathfrak{z}_1, \mathfrak{z}_2))],\end{aligned} \tag{41}$$

where

$$\zeta(\mathfrak{z}_1, \mathfrak{z}_2) = \max\left\{\left|\vartheta f \mathfrak{z}_2 - \vartheta g \mathfrak{z}_2\right|, \left|\vartheta f \mathfrak{z}_1 - \vartheta g \mathfrak{z}_1\right|\right\}. \tag{42}$$

Then, $\zeta(\mathfrak{z}_1, \mathfrak{z}_2) = 0$. Therefore, $\eta(\delta(\mathfrak{z}_1, \mathfrak{z}_2)) = 0$. Consequently, $\mathfrak{z}_1 = \mathfrak{z}_2$.

Thus, we have proven the uniqueness of the FP that we have found, and by this, we have completed proving our result.

Now, we will present the following corollaries, which are derived from our main result.

Corollary 1. *Let (X, δ) be a HMS let $f, g: X \longrightarrow X$ are weakly compatible fulfilling σ-AMs with respect to γ. Let $\psi \in \mathscr{C}$ and $\eta, \xi \in \upsilon$ such that for all $\mathfrak{u}, \mathfrak{v} \in X$.*

$$\begin{aligned}\sigma(\mathfrak{u}, \mathfrak{v}) \geq \gamma(\mathfrak{u}, \mathfrak{v}) \Rightarrow \eta(\delta(f\mathfrak{u}, f\mathfrak{v})) \\ \leq [\eta(\zeta(\mathfrak{u}, \mathfrak{v})) - \xi(\zeta(\mathfrak{u}, \mathfrak{v}))],\end{aligned} \tag{43}$$

*where $\zeta(\mathfrak{u}, \mathfrak{v})$ and the conditions **a**, **b**, and **c** are the same as in Theorem 1. Then, there exists $\mathfrak{u} \in X$ such that $f^n\mathfrak{u} = g^n\mathfrak{u} = \mathfrak{u}$, for some $n \in \mathscr{N}$; that is, $\mathfrak{u}$ is a periodic point, but if for each periodic point $\mathfrak{u}$ satisfying $\sigma(f\mathfrak{u}, g\mathfrak{u}) \geq \gamma(f\mathfrak{u}, g\mathfrak{u})$, then f and g have a common FP. Moreover, the FP is an unique if for all $\mathfrak{u}, \mathfrak{v} \in \psi(f) = \{\mathfrak{u} \in V: f\mathfrak{u} = g\mathfrak{u} = \mathfrak{u}\}$, such that $\sigma(\mathfrak{u}, \mathfrak{v}) \geq \gamma(\mathfrak{u}, \mathfrak{v})$.*

Proof. Take $\psi(\mathfrak{u}_1 - \mathfrak{u}_2) = \mathfrak{u}_1 - \mathfrak{u}_2$ and follow the same method proving Theorem 1.

To exemplify our findings, we will introduce the following example. □

Example 2 Let $X = [0, 1]$. Suppose f and g are weakly compatible self-mappings on X such that

$$f\mathfrak{u} = \begin{cases} \dfrac{2\mathfrak{u} + 1}{2}, & \mathfrak{u} \in \left[0, \dfrac{1}{2}\right), \\ & \qquad g\mathfrak{u} = \mathfrak{u}, \quad \forall \mathfrak{u} \in [0, 1]. \\ \dfrac{1}{2}, & \mathfrak{u} \in \left[\dfrac{1}{2}, 1\right], \end{cases} \tag{44}$$

Let $\sigma, \gamma: X \times X \longrightarrow [0, \infty)$, such that $\sigma(\mathfrak{u}, \mathfrak{v}) = 3$, $\gamma(\mathfrak{u}, \mathfrak{v}) = 2$, $\forall \mathfrak{u}, \mathfrak{v} \in X$. Suppose $\delta: X \times X \longrightarrow [0, 1)$ be a MS. Consider $\psi: [0, \infty) \times [0, \infty) \longrightarrow \mathscr{R}$ and $\eta, \xi: [0, \infty) \longrightarrow [0, \infty)$ defined as $\psi(\mathfrak{u}_1, \mathfrak{u}_2) = \mathfrak{u}_2 - \mathfrak{u}_1$, $3\eta(\mathfrak{u}) = \mathfrak{u}$ and $5\xi(\mathfrak{u}) = 4\mathfrak{u}$. Then, all the terms of Theorem 1 are satisfied. Hence, $\mathfrak{u} = 1/2$ be a unique common FP of f and g on X.

Data Availability

No data were used in this investigation.

Conflicts of Interest

All authors declare that they have no conflicts of interest.

