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Research Article

Existence and Nonexistence of Positive Solutions for a Weighted Quasilinear Elliptic System

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This paper deals with the existence and nonexistence of solutions for the following weighted quasilinear elliptic system, $(\mathcal{S}^{q_1,q_2}_{\mu_1,\mu_2})\begin{cases} -\text{div}(q_1(x)|\nabla u|^{p-2}\nabla u) = \mu_1|u|^{p-2}u + (\alpha+1)|u|^{\alpha-1}u|v|^{\beta+1} & \text{in } \Omega \\ -\text{div}(q_2(x)|\nabla v|^{p-2}\nabla v) = \mu_2|v|^{p-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega \\ u>0, \quad v>0 & \text{in } \Omega \\ u=v=0 & \text{on } \partial\Omega \end{cases}$, where $\Omega \subset \mathfrak{R}^N\,(N \geq 3), 2 \leq p < N$, q_1, $q_2 \in W^{1,p}(\Omega)\cap C(\overline{\Omega})$, $\alpha, \beta \geq 0$, $\mu_1, \mu_2, \geq 0$ and $\alpha, \beta > 0$ satisfy $\alpha+\beta = p^* - 2$ with $p^* = pN/N-p$ is the critical Sobolev exponent. By means of variational methods we prove the existence of positive solutions which depends on the behavior of the weights q_1, q_2 near their minima and the dimension N. Moreover, we use the well known Pohozaev identity for prove the nonexistence result.

1. Introduction and Main Results

Let Ω be a bounded smooth domain in $\mathbb{R}^N\,(N \geq 3)$ and consider the following weighted quasilinear elliptic system:

$$\left(\mathcal{S}^{q_1,q_2}_{\mu_1,\mu_2}\right)\begin{cases} -\text{div}\left(q_1(x)|\nabla u|^{p-2}\nabla u\right) = \mu_1|u|^{p-2}u + (\alpha+1)|u|^{\alpha-1}u|v|^{\beta+1}, & \text{in } \Omega, \\ -\text{div}\left(q_2(x)|\nabla v|^{p-2}\nabla v\right) = \mu_2|v|^{p-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v, & \text{in } \Omega, \\ u>0, \quad v>0, & \text{in } \Omega, \\ u=v=0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $2 \leq p < N$, and q_1 and q_2 are given positive weights defined on $\overline{\Omega}$ such that q_1 and $q_2 \in W^{1,p}(\Omega)\cap C(\overline{\Omega})$, $p^* = pN/N-p$ is the critical Sobolev exponent of the noncompact embedding $W^{1,p}_0(\Omega)$ into $L^{p^*}(\Omega)$, $\alpha, \beta \geq 0$ satisfy $\alpha+\beta = p^*-2$, and the parameters $\mu_1$and μ_2 satisfy the following assumption:

$$(\mathscr{H}_0)0<\mu_1\le\mu_2<\lambda_1, \tag{2}$$

where $\lambda_1=\min\{\lambda_1^{q_i},1\le i\le 2\}$ and $\lambda_1^{q_i}$ denotes the first eigenvalue of $-\mathrm{div}(q_i(x)\nabla.)$ in $W_0^{1,p}(\Omega)$.

Note that $|u|^{\alpha-1}u|v|^{\beta+1}$ and $|u|^{\alpha+1}|v|^{\beta-1}v$ are called strongly-coupled terms and $|u|^{p-2}u$, $|v|^{p-2}v$ are called weakly coupled terms.

The problem $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ is important in many fields of sciences; it arises in biological applications (e.g., population dynamics) or physical applications (e.g., models of a nuclear reactor) and have drawn a lot of attention (see [1–4] and references therein).

Our system is posed in the framework of the Sobolev space $E=W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$, endowed with the following norm:

$$\|(u,v)\|:=\left(\int_\Omega q_1(x)|\nabla u(x)|^p\mathrm{d}x+\int_\Omega q_2(x)|\nabla v(x)|^p\mathrm{d}x\right)^{1/p}. \tag{3}$$

The energy functional of $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ is defined on E by the following equation:

$$J(u,v):=\frac{1}{p}\|(u,v)\|^p-\frac{1}{p}\int_\Omega(\mu_1|u|^p+\mu_2|v|^p)\mathrm{d}x-\int_\Omega|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x. \tag{4}$$

It is clear that $J\in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle J'(u,v),(\varphi,\phi)\rangle:=&\int_\Omega\left(q_1(x)|\nabla u|^{p-2}\nabla u\nabla\varphi+q_2(x)|\nabla v|^{p-2}\nabla v\nabla\phi\right)\mathrm{d}x+\\&-\mu_1\int_\Omega|u|^{p-2}u\varphi\mathrm{d}x-\mu_2\int_\Omega|v|^{p-2}v\phi\,dx-(\alpha+1)\int_\Omega|u|^{\alpha-1}u|v|^{\beta+1}\varphi\mathrm{d}x+\\&-(\beta+1)\int_\Omega|u|^{\alpha+1}|v|^{\beta-1}v\phi\mathrm{d}x=0,\end{aligned} \tag{5}$$

where $(u,v),(\varphi,\phi)\in E$, and $J'(u,v)$ denote the Fréchet derivative of J at (u,v).

A pair of functions $(u,v)\in E$ is said to be a weak solution of $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ if $u>0$ and $v>0$ on Ω satisfy $\langle J'(u,v),(\varphi,\phi)\rangle=0$ for all $(\varphi,\phi)\in E$. Therefore, the weak solutions of $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ are the critical points of J.

Before stating our main results, let us recall a brief history.

