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Research Article

A New Approach to Timelike Hypersurfaces of Constant Ratio in IE_1^4

Sezgin Büyükkütük

Kocaeli University, Gölcük Vocational School of Higher Education, Kocaeli 41650, Turkey

Correspondence should be addressed to Sezgin Büyükkütük; sezgin.buyukkutuk@kocaeli.edu.tr

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In this study, we consider timelike revolution hypersurfaces of constant ratio in Minkowski space-time. At first, we exhibit the representations of revolution hypersurfaces given by three different forms. Then, we yield the conditions for such hypersurfaces to correspond to constant ratio surface. As a result of these conditions, we present the position vectors of constant ratio timelike rotational hypersurfaces in IE_1^4.

1. Introduction

The concept of constant ratio submanifolds was first discussed by Chen in 2001, and then many researchers evaluated this concept on curves and surfaces from different perspectives [1–6].

Let S: $\psi(u, v, w)$: $(u, v, w) \in D(D \subset E^3)$ be a hypersurface in Minkowski 4–space. The parameterization of the hypersurface can be separated into tangent component and normal component as

$$\psi = \psi^T + \psi^N. \tag{1}$$

The name “constant ratio” comes from the ratio of this tangent component and the normal component. Denoting the orthonormal frame $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and the distance function $\rho = \|\psi\|$, the gradient of ρ is known as

$$\operatorname{grad}(\rho) = \sum_{k=1}^{4} \sigma_k(\rho)\sigma_k. \tag{2}$$

Moreover, by the use of

$$\sigma_k(\rho) = \frac{g(\sigma_k, \psi)}{\|\psi\|}, \tag{3}$$

equation (2) becomes

$$\operatorname{grad}(\rho) = \sum_{k=1}^{4} \frac{g(\sigma_k, \psi)}{\|\psi\|}\sigma_k. \tag{4}$$

Here, g indicates the Lorentzian metric in IE_1^4. Hence, the norm of gradient function is congruent to the equality

$$\|\operatorname{grad}(\rho)\|^2 = \sum_{k=1}^{4} \frac{(g(\sigma_k, \psi))^2}{\|\psi\|^2}. \tag{5}$$

If this relation is equal to a positive constant, then related surface is called as constant ratio surface, i.e.,

$$\|\operatorname{grad}(\rho)\| = k, \ldots k \in \mathrm{IR}^+. \tag{6}$$

Revolution surface that has many applications in multidisciplinary sciences are also used theoretically in geometry with the forms catenoid, tube surface, canal surface, ruled, and developable surface. Some of them have characteristic features as being minimal (catenoid) and being flat (developable surface) [7–9].

In the present work, we evaluate the timelike constant ratio hypersurfaces of revolution in four-dimensional Minkowski space. Firstly, we present the three types of parameterizations of rotational hypersurfaces. Then, we yield the conditions for them to become constant ratio surface. We classify these types of hypersurfaces with respect to satisfying $\|\operatorname{grad}(\rho)\| = 0$, $\|\operatorname{grad}(\rho)\| = 1$, and $\|\operatorname{grad}(\rho)\| = k$ [1, 2].

2. Preliminaries

In Minkowski space-time, the Lorentzian metric is given by

$$g(u,v) = -u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4, \tag{7}$$

and the vector product is known as

$$u\times v\times w = \begin{vmatrix} -\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}, \tag{8}$$

where $u = (u_1,u_2,u_3,u_4)$, $v = (v_1,v_2,v_3,v_4)$, and $w = (w_1,w_2,w_3,w_4)$.