Acknowledgments

Their authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Larg Groups (project under grant number: RGP.2/4/43). This research was financially supported by the authors.

References

[1] S. Banach, “Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,” *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.
[2] R. Kannan, “Some results on fixed points,” *Bulletin of the Calcutta Mathematical Society*, vol. 60, pp. 71–76, 1968.
[3] S. Chatterjee, “Fixed point theorem,” *C. R. Acad, Bulgar Sci*, vol. 25, pp. 727–730, 1972.
[4] S. Reich, “Fixed points of contractive functions,” *Bollettino Della Unione Matematica Italiana - A*, vol. 5, pp. 26–42, 1972.
[5] H. Isik, B. Mohammadi, M. R. Haddadi, and V. Parvaneh, “On a new generalization of banach contraction principle with application,” *Mathematics*, vol. 7, no. 9, p. 862, 2019.
[6] B. Samet, “Discussion on A fixed point theorem of Banach--Caccioppoli type on a class of generalized metric spaces by A. Branciari,” *Publicationes Mathematicae Debrecen*, vol. 76, no. 4, pp. 493-494, 2010.
[7] A. Biahdillah and S. Surjanto, “Application of banach contraction principle in complex valued rectangular b-metric space,” *Journal of Physics: Conference Series*, vol. 1490, no. 1, pp. 1–11, 2020.
[8] M. Jleli, E. Karapinar, and B. Samet, “Further generalizations of the Banach contraction principle,” *Journal of Inequalities and Applications*, vol. 2014, no. 1, pp. 439–9, 2014.
[9] J. Patil, B. Hardan, A. A. Hamoud, and A. Bachhav, “Common fixed point theorem for Hardy-Rogers contractive type in cone 2-metric spaces and its results,” *The interdisciplinary journal of Discontinuity, Nonlinearity, and Complexity*, vol. 12, no. 1, pp. 197–206, 2023.
[10] J. Patil, B. Hardan, A. A. Hamoud, A. Bachhav, H. Emadifar, and H. Günerhan, “Corrigendum to “generalization contractive mappings on rectangular b-metric space”,” *Advances in Mathematical Physics*, vol. 2022, Article ID 7291001, 1 page, 2022.
[11] J. Patil, B. Hardan, A. A. Hamoud, A. Bachhav, and H. Emadifar, “A new result on Branciari metric space using (α, γ)-contractive mappings,” *Topological Algebra and Its Applications*, vol. 10, no. 1, pp. 103–112, 2022.
[12] J. Patil and B. Hardan, “On fixed point theorems in complete metric space,” *Journal of Computer and Mathematical Sciences*, vol. 10, no. 7, pp. 1419–1425, 2019.
[13] J. Caristi, “Fixed point theorems for mappings satisfying inwardness conditions,” *Transactions of the American Mathematical Society*, vol. 215, no. 0, pp. 241–251, 1976.

[14] J. Caristi, "Fixed point theory and inwardness conditions," *Applied Nonlinear Analysis*, pp. 479–483, 1979.

[15] T. Abdeljawad and E. Karapinar, "Quasi-cone metric spaces and generalizations of Caristi Kirk's theorem," *Fixed Point Theory and Applications*, vol. 2009, Article ID 574387, 9 pages, 2009.

[16] M. Aamri, K. Chaira, S. Lazaiz, and El-M. Marhrani, "Caristi type fixed point theorems using Száz principle in quasi-metric spaces," *Carpathian Journal of Mathematics*, vol. 36, no. 2, pp. 179–188, 2020.

[17] I. Altun, H. Sahin, and D. Turkoglu, "Caristi-type fixed point theorems and some generalizations on M-metric space," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 3, pp. 2647–2657, 2020.

[18] M. Aslantas, H. Sahin, and D. Turkoglu, "Some Caristi type fixed point theorems," *The Journal of Analysis*, vol. 29, pp. 89–103, 2020.

[19] N. Chuensupantharat and D. Gopal, "On Caristi's fixed point theorem in metric spaces with a graph," *Carpathian Journal of Mathematics*, vol. 36, no. 2, pp. 259–268, 2020.

[20] W. S. Du, "A direct proof of Caristi's fixed point theorem," *Applied Mathematical Sciences*, vol. 10, no. 10, pp. 2289–2294, 2016.

[21] F. V. Kuhlmann, K. Kuhlmann, and M. Paulsen, "The caristi–kirk fixed point theorem from the point of view of ball spaces," *Journal of Fixed Point Theory and Applications*, vol. 20, no. 3, pp. 107–109, 2018.