For the scalar case, that is, when $\alpha=\beta$, $\alpha+\beta=p^*-2$, $\mu_1=\mu_2=\mu$, $q_1=q_2=q$, and $u=v$, then the system $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ reduces to the single elliptic equation as follows:

$$(\mathscr{P}^q_\mu)\begin{cases}-\mathrm{div}\left(q(x)|\nabla u|^{p-2}\nabla u\right)=\mu|u|^{p-2}u+|u|^{p^*-2}u, & \text{in }\Omega,\\ u>0, & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega.\end{cases} \tag{6}$$

In the special case $q\equiv 1$, much interest has grown on this critical problem, starting from the celebrated paper by Brézis and Nirenberg [5] for the semilinear equation ($p=2$). They established existing results in dimension $N=3$ when Ω is a ball, namely, they ensured the existence of a positive constant λ_0 such that problem $(\mathscr{P}^1_\mu)$ admits a positive solution for all $\mu\in]\lambda_0,\lambda_1^1[$, where λ_1^1 is the first eigenvalue of the operator $-\Delta$. In higher dimensions, $N\ge 4$, they proved the existence of a positive solution for all $\mu\in]0,\lambda_1^1[$ and no positive solution for $\mu>\lambda_1^1$ or $\mu\le 0$ and Ω is a star-shaped domain. After that, many authors generalized the results of reference [5] for the quasilinear case, for example, see [6–9] and the references therein.

In the case where q is not constant, Hadiji and Yazidi [10] extended the results of [5] to the weighted problem $(\mathscr{P}^q_\mu)$ with $p=2$ and $q\in H^1(\Omega)\cap C(\overline{\Omega})$ such that

$$q(x)=q(x_0)+a_k|x-x_0|^k+|x-x_0|^k\theta(x), \tag{7}$$

where $x_0\in\Omega$, $q(x_0)=\min\{q(x),\quad x\in\overline{\Omega}\}$, $k>0$, $a_k>0$, and $\theta(x)\longrightarrow 0$ when $x\longrightarrow x_0$. They showed that the existence of solutions depends not only on parameter λ but also on the behavior of q near its minima. More recently, Benhamida and Yazidi [11] have generalized the results of reference [10] for the quasilinear case $(2\le p<N)$.

Concerning the vectorial case and without weights, a lot of papers have appeared in recent years dealing with system involving Laplacian or p-Laplacian operator, see for instance [1, 12–15] and the references therein. On the other hand, it should be mentioned that when $p=2$, Bouchekif and Hamzaoui [13] studied the following weighted system:

$$\left(\mathscr{P}^{q_1,q_2}_{a,b,c}\right)\begin{cases} -\mathrm{div}\left(q_1(x)\nabla u\right) = au + bv + (\alpha+1)|u|^{\alpha-1}u|v|^{\beta+1} & \text{in } \Omega, \\ -\mathrm{div}\left(q_2(x)\nabla v\right) = bu + cv + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where a, b, c are real parameters and $2^* = 2N/N-2$ denotes the critical Sobolev's exponent of the embedding $H^1_0(\Omega)$ into $L^{2^*}(\Omega)$. They proved the existence of at least one positive solution under suitable assumptions on the data.

A natural interesting question is whether the results concerning the solutions of $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ with $p=2$ in [13] remain true for $2 \le p < N$. By using [11, 13], we gave some positive answers. To the best of our knowledge, the results are new in the case when $p \ne 2$. Note that this quasilinear problem creates many difficulties in applying variational methods in the fact that $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ contains the critical exponent p^* and weights q_1 and q_2, then the functional J does not satisfy the Palais–Smale condition in all the range. To overcome the lack of compactness, we need to determine a good level of the Palais–Smale condition. On the other hand, it is very difficult to prove that the critical value is contained in the range of this level, so we need more delicate estimates where q_1 and q_2 play an essential role.

Now, we introduced some notations and hypotheses.

We assume the existence of x_0 in Ω such that, in a neighborhood of x_0, the weighted q_1 and q_2 behave similar to

$$q_1(x) = q_1(x_0) + a_{k_1}|x-x_0|^{k_1} + |x-x_0|^{k_1}\theta_{q_1}(x), \tag{9}$$

and

$$q_2(x) = q_2(x_0) + \tilde{a}_{k_2}|x-x_0|^{k_2} + |x-x_0|^{k_2}\theta_{q_2}(x), \tag{10}$$

where $x_0 \in \overline{\Omega}$, $q_i(x_0) = \min_{x\in\overline{\Omega}} q_i(x)$, $1 \le i \le 2$, k_1, k_2, a_{k_1}, and $\tilde{a}_{k_2}$ are positive constants and $\theta_{q_i}(x)$ tends to 0 as x goes to x_0.

The parameters k_1 and k_2 will play an essential role in the study of our system. In fact, if $N \ge p^2$, the case $k_1 > p$ and $k_2 > p$ is treated by a classical procedure. For the other cases, we restricted ourself to the case where q_1 and q_2 satisfy the following additional conditions:

$$k_1 a_{k_1} \le \frac{\tilde{q}_1(x)}{|x-x_0|^{k_1}}, \quad \text{a.e. } x \in \Omega, \tag{11}$$

and

$$k_2 \tilde{a}_{k_2} \le \frac{\tilde{q}_2(x)}{|x-x_0|^{k_2}}, \quad \text{a.e. } x \in \Omega, \text{ respectively} \tag{12}$$

where

$$\begin{aligned} \tilde{q}_1(x) &:= \nabla q_1(x).(x-x_0) \text{ and} \\ \tilde{q}_2(x) &:= \nabla q_2(x).(x-x_0). \end{aligned} \tag{13}$$

Let

$$\gamma(q_1,q_2) := \inf_{(u,v)\in E\setminus\{(0,0)\}} I(u,v), \tag{14}$$

where $I(u,v) := (1/p)((\int_\Omega(\tilde{q}_1(x)|\nabla u|^p + \tilde{q}_2(x)|\nabla v|^p)dx)/(\int_\Omega(|u|^p+|v|^p)dx))$.

Now, we are in a position to state the results of our paper.

For the nonexistence results, we have the following theorem.

Theorem 1. *Assume that $\mu_2 \le \gamma(q_1,q_2)$ and Ω is a star-shaped domain with respect to x_0. Then, $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ has no nontrivial solution.*

For the existence results, we have the following theorems.