A vector u in IE_1^4 is called as timelike, null, or spacelike with respect to satisfying $g(u,u) < 0$, $g(u,u) = 0$, or $g(u,u) > 0$, respectively. Also, the norm of this vector is presented by

$$\|u\| = \sqrt{|g(u,u)|}. \tag{9}$$

In Minkowski space-time, a hypersurface

$$\psi(u,v,w) = (\psi_1(u,v,w), \psi_2(u,v,w), \psi_3(u,v,w), \psi_4(u,v,w)), \tag{10}$$

is named as timelike (spacelike), based on its unit normal vector (or Gauss map) being spacelike (timelike), and the normal vector field is calculated by

$$N = \frac{\psi_u\times\psi_v\times\psi_w}{\|\psi_u\times\psi_v\times\psi_w\|}. \tag{11}$$

The matrix that corresponds to the first fundamental form is [10]

$$I = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \tag{12}$$

where the coefficients are

$$\begin{aligned} &E = g(\psi_u,\psi_u), F = g(\psi_u,\psi_v), G = g(\psi_v,\psi_v), \\ &A = g(\psi_u,\psi_w), B = g(\psi_v,\psi_w), C = g(\psi_w,\psi_w). \end{aligned} \tag{13}$$

For a timelike hypersurface, the coefficient E, G, or C is negative definite.

In four-dimensional Minkowski space, with the help of spacelike, timelike, and lightlike axis spanned by (0, 0, 0, 1), (1, 0, 0, 0), and (1, 1, 0, 0), the three rotation matrices are given by [11]

$$\begin{aligned} R_1 &= \begin{pmatrix} \cosh v\cosh w & \sinh v\cosh w & \sinh w & 0 \\ \sinh v & \cosh v & 0 & 0 \\ \cosh v\sinh w & \sinh v\sinh w & \cosh w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ R_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos w & -\sin v\sin w & -\cos v\sin w \\ 0 & 0 & \cos v & -\sin v \\ 0 & \sin w & \sin v\cos w & \cos v\cos w \end{pmatrix}, \\ R_3 &= \begin{pmatrix} \dfrac{v^2+w^2}{2}+1 & -\dfrac{v^2+w^2}{2} & v & w \\ \dfrac{v^2+w^2}{2} & 1-\dfrac{v^2+w^2}{2} & v & w \\ v & -v & 1 & 0 \\ w & -w & 0 & 1 \end{pmatrix}. \end{aligned} \tag{14}$$

3. Hypersurfaces of Constant Ratio in Four-Dimensional Minkowski Space

Definition 1. Let S: $\psi(u,v,w)$: $(u,v,w)\in D(D\subset E^3)$ be a hypersurface in Minkowski space-time. In case of the norm of $\operatorname{grad}(\rho)$ being positive real constant, S is said to be constant ratio surface:

$$\|\operatorname{grad}(\rho)\| = k, \ldots k\in \mathrm{IR}. \tag{15}$$

As it can be understood from the definition, satisfying the condition $\|\operatorname{grad}(\rho)\| = k$ means that

$$\|\psi^T\| = k\|\psi\|. \tag{16}$$

By the use of (16) and the inequality $\|\psi^T\|\leq\|\psi\|$, we can say $k\leq 1$.

Let $\{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$ be the orthonormal frame in IE_1^4. σ_1 can be considered as parallel to ψ^T. Therefore, the following relations can be written:

$$\begin{aligned} \psi &= \psi^T + \psi^N, \\ \psi^T &= a\sigma_1, \\ \psi^N &= b\sigma_4, \end{aligned} \tag{17}$$

where a and b are differentiable functions.

In case the hypersurface is of constant ratio, we get

$$\left\|\psi^T\right\| = k\|\psi\|. \tag{18}$$

Therefore,

$$\begin{aligned} a &= k\|\psi\|, \\ b &= \sqrt{1-k^2}\|\psi\|. \end{aligned} \tag{19}$$

3.1. Timelike Revolution Hypersurfaces of Type I

Definition 2. Let r be a smooth function and $C\colon I \subset \mathrm{IR} \to \pi$ be a curve on a plane parameterized by $C(u) = (u, 0, 0, r(u))$ in IE_1^4. The surface S formed by the rotation of the curve C around the spacelike axis $(0, 0, 0, 1)$ is called as revolution hypersurface of type I. Therefore, with the help of the matrix R_1, the parameterization of S is given by

$$\psi(u, v, w) = (u\cosh v\cosh w, u\sinh v, u\cosh v\sinh w, r(u)). \tag{20}$$

The tangent vector fields are

$$\psi_u = \left(\cosh v\cosh w, \sinh v, \cosh v\sinh w, r'(u)\right), \tag{21}$$