[22] B. Hardan, J. Patil, A. Chaudhari, and A. Bachhav, "Caristi type fixed point theorems of contractive mapping with application," in *Proceedings of the One Day National Conference on Recent Advances In Sciences*, pp. 1–6, Department of Physics, Chemistry, Mathematics, Botany & Zoology, Shivaji Arts, Commerce and Science College, Kannad, Aurangabad, India, February 2020.

[23] Ö. Acar, I. Altun, and S. Romaguera Bonilla, "Caristi's type mappings on complete partial metric spaces," *Fixed Point Theory*, vol. 14, no. 1, pp. 3–10, 2013.

[24] Ö. Acar and I. Altun, "Some generalizations of Caristi type fixed point theorem on partial metric spaces," *Filomat*, vol. 26, no. 4, pp. 833–837, 2012.

[25] E. Karapinar, "Generalizations of Caristi Kirk's theorem on partial metric spaces," *Fixed Point Theory and Applications*, vol. 2011, no. 1, pp. 1–7, 2011.

[26] M. Nazam and Ö. Acar, "Fixed points of (α, ψ)-contractions in Hausdorff partial metric spaces," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 16, pp. 5159–5173, 2019.

[27] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for σ, γ-contractive type mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[28] H. Alsulami, S. Chandok, M. Taoudi, and I. M. Erhan, "Some fixed point theorems for (ϕ, ψ)-rational type contractive mappings," *Fixed Point Theory and Applications*, vol. 2015, pp. 1–17, 2015.

[29] D. Gopal, P. Kumam, and M. Abbas, *Background and Recent Developments of Metric Fixed Point Theory*, CRC Press, Boca Raton, Florida, 2017.

[30] E. Karapınar, "Discussion on α, ψ-contractions on generalized metric spaces," *Abstract and Applied Analysis*, vol. 2014, pp. 1–7, 2014.

[31] P. Salimi, A. Latif, and N. Hussain, "Modified $\alpha - \psi$-contractive mappings with applications," *Fixed Point Theory and Applications*, vol. 2013, no. 1, pp. 1–19, 2013.

[32] L. Budhia, M. Kir, D. Gopal, and H. Kiziltunç, "New fixed point results in rectangular metric space and application to fractional calculus," *Tbilisi Mathematical Journal*, vol. 10, no. 1, pp. 91–104, 2017.

[33] A. Ansari, "Note on α, ψ-contractive type mappings and related fixed point,"vol. 2014, pp. 377–380, in *Proceedings of the The 2nd regional conference on mathematics and applications*, vol. 2014, pp. 377–380, Noor Univ, Tonekabon, Iran, September 2014.

[34] R. P. Agarwal and E. Karapinar, "Remarks on some coupled fixed point theorems in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2013, no. 1, pp. 2–33, 2013.

[35] S. A. Al-Mezel, H. H. Alsulami, E. Karapinar, and A. F. R. López-de-Hierro, "Discussion on Multidimensional coincidence points via recent publications," *Abstract and Applied Analysis*, vol. 2014, pp. 1–13, 2014.

[36] Y. Cho, J. Martínez-Moreno, A. Roldán, and C. Roldán, "Multidimensional fixed point theorems under (ψ, ϕ)-contractive conditions in partially ordered complete metric spaces," *Journal of Computational and Applied Mathematics*, vol. 273, pp. 76–87, 2015.

[37] A. Roldán, J. Martínez-Moreno, C. Roldán, and Y. J. Cho, "Multidimensional coincidence point results for compatible mappings in partially ordered fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 251, pp. 71–82, 2014.

[38] E. Karapinar, A. Roldán, N. Shahzad, and W. Sintunaravat, "Discussion on coupled and tripled coincidence point theorems for ϕ-contractive mappings without the mixed g-monotone property," *Fixed Point Theory and Applications*, vol. 2014, p. 92, 2014.

[39] F. Khojasteh and V. Rakoćević, "Some new common fixed point results for generalized contractive multi-valued non-self-mappings," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 287–293, 2012.

[40] L. Budhia, H. Aydi, A. H. Ansari, and D. Gopal, "Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations," *Nonlinear Analysis Modelling and Control*, vol. 25, no. 4, pp. 580–597, 2020.

[41] M. S. Khan, M. Swaleh, and S. Sessa, "Fixed point theorems by altering distances between the points," *Bulletin of the Australian Mathematical Society*, vol. 30, no. 1, pp. 1–9, 1984.

[42] M. Arshad, J. Ahmad, and E. Karapınar, "Some common fixed point results in rectangular metric spaces," *International Journal of Analysis*, vol. 2013, Article ID 307234, 7 pages, 2013.