Theorem 2. *Suppose that $N = p^2$, $(\mathscr{H}_0)$ holds and q_1 and q_2 satisfy equations (9) and (10), respectively. Then, there exist constants $\nu_1, \nu_2 > 0$ such that $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ has a positive solution, under one of the following hypotheses:*

$(\mathscr{H}_1) k_1 > N-p/p-1, k_2 > N-p/p-1$, *and* $\mu_1 > 0$.
$(\mathscr{H}_2) k_1 = N-p/p-1, k_2 > p$, *and* $\mu_1 > \nu_1$.
$(\mathscr{H}_3) k_1 > p, k_2 = N-p/p-1$, *and* $\mu_1 > \nu_2$.
$(\mathscr{H}_4) k_1 = k_2 = N-p/p-1$, *and* $\mu_1 > \nu_1+\nu_2$.

Theorem 3. *Suppose that $N \ne p^2$, $(\mathscr{H}_0)$ holds and q_1 and q_2 satisfy equations (11) and (12), respectively. Then, there exist constants $\nu_3, \nu_4, \nu_5 > 0$ such that $(\mathscr{S}^{q_1,q_2}_{\mu_1,\mu_2})$ has a positive solution, under one of the following hypotheses:*

$(\mathscr{H}_5) N > p^2, k_1 > p, k_2 > p$, *and* $\mu_1 > 0$.
$(\mathscr{H}_6) N > p^2, k_1 > p, k_2 > p$, *and* $\mu_1 > \nu_3$.
$(\mathscr{H}_7) N > p^2, k_1 = p, k_2 = p$, *and* $\mu_1 > \nu_4$.
$(\mathscr{H}_8) N > p^2, k_1 = p, k_2 = p$, *and* $\mu_1 > \nu_3+\nu_4$.
$(\mathscr{H}_9) p < N < p^2, k_1 > N-p/p-1, k_2 > N-p/p-1$, *and* $\mu_1 > \nu_5$.

This paper is organized as follows: In Section 2, we collected some preliminaries results that will be used throughout the work. In Section 3, we proved Theorem 1 (nonexistence result). In Section 4, we proved Theorems 2 and 3 (existence results) by using the mountain pass theorem.

2. Some Preliminary Results

Throughout this paper, we shall denote by C and C_i, $(i = 0, 1, 2, \ldots)$ for the various positive constants. The diameter of Ω will be denoted by diam (Ω), we use $\longrightarrow$ and ¶ to denote the strong and weak convergence in the related function spaces, respectively, and $u^+ = \max\{u, 0\}$ and B, (x, r) represents the ball of radius r centered at x.

We define the following equation:

$$S_{\mu}^{q} := \inf_{u \in W_0^{1,p}(\Omega)\backslash\{0\}} Q_{\mu}^{q}(u) \text{ and } S_{\mu_1,\mu_2}^{q_1,q_2} := \inf_{(u,v) \in E\backslash\{(0,0)\}} Q_{\mu_1,\mu_2}^{q_1,q_2}(u,v), \tag{15}$$

where

$$Q_{\mu}^{q}(u) = \frac{\int_{\Omega} q(x)|\nabla u|^{p}\mathrm{d}x - \mu\int_{\Omega}|u|^{p}\mathrm{d}x}{\left(\int_{\Omega}|u|^{p^*}\mathrm{d}x\right)^{p/p^*}},$$
$$Q_{\mu_1,\mu_2}^{q_1,q_2}(u,v) = \frac{\int_{\Omega}\left(q_1(x)|\nabla u|^{p} + q_2(x)|\nabla v|^{p}\right)\mathrm{d}x - \int_{\Omega}\left(\mu_1|u|^{p} + \mu_2|v|^{p}\right)\mathrm{d}x}{\left(\int_{\Omega}|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x\right)^{p/p^*}}. \tag{16}$$

First, we recall the following Hardy's inequality, see for example [16].

Lemma 1. *Let $t \in \mathbb{R}$ such that $N + t > 0$, we have the following equation:*

$$\int_{\Omega}|x|^{t}|u|^{p}\mathrm{d}x \leq \left(\frac{p}{N+t}\right)^{p}\int_{\Omega}|x|^{t}|x.\nabla u|^{p}\mathrm{d}x, \text{ for all } u \in W_0^{1,p}(\Omega). \tag{17}$$

Moreover, the constant $(p/N+t)^{p}$ is optimal and not achieved.

We note that direct calculations imply that if $t = 0$, Lemma 1 applies even if we replace x with $x - x_0$.

Lemma 2

(1) Assume that q_1 and $q_2 \in C^1(\Omega)$, and there exists $b \in \Omega$ such that $\widetilde{q}_1(b) + \widetilde{q}_2(b) < 0$, then $\gamma(q_1,q_2) = -\infty$.

(2) Suppose that q_1 and $q_2 \in W^{1,p}(\Omega)\cap C(\overline{\Omega})$ satisfy equations (9) and (10), respectively, and $\widetilde{q}_1(x) \geq 0$ and $\widetilde{q}_2(x) \geq 0$ a.e. $x \in \Omega$, we have

(i) If $k_1 > p$ and $k_2 > p$ and q_1 and $q_2 \in C^1(\Omega)$, then $\gamma(q_1,q_2) = 0$.

(ii) If $k_1 = p$ and $k_2 > p$ or $k_1 > p$ and $k_2 = p$, then

$$0 \leq \gamma(q_1,q_2) \leq \frac{a_p}{p}\lambda_1^1(\mathrm{diam}(\Omega))^{p} \text{ or } 0 \leq \gamma(q_1,q_2) \leq \frac{\widetilde{a}_p}{p}\lambda_1^1(\mathrm{diam}(\Omega))^{p}, \tag{18}$$

respectively.