$$\psi_v = (u\sinh v\cosh w, u\cosh v, u\sinh v\sinh w, 0), \tag{22}$$

$$\psi_w = (u\cosh v\sinh w, 0, u\cosh v\cosh w, 0). \tag{23}$$

Using vector product (11), the unit normal vector is calculated by

$$N = -\frac{1}{\sqrt{1-(r')^2}}\left(r'\cosh v\cosh w, r'\sinh v, r'\cosh v\sinh w, 1\right), \tag{24}$$

where $r = r(u)$. Since we suppose the surface is timelike, $1 - (r')^2 > 0$. Let the first unit tangent vector σ_1 be parallel to ψ^T and timelike ($\langle\sigma_1, \sigma_1\rangle$ is negative definite). Then, by the use of (9) and (21), we write

$$\sigma_1 = \frac{1}{\sqrt{1-(r')^2}}\left(\cosh v\cosh w, \sinh v, \cosh v\sinh w, r'(u)\right), \tag{25}$$

and denote $\sigma_4 = N$. With the help of the relation

$$\psi = a\sigma_1 + b\sigma_4, \tag{26}$$

we get

$$\begin{aligned} u &= \frac{a - \mathrm{br}'}{\sqrt{1-(r')^2}}, \\ r &= \frac{\mathrm{ar}' - b}{\sqrt{1-(r')^2}}. \end{aligned} \tag{27}$$

Thus, the functions a and b are

$$\begin{aligned} a &= \frac{u - r'(u)r(u)}{\sqrt{1-(r'(u))^2}}, \\ b &= \frac{r'(u)u - r(u)}{\sqrt{1-(r'(u))^2}}. \end{aligned} \tag{28}$$

3.2. Timelike Revolution Hypersurfaces of Type II

Definition 3. Let r be a smooth function and $C\colon I \subset \mathrm{IR} \to \pi$ be a curve on a plane parameterized by $C(u) = (r(u), 0, 0, u)$ in IE_1^4. The surface S formed by the rotation of the curve C around the timelike axis $(1, 0, 0, 0)$ is called as revolution hypersurface of type II. Therefore, with the help of the matrix R_2, the parameterization of S is given by

$$\psi(u, v, w) = (r(u), -u\cos v\sin w, -u\sin v, u\cos v\cos w). \tag{29}$$

The tangent vector fields are

$$\begin{aligned} \psi_u &= \left(r'(u), -\cos v\sin w, -\sin v, \cos v\cos w\right), \\ \psi_v &= (0, u\sin v\sin w, -u\cos v, -u\sin v\cos w), \\ \psi_w &= (0, -u\cos v\cos w, 0, -u\cos v\sin w). \end{aligned} \tag{30}$$

Using vector product (14), the unit normal vector is calculated by

$$N = \frac{1}{\sqrt{(r')^2 - 1}}\left(1, -r'\cos v\sin w, -r'\sin v, r'\cos v\cos w\right), \tag{31}$$

where $r = r(u)$. Since we suppose the hypersurface is timelike, the unit normal vector field is spacelike ($(r')^2 - 1 > 0$). Let the first unit tangent vector σ_1 be parallel to ψ^T and timelike. Using ψ_u and (9), we write

$$\sigma_1 = \frac{1}{\sqrt{(r')^2 - 1}}\left(r'(u), -\cos v\sin w, -\sin v, \cos v\cos w\right), \tag{32}$$

and denote $\sigma_4 = N$. With the help of the relation

$$\psi = a\sigma_1 + b\sigma_4, \tag{33}$$

we get

$$r(u) = \frac{ar' + b}{\sqrt{(r')^2 - 1}},$$
$$u = \frac{a + br'}{\sqrt{(r')^2 - 1}}. \tag{34}$$

Thus, the functions a and b are

$$a = \frac{r'(u)r(u) - u}{\sqrt{(r'(u))^2 - 1}},$$
$$b = \frac{r'(u)u - r(u)}{\sqrt{(r'(u))^2 - 1}}. \tag{35}$$