(iii) If $0 < k_1 \leq p$, $0 < k_2 \leq p$, q_1, and q_2 satisfy the conditions (11) and (12), respectively, then

$$\gamma(q_1,q_2) \geq \frac{N^{p}}{p^{p+1}}\min\left\{k_1 a_{k_1}(\mathrm{diam}(\Omega))^{k_1-p}, \quad k_2\widetilde{a}_{k_2}(\mathrm{diam}(\Omega))^{k_2-p}\right\}. \tag{19}$$

Remark 1. If $k_1 = k_2 = p$ and $q_1, q_2 \in C^1(\Omega)$, we obtain the following estimate:

$$\left(\frac{N}{p}\right)^{p}\min\left\{a_p, \widetilde{a}_p\right\} \leq \gamma(q_1,q_2) \leq \frac{\lambda_1^1}{p}\left(a_p + \widetilde{a}_p\right)(\mathrm{diam}(\Omega))^{p}. \tag{20}$$

Proof

(1) Let $\varphi \in C_0^{\infty}(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$ on $\mathbb{R}^N$, $\varphi \equiv 1$ on $B(0,r)$ and $\varphi \equiv 0$ on $\mathbb{R}^N/B(0,2r)$, where $0 < r < 1$.

Let $\varphi_j(x) = \varphi(j(x-b))$ for $j \in \mathbb{N}^*$. We have the following equation:

$$\gamma(q_1,q_2)\le \frac{1}{2p}\frac{\int_\Omega (\tilde{q}_1(x)+\tilde{q}_2(x))\left|\nabla\varphi_j(x)\right|^p\mathrm{d}x}{\int_\Omega\left|\varphi_j(x)\right|^p\mathrm{d}x}$$
$$\le\frac{1}{2p}\frac{\int_{B(b,(2r/j))}(\tilde{q}_1(x)+\tilde{q}_2(x))\left|\nabla\varphi_j(x)\right|^p\mathrm{d}x}{\int_{B(b,(2r/j))}\left|\varphi_j(x)\right|^p\mathrm{d}x}. \tag{21}$$

Using the change of variable $y=j(x-b)$, we obtain the following equation:

$$\gamma(q_1,q_2)\le\frac{j^p}{2p}\frac{\int_{B(0,2r)}(\tilde{q}_1(y/j+b)+\tilde{q}_2(y/j+b))|\nabla\varphi(y)|^p\mathrm{d}x}{\int_{B(0,2r)}|\varphi(y)|^p\mathrm{d}x}. \tag{22}$$

Letting $j\longrightarrow\infty$, then by the Dominated Convergence theorem we deduce the desired result.

(2) First, we proof 2.i). Since q_1 and $q_2\in C^1(\Omega)$ in a neighborhood V of x_0, then by equations (9) and (10), we can write the following equations:

$$q_1(x)=q_1(x_0)+a_{k_1}\left|x-x_0\right|^{k_1}+\theta_{q_1}(x), \tag{23}$$

and

$$q_2(x)=q_2(x_0)+\tilde{a}_{k_2}\left|x-x_0\right|^{k_2}+\theta_{q_2}(x), \tag{24}$$

where $\theta_{q_1},\theta_{q_2}\in C^1(V)$ such that

$$\lim_{x\to x_0}\frac{\left|\theta_{q_1}(x)\right|}{\left|x-x_0\right|^{k_1}}=0,$$
$$\lim_{x\to x_0}\frac{\left|\theta_{q_2}(x)\right|}{\left|x-x_0\right|^{k_2}}=0. \tag{25}$$

From equation (25), we obtain the existence of r, $0<r<1$, such that

$$\left|\theta_{q_1}(x)\right|\le\left|x-x_0\right|^{k_1}\text{ and}$$
$$\left|\theta_{q_2}(x)\right|\le\left|x-x_0\right|^{k_2},\text{ for all } x\in B(x_0,2r)\subset V. \tag{26}$$

Defining $\varphi_j(x)=\varphi(j(x-x_0))$, then

$$0\le\gamma(q_1,q_2)\le\frac{1}{2p}\frac{\int_\Omega(\tilde{q}_1(x)+\tilde{q}_2(x))\left|\nabla\varphi_j(x)\right|^p\mathrm{d}x}{\int_\Omega\left|\varphi_j(x)\right|^p\mathrm{d}x}, \tag{27}$$

and from equations (23) and (24), we deduce the following equation:

$$0\le\gamma(q_1,q_2)\le\frac{1}{2p}\frac{\int_{B(x_0,2r/j)}\left(k_1a_{k_1}\left|x-x_0\right|^{k_1}+k_2\tilde{a}_{k_2}\left|x-x_1\right|^{k_2}\right)\left|\nabla\varphi_j(x)\right|^p\mathrm{d}x}{\int_{B(x_0,2r/j)}\left|\varphi_j(x)\right|^p\mathrm{d}x}+$$
$$+\frac{1}{2p}\frac{\int_{B(x_0,2r/j)}\left(\nabla\theta_{q_1}(x).(x-x_0)+\nabla\theta_{q_2}(x).(x-x_0)\right)\left|\nabla\varphi_j(x)\right|^p\mathrm{d}x}{\int_{B(x_0,2r/j)}\left|\varphi_j(x)\right|^p\mathrm{d}x}. \tag{28}$$

Using the change of variable $y=j(x-x_0)$, and integrating by parts, we obtain the following equation:

$$0\le\gamma(q_1,q_2)\le\frac{k_1a_{k_1}}{2pj^{k_1-p}}\frac{\int_{B(0,2r)}|y|^{k_1}|\nabla\varphi(y)|^p\mathrm{d}y}{\int_{B(0,2r)}|\varphi(y)|^p\mathrm{d}y}-\frac{j^{p-1}}{2p}\frac{\int_{B(0,2r)}\theta_{q_1}(y/j+x_0)\mathrm{div}(y|\nabla\varphi(y)|^p)\mathrm{d}y}{\int_{B(0,2r)}|\varphi(y)|^p\mathrm{d}y}+$$
$$+\frac{k_2\tilde{a}_{k_2}}{2pj^{k_2-p}}\frac{\int_{B(0,2r)}|y|^{k_2}|\nabla\varphi(y)|^p\mathrm{d}y}{\int_{B(0,2r)}|\varphi(y)|^p\mathrm{d}y}-\frac{j^{p-1}}{2p}\frac{\int_{B(0,2r)}\theta_{q_2}(y/j+x_0)\mathrm{div}(y|\nabla\varphi(y)|^p)\mathrm{d}y}{\int_{B(0,2r)}|\varphi(y)|^p\mathrm{d}y}. \tag{29}$$