3.3. Timelike Revolution Hypersurfaces of Type III

Definition 4. Let r be a smooth function and $C: I \subset \mathrm{IR} \to \pi$ be a curve on a plane parameterized by $C(u) = (u, r(u), 0, 0)$ in IE_1^4. The surface S formed by the rotation of the curve C around the lightlike axis $(1, 1, 0, 0)$ is called as revolution hypersurface of type III. Therefore, with the help of the matrix R_3, the parameterization of S is given by

$$\psi(u, v, w) = \left(\left(\frac{v^2 + w^2}{2} + 1\right)u - \frac{v^2 + w^2}{2} r(u), \frac{v^2 + w^2}{2} u + \left(1 - \frac{v^2 + w^2}{2}\right) r(u), \mathrm{uv} - r(u)v, \mathrm{uw} - r(u)w\right), \tag{36}$$

where $u \in R - \{0\}$.

This parameterization can be written as

$$\psi(u, v, w) = \left(\left(\frac{v^2 + w^2}{2}\right)(u - r(u)) + u, \left(\frac{v^2 + w^2}{2}\right)(u - r(u)) + r(u), v(u - r(u)), w(u - r(u))\right). \tag{37}$$

The tangent vector fields are

$$\psi_u = \left(\frac{v^2 + w^2}{2}(1 - r'(u)) + 1, \frac{v^2 + w^2}{2}(1 - r'(u)) + r'(u), v(1 - r'(u)), w(1 - r'(u))\right),$$
$$\psi_v = (u - r(u))(v, v, 1, 0),$$
$$\psi_w = (u - r(u))(w, w, 0, 1). \tag{38}$$

Using vector product (11), the unit normal vector is calculated by

$$N = \frac{1}{2\sqrt{1 - (r')^2}}\left(v^2 + w^2 - (v^2 + w^2 + 2)r', v^2 + w^2 - 2 - (v^2 + w^2)r', 2v(1 - r'), 2w(1 - r')\right), \tag{39}$$

where $r = r(u)$. Since we suppose the surface is timelike, $1 - (r')^2 > 0$. Let the first unit tangent vector σ_1 be parallel to ψ^T and timelike. Then, by the use of ψ_u and (9), we note

$$\sigma_1 = \frac{1}{\sqrt{1 - (r')^2}}\left(\frac{v^2 + w^2}{2}(1 - r'(u)) + 1, \frac{v^2 + w^2}{2}(1 - r'(u)) + r'(u), v(1 - r'(u)), w(1 - r'(u))\right), \tag{40}$$

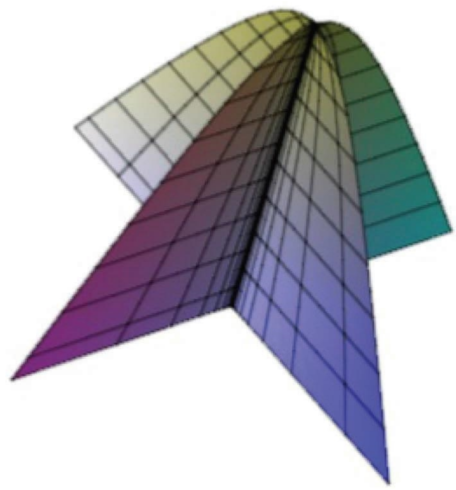

Figure 1: Projection of constant ratio timelike hypersurface.

and denote $\sigma_4 = N$. With the help of the relation

$$\psi = a\sigma_1 + b\sigma_4, \tag{41}$$

we get the functions a and b as

$$a = \frac{u - r'(u)r(u)}{\sqrt{1-(r'(u))^2}},$$
$$b = \frac{r'(u)u - r(u)}{\sqrt{1-(r'(u))^2}}. \tag{42}$$

3.4. Results for Timelike Revolution Hypersurfaces of Constant Ratio in $\mathbb{E}_1^4$

Theorem 5. *Let S be a hypersurface of revolution given by (20), (29), or (37). Then, S corresponds to a constant ratio surface satisfying $\|grad(d)\| = 0$ if and only if the differentiable function $r(u)$ is presented by*

$$r(u) = \pm\sqrt{u^2 + c_1}, \tag{43}$$

where $c_1 \in (0, \infty)$ for (20) and (37), $c_1 \in (-\infty, 0)$ for (29).