Hence, by using equation (26), we obtain the following equation:

$$0 \le \gamma(q_1, q_2) \le \frac{k_1 a_{k_1}}{2pj^{k_1-p}} \frac{\int_{B(0,2r)} |y|^{k_1} |\nabla\varphi(y)|^p \mathrm{d}y}{\int_{B(0,2r)} |\varphi(y)|^p \mathrm{d}y} + \frac{C}{2pj^{k_1-p+1}} \frac{\int_{B(0,2r)} |y|^{k_1} \mathrm{d}y}{\int_{B(0,2r)} |\varphi(y)|^p \mathrm{d}y} + \frac{k_2 \tilde{a}_{k_2}}{2pj^{k_2-p}} \frac{\int_{B(0,2r)} |y|^{k_2} |\nabla\varphi(y)|^p \mathrm{d}y}{\int_{B(0,2r)} |\varphi(y)|^p \mathrm{d}y} + \frac{C}{2pj^{k_2-p+1}} \frac{\int_{B(0,2r)} |y|^{k_2} \mathrm{d}y}{\int_{B(0,2r)} |\varphi(y)|^p \mathrm{d}y}, \tag{30}$$

where $C = \max_{y \in B}(0,2r) |\mathrm{div}(y|\nabla\varphi(y)|^p)|$.

Therefore, for $k_1 > p$ and $k_2 > p$, we reach that $\gamma(q_1, q_2) = 0$. This concludes the proof of 2.*i*.

To prove 2.*ii*, first we start by the case $k_1 = p$ and $k_2 > p$.

Let $\xi_j(x) = \varphi_1(j(x - x_0))$ for $j \in \mathbb{N}$ is large enough, where φ_1 is the positive eigenfunction corresponding to the first eigenvalue λ_1^1 of the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$.

We have the following equation:

$$0 \le \gamma(q_1, q_2) \le \frac{1}{2p} \frac{\int_\Omega (\tilde{q}_1(x) + \tilde{q}_2(x)) |\nabla\xi_j(x)|^p \mathrm{d}x}{\int_\Omega |\xi_j(x)|^p \mathrm{d}x}, \tag{31}$$

Using equations (23) and (24), we obtain the following equation:

$$0 \le \gamma(q_1, q_2) \le \frac{1}{2p} \frac{\int_{x_0 - 1/j\Omega} \left(k_1 a_{k_1} |x - x_0|^{k_1} + k_2 \tilde{a}_{k_2} |x - x_0|^{k_2}\right) |\nabla\xi_j(x)|^p \mathrm{d}x}{\int_{x_0 + 1/j\Omega} |\xi_j(x)|^p \mathrm{d}x} + \frac{1}{2p} \frac{\int_{x_0 + 1/j\Omega} \left(\nabla\theta_{q_1}(x).(x - x_0) + \nabla\theta_{q_2}(x).(x - x_0)\right) |\nabla\xi_j(x)|^p \mathrm{d}x}{\int_{x_0 + 1/j\Omega} |\xi_j(x)|^p \mathrm{d}x}. \tag{32}$$

By a simple change of variable $y = j(x - x_0)$ and integrating by parts, we obtain the following equation by equation (28):

$$0 \le \gamma(q_1, q_2) \le \frac{a_p}{2} \frac{\int_\Omega |y|^p |\nabla\varphi_1(y)|^p \mathrm{d}y}{\int_\Omega |\varphi_1(y)|^p \mathrm{d}y} + \frac{C}{2pj} \frac{\int_\Omega |y|^{k_1} \mathrm{d}y}{\int_\Omega |\varphi_1(y)|^p \mathrm{d}y} + \frac{k_2 \tilde{a}_{k_2}}{2pj^{k_2-p}} \frac{\int_\Omega |y|^{k_2} |\nabla\varphi_1(y)|^p \mathrm{d}y}{\int_\Omega |\varphi_1(y)|^p \mathrm{d}y} + \frac{C}{2pj^{k_2-p+1}} \frac{\int_\Omega |y|^{k_2} \mathrm{d}y}{\int_\Omega |\varphi_1(y)|^p \mathrm{d}y}, \tag{33}$$

where $C = \max_{y \in \Omega} |\mathrm{div}(y|\nabla\varphi_1(y)|^p)|$. Letting $j \longrightarrow \infty$, we obtain the following equation:

$$0 \le \gamma(q_1, q_2) \le \frac{a_p}{2} \frac{\int_\Omega |y|^p |\nabla\varphi_1(y)|^p \mathrm{d}y}{\int_\Omega |\varphi_1(y)|^p \mathrm{d}y}, \tag{34}$$

thus

$$0 \le \gamma(q_1, q_2) \le \frac{a_p}{2} \lambda_1^1 (\mathrm{diam}(\Omega))^p. \tag{35}$$

Similarly, we deduce in the case $k_1 > p$ and $k_2 = p$, that

$$0 \le \gamma(q_1, q_2) \le \frac{\tilde{a}_p}{2} \lambda_1^1 (\mathrm{diam}(\Omega))^p. \tag{36}$$

Now, we proof 2.iii). Since q_1 and q_2 satisfy equations (11) and (12), respectively, for all $(u, v) \in E/\{(0,0)\}$ we have the following equation:

$$\begin{aligned} I(u,v) &\geq \frac{k_1}{p}a_{k_1}\frac{\int_\Omega |x-x_0|^{k_1-p}\left(|x-x_0||\nabla u|\right)^p \mathrm{d}x}{\int_\Omega (|u|^p+|v|^p)\mathrm{d}x} + \frac{k_2}{p}\tilde{a}_{k_2}\frac{\int_\Omega |x-x_0|^{k_2-p}\left(|x-x_0||\nabla v|\right)^p \mathrm{d}x}{\int_\Omega (|u|^p+|v|^p)\mathrm{d}x} \\ &\geq \frac{k_1}{p}a_{k_1}(\mathrm{diam}(\Omega))^{k_1-p}\frac{\int_\Omega |(x-x_0).\nabla u|^p \mathrm{d}x}{\int_\Omega (|u|^p+|v|^p)\mathrm{d}x} + \frac{k_2}{p}\tilde{a}_{k_2}(\mathrm{diam}(\Omega))^{k_2-p}\frac{\int_\Omega |(x-x_0).\nabla v|^p \mathrm{d}x}{\int_\Omega (|u|^p+|v|^p)\mathrm{d}x}. \end{aligned} \tag{37}$$

By applying Lemma 1 for $t = 0$, we obtained the following equation:

$$I(u,v) \geq \frac{k_1}{p}a_{k_1}(\mathrm{diam}(\Omega))^{k_1-p}\left(\frac{N}{p}\right)^p \frac{\int_\Omega |u|^p \mathrm{d}x}{\int_\Omega (|u|^p+|v|^p)\mathrm{d}x} + \frac{k_2}{p}\tilde{a}_{k_2}(\mathrm{diam}(\Omega))^{k_2-p}\left(\frac{N}{p}\right)^p \frac{\int_\Omega |v|^p \mathrm{d}x}{\int_\Omega (|u|^p+|v|^p)\mathrm{d}x}. \tag{38}$$

Thus,

$$\gamma(q_1,q_2) \geq \frac{N^p}{p^{p+1}}\min\left\{k_1 a_{k_1}(\mathrm{diam}(\Omega))^{k_1-p}, k_2\tilde{a}_{k_2}(\mathrm{diam}(\Omega))^{k_2-p}\right\}. \tag{39}$$

The proof is complete.

Inspired by [1], we obtain the following result. □

Lemma 3. *We have the following equation:*

$$K(\alpha,\beta)\left(S_0^{q_1}\right)^{\alpha+1/\alpha+\beta+2}\left(S_0^{q_2}\right)^{\beta+1/\alpha+\beta+2} \leq S_{0,0}^{q_1,q_2} \leq K(\alpha,\beta)S_0^h, \tag{40}$$

with

$$\begin{aligned} K(\alpha,\beta) &= \left(\frac{\alpha+1}{\beta+1}\right)^{\beta+1/\alpha+\beta+2} + \left(\frac{\alpha+1}{\beta+1}\right)^{-\alpha+1/\alpha+\beta+2}, \text{ and} \\ h(x) &= q_1(x)+q_2(x). \end{aligned} \tag{41}$$

Proof. Consider a minimizing sequence w_n for S_0^h. Let $s_n, t_n > 0$ be chosen later. Taking $u_n = s_n w_n$ and $v_n = t_n w_n$ in quotient (15), we obtain the following equation:

$$\begin{aligned} S_{0,0}^{q_1,q_2} &\leq \frac{\int_\Omega \left(s_n^p q_1(x) + t_n^p q_2(x)\right)|\nabla w_n|^p \mathrm{d}x}{s_n^{p(\alpha+1)/\alpha+\beta+2} t_n^{p(\beta+1)/\alpha+\beta+2}\left(\int_\Omega |w_n|^{\alpha+\beta+2}\mathrm{d}x\right)^{p/\alpha+\beta+2}} \\ &\leq \frac{s_n^p + t_n^p}{s_n^{p(\alpha+1)/\alpha+\beta+2} t_n^{p(\beta+1)/\alpha+\beta+2}} \frac{\int_\Omega \left(q_1(x)+q_2(x)\right)|\nabla w_n|^p \mathrm{d}x}{\left(\int_\Omega |w_n|^{\alpha+\beta+2}\mathrm{d}x\right)^{p/\alpha+\beta+2}}. \end{aligned} \tag{42}$$

Observe that

$$\frac{s_n^p + t_n^p}{s_n^{p(\alpha+1)/\alpha+\beta+2} t_n^{p(\beta+1)/\alpha+\beta+2}} = \left(\frac{s_n}{t_n}\right)^{p(\beta+1)/\alpha+\beta+2} + \left(\frac{s_n}{t_n}\right)^{-p(\alpha+1)/\alpha+\beta+2}. \tag{43}$$

Let $r = (s_n/t_n)^p$ and define the following function:

$$f(r) = r^{(\beta+1)/\alpha+\beta+2} + r^{-(\alpha+1)/\alpha+\beta+2}, \quad r > 0. \tag{44}$$

The minimum of the function f is achieved at the point $r_0 = \alpha+1/\beta+1$ with minimum value

$$f(r_0) = K(\alpha,\beta). \tag{45}$$

Choosing s_n and t_n in equation (42) such that $s_n^p(\beta+1) = t_n^p(\alpha+1)$, we obtain the following equation:

$$S_{0,0}^{q_1,q_2} \leq K(\alpha,\beta)\frac{\int_\Omega h(x)|\nabla w_n|^p \mathrm{d}x}{\left(\int_\Omega |w_n|^{\alpha+\beta+2}\mathrm{d}x\right)^{p/\alpha+\beta+2}}, \tag{46}$$

hence,

$$S_{0,0}^{q_1,q_2} \leq K(\alpha,\beta)S_0^h. \tag{47}$$

To complete the proof, let (u_n, v_n) be a minimizing sequence for $S_{0,0}^{q_1,q_2}$ and define $z_n = s_n v_n$ with the following equation:

$$(s_n)^{\alpha+\beta+2} = \frac{\int_\Omega |u_n|^{\alpha+\beta+2}\mathrm{d}x}{\int_\Omega |v_n|^{\alpha+\beta+2}\mathrm{d}x}. \tag{48}$$

Then,

$$\int_\Omega |z_n|^{\alpha+\beta+2}\mathrm{d}x = \int_\Omega |u_n|^{\alpha+\beta+2}\mathrm{d}x. \tag{49}$$