Proof. Let S: $\psi(u, v, w)$ be a constant ratio hypersurface of revolution with $\|\mathrm{grad}(d)\| = 0$. Using (6) and (19), $k = a = 0$. In this case, $\|\psi\|$ must be constant (see [1]). Thus, one can write

$$b = \sqrt{1-k^2}\|\psi\| = \|\psi\| = \sqrt{|r^2(u) - u^2|} = \text{const}. \tag{44}$$

By the use of (28) or (35) or (42), we obtain the differential equations

$$u - r'(u)r(u) = 0,$$
$$\frac{r'(u)u - r(u)}{\sqrt{1-(r'(u))^2}} = \text{const}, \tag{45}$$

which have the solution

$$r(u) = \pm\sqrt{u^2 + c_1}. \tag{46}$$

This completes the proof. □

Theorem 6. *Let S be a hypersurface of revolution given by (20), (29), or (37). Then, S corresponds to a constant ratio surface satisfying $\|grad(d)\| = 1$ if and only if the differentiable function $r(u)$ is presented by*

$$r(u) = c_1 u, \tag{47}$$

where c_1 is a real constant.

Proof. Let S: $\psi(u, v, w)$ be a constant ratio hypersurface of revolution with $\|\mathrm{grad}(d)\| = 1$. Using (6) and (19), $k = 1$ and

$$b = 0, a = \|\psi\| = \sqrt{|r^2(u) - u^2|}. \tag{48}$$

By the use of (28) or (35) or (42), we obtain the differential equations

$$\frac{u - r'(u)r(u)}{\sqrt{1-(r'(u))^2}} = \sqrt{|r^2(u) - u^2|},$$
$$r'(u)u - r(u) = 0, \tag{49}$$

which have the solution

$$r(u) = c_1 u, \tag{50}$$

where $c_1 \in R$. This completes the proof. □

Theorem 7. *Let S be a hypersurface of revolution given by (20), (29), or (37). Then, S corresponds to a constant ratio surface satisfying $\|grad(d)\| = k$, $0 < k < 1$, if and only if*

$$u=\int \frac{-k^2+\sqrt{-r^2(u)+1}}{r(u)\left(\sqrt{-r^2(u)+1}\,k^2+r^2(u)-1\right)\sqrt{-r^2(u)+1}}\mathrm{d}r(u), \tag{51}$$

holds.

Proof. Let S: $\psi(u,v,w)$ be a constant ratio hypersurface of revolution with $\|\mathrm{grad}(d)\|=\lambda$, $0<\lambda<1$. Then, for the length of the position vector of S,

$$\|\psi\|=k.u, \tag{52}$$

is satisfied (see, [1, 2]). With the help of this equation and (16), we get

$$\frac{\|\psi^T\|}{k.u}=k, \tag{53}$$

$$a=\|\psi^T\|=k^2u. \tag{54}$$

Combining (28) or (35) or (42) with (43) and (54), we get

$$\frac{u-r'(u)r(u)}{\sqrt{1-(r'(u))^2}}=k^2u, \tag{55}$$

which indicates (51). This completes the proof. □

Example 8. Taking $w=\pi$ and $r(u)=\sqrt{u^2+4}$ in the parameterization (20), we can plot the projection of the hypersurface of constant ratio shown in Figure 1 by using Maple command:

$$\mathrm{plot3}d\left([u*\cosh v*(\cosh(Pi)+\sinh(Pi)),u*\sinh v,\mathrm{sqrt}(u942+4)],u=-2*\mathrm{Pi..}2*\mathrm{Pi},v=-2*\mathrm{Pi..}2*\mathrm{Pi}\right). \tag{56}$$

4. Conclusion

Constant ratio submanifolds are among the significant classifications in differential geometry. In this work, constant ratio hypersurfaces in Minkowski space-time are discussed on the parameterizations of revolution hypersurfaces according to three rotations. Some different characterizations of these types of hypersurfaces can be investigated in future studies.

Data Availability

No data were used to support the findings of this article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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