By Young's inequality, it follows that

$$\int_\Omega |u_n|^{\alpha+1}|z_n|^{\beta+1}\mathrm{d}x \leq \frac{\alpha+1}{\alpha+\beta+2}\int_\Omega |u_n|^{\alpha+\beta+2}\mathrm{d}x + \frac{\beta+1}{\alpha+\beta+2}\int_\Omega |z_n|^{\alpha+\beta+2}\mathrm{d}x. \tag{50}$$

By equation (49), we have the following equation:

$$\left(\int_\Omega |u_n|^{\alpha+1}|z_n|^{\beta+1}\mathrm{d}x\right)^{p/\alpha+\beta+2} \leq \left(\int_\Omega |u_n|^{\alpha+\beta+2}\mathrm{d}x\right)^{p/\alpha+\beta+1} = \left(\int_\Omega |z_n|^{\alpha+\beta+2}\mathrm{d}x\right)^{p/\alpha+\beta+2}. \tag{51}$$

Consequently,

$$\begin{aligned} I_n &= \frac{\int_\Omega \left(q_1(x)|\nabla u_n|^p + q_2(x)|\nabla v_n|^p\right)\mathrm{d}x}{\left(\int_\Omega |u_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x\right)^{p/\alpha+\beta+2}} \\ &= s_n^{p(\beta+1)/\alpha+\beta+2}\frac{\int_\Omega \left(q_1(x)|\nabla u_n|^p + q_2(x)|\nabla v_n|^p\right)\mathrm{d}x}{\left(\int_\Omega |u_n|^{\alpha+1}|z_n|^{\beta+1}\mathrm{d}x\right)^{p/\alpha+\beta+2}} \\ &\geq s_n^{p(\beta+1)/\alpha+\beta+2}\frac{\int_\Omega q_1(x)|\nabla u_n|^p\mathrm{d}x}{\left(\int_\Omega |u_n|^{\alpha+\beta+2}\mathrm{d}x\right)^{p/\alpha+\beta+2}} + s_n^{p(\beta+1)/\alpha+\beta+2}s_n^{-p}\frac{\int_\Omega q_2(x)|\nabla z_n|^p\mathrm{d}x}{\left(\int_\Omega |z_n|^{\alpha+\beta+2}\mathrm{d}x\right)^{p/\alpha+\beta+2}} \\ &\geq S_0^{q_1}s_n^{p(\beta+1)/\alpha+\beta+2} + S_0^{q_2}s_n^{-p(\alpha+1)/\alpha+\beta+2}, \end{aligned} \tag{52}$$

we know that

$$\min_{t>0}\left(S_0^{q_1}t^{p(\beta+1)/\alpha+\beta+2} + S_0^{q_2}t^{-p(\alpha+1)/\alpha+\beta+2}\right) = K(\alpha,\beta)\left(S_0^{q_1}\right)^{(\alpha+1)/\alpha+\beta+2}\left(S_0^{q_2}\right)^{(\beta+1)/\alpha+\beta+2}, \tag{53}$$

then

$$I_n \geq K(\alpha,\beta)\left(S_0^{q_1}\right)^{(\alpha+1)/\alpha+\beta+2}\left(S_0^{q_2}\right)^{(\beta+1)/\alpha+\beta+2}, \tag{54}$$

Thus,

$$S_{0,0}^{q_1,q_2} \geq K(\alpha,\beta)\left(S_0^{q_1}\right)^{(\alpha+1)/\alpha+\beta+2}\left(S_0^{q_2}\right)^{(\beta+1)/\alpha+\beta+2}. \tag{55}$$

We know that S_0^q is achieved if and only if $\Omega = \mathbb{R}^N$ by the following function:

$$U_\varepsilon(x) = C_\varepsilon\left(\frac{1}{\varepsilon + |x-x_0|^{p/p-1}}\right)^{(N-p)/p}, \tag{56}$$

where C_ε is a normalization constant and ε is a small positive constant; for more details, see [17, 18].

Set $u_\varepsilon(x) = \xi(x)U_\varepsilon(x)$, where $\xi \in C_0^\infty(\overline{\Omega})$ is a fixed function such that $0 \leq \xi \leq 1$ and $\xi \equiv 1$ in some neighborhood of x_0. We have, from [8], that

$$\int_{\Omega}|\nabla u_{\varepsilon}|^{p}\mathrm{d}x = K_1 + o\left(\varepsilon^{N-p/p}\right),$$
$$\left(\int_{\Omega}|u_{\varepsilon}|^{p^*}\mathrm{d}x\right)^{\frac{p}{p^*}} = K_2 + o\left(\varepsilon^{N-p/p}\right), \text{ and}$$
$$\int_{\Omega}|u_{\varepsilon}|^{p}\mathrm{d}x = \begin{cases} K_{3,1}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right), & \text{if } N > p^2, \\ K_{3,2}\varepsilon^{p-1}|\ln(\varepsilon)| + o\left(\varepsilon^{p-1}|\ln(\varepsilon)|\right), & \text{if } N = p^2, \\ K_{3,3}\varepsilon^{N-p/p} + o\left(\varepsilon^{N-p/p}\right), & \text{if } N < p^2, \end{cases} \tag{57}$$

where K_1, K_2, and $K_{3,i}$ are positive constants. □

Lemma 4. *Assume that* $(\mathscr{H}_0)$ *hold and one of the hypotheses* $(\mathscr{H}_j)_{1\le j\le 9}$ *is satisfied. Then,*

$$S^{q_1,q_2}_{\mu_1,\mu_2} < S^{q_1,q_2}_{0,0}. \tag{58}$$

Proof. Let $A, B > 0$ such that

$$\left(\frac{A}{B}\right)^{p} = \frac{(\alpha+1)S_0^{q_2}}{(\beta+1)S_0^{q_1}} = s^p \text{ and}$$
$$\eta \coloneqq \eta(\alpha,\beta,q_1,q_2) = s^{p(\beta+1/p^*)} + s^{-p\alpha+1/p^*}. \tag{59}$$

Then, by $(\mathscr{H}_0)$ and equations (15) and (59), we have the following equation:

$$\begin{aligned} Q^{q_1,q_2}_{\mu_1,\mu_2}(Au_{\varepsilon}, Bu_{\varepsilon}) &\le \frac{\int_{\Omega}(q_1(x)A^p + q_2(x)B^p)|\nabla u_{\varepsilon}|^p\mathrm{d}x - \mu_1\int_{\Omega}(A^p+B^p)|u_{\varepsilon}|^p\mathrm{d}x}{\left(A^{\alpha+1}B^{\beta+1}\right)^{p/p^*}\left(\int_{\Omega}|u_{\varepsilon}|^{p^*}dx\right)^{p/p*}} \\ &\le s^{p(\beta+1/p^*)}\frac{\int_{\Omega}\left(q_1(x)|\nabla u_{\varepsilon}|^p - \mu_1|u_{\varepsilon}|^p\right)\mathrm{d}x}{\left(\int_{\Omega}|u_{\varepsilon}|^{p^*}\mathrm{d}x\right)^{p/p*}} + \\ &\quad + s^{-p\alpha+1/p^*}\frac{\int_{\Omega}\left(q_2(x)|\nabla u_{\varepsilon}|^p - \mu_1|u_{\varepsilon}|^p\right)\mathrm{d}x}{\left(\int_{\Omega}|u_{\varepsilon}|^{p^*}\mathrm{d}x\right)^{p/p*}} \\ &\le s^{p\beta+1/p^*}Q^{q_1}_{\mu_1}(u_{\varepsilon}) + s^{-p\alpha+1/p^*}Q^{q_2}_{\mu_1}(u_{\varepsilon}). \end{aligned} \tag{60}$$

We know by [11] $Q_{\mu_1}^{q_1}(u_\varepsilon) \leq q_1(x_0) S_0^1 +$

$$\begin{cases} -\mu_1 \dfrac{K_{3,1}}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right), & \text{if } N > p^2,\ k_1 > p, \\ -\left(\mu_1 - \dfrac{C_{p,1}\left(a_p\right)}{K_{3,1}}\right)\dfrac{K_{3,1}}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right), & \text{if } N > p^2,\ k_1 = p, \\ -\left(\mu_1 - \dfrac{C_{k_1,2}\left(a_{k_1}\right)}{K_{3,2}}\right)\dfrac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right), & \text{if } N = p^2,\ k_1 = \dfrac{N-p}{p-1}, \\ -\left(\mu_1 - \dfrac{C_{k_1,3}\left(a_{k_1}\right)}{K_{3,3}}\right)\dfrac{K_{3,3}}{K_2}\varepsilon^{N-p/p} + o\left(\varepsilon^{(N-p/p)}\right), & \text{if } N < p^2,\ k_1 > \dfrac{N-p}{p-1}, \\ -\mu_1 \dfrac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right), & \text{if } N = p^2,\ k_1 > \dfrac{N-p}{p-1}, \\ \dfrac{C_{k_1,1}\left(a_{k_1}\right)}{K_2}\varepsilon^{k_1(p-1)/p} + o\left(\varepsilon^{k_1(p-1)/p}\right), & \text{if } N > p^2,\ 0 < k_1 < p, \\ \dfrac{C_{k_1,1}\left(a_{k_1}\right)}{K_2}\varepsilon^{k_1(p-1)/p} + o\left(\varepsilon^{k_1(p-1)/p}\right), & \text{if } N < p^2,\ p < k_1 < \dfrac{N-p}{p-1}, \\ \dfrac{C_{k_1,2}\left(a_{k_1}\right)}{K_2}\varepsilon^{N-p/p}|\log(\varepsilon)| + o\left(\varepsilon^{N-p/p}|\log(\varepsilon)|\right), & \text{if } N < p^2,\ k_1 = \dfrac{N-p}{p-1}, \end{cases} \tag{61}$$

and $Q_{\mu_1}^{q_2}(u_\varepsilon) \leq q_2(x_0) S_0^1 +$

$$\begin{cases} -\mu_1 \dfrac{K_{3,1}}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right), & \text{if } N > p^2,\ k_2 > p, \\ -\left(\mu_1 - \dfrac{C_{p,1}\left(\tilde{a}_p\right)}{K_{3,1}}\right)\dfrac{K_{3,1}}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right) & \text{if } N > p^2,\ k_2 = p, \\ -\left(\mu_1 - \dfrac{C_{k_2,2}\left(\tilde{a}_{k_2}\right)}{K_{3,2}}\right)\dfrac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right) & \text{if } N = p^2,\ k_2 = \dfrac{N-p}{p-1}, \\ -\left(\mu_1 - \dfrac{C_{k_2,3}\left(\tilde{a}_{k_2}\right)}{K_{3,3}}\right)\dfrac{K_{3,3}}{K_2}\varepsilon^{N-p/p} + o\left(\varepsilon^{N-p/p}\right) & \text{if } N < p^2,\ k_2 > \dfrac{N-p}{p-1}, \\ -\mu_1 \dfrac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right) & \text{if } N = p^2,\ k_2 > \dfrac{N-p}{p-1}, \\ \dfrac{C_{k_2,1}\left(\tilde{a}_{k_2}\right)}{K_2}\varepsilon^{k_2(p-1)/p} + o\left(\varepsilon^{k_2(p-1)/p}\right) & \text{if } N > p^2,\ 0 < k_2 < p, \\ \dfrac{C_{k_2,1}\left(\tilde{a}_{k_2}\right)}{K_2}\varepsilon^{k_2(p-1)/p} + o\left(\varepsilon^{k_2(p-1)/p}\right) & \text{if } N < p^2,\ p < k_2 < \dfrac{N-p}{p-1}, \\ \dfrac{C_{k_2,2}\left(\tilde{a}_{k_2}\right)}{K_2}\varepsilon^{N-p/p}|\log(\varepsilon)| + o\left(\varepsilon^{N-p/p}|\log(\varepsilon)|\right), & \text{if } N < p^2,\ k_2 = \dfrac{N-p}{p-1}, \end{cases} \tag{62}$$

with

$$C_{k_1,1}(a_{k_1}) = a_{k_1}\left(\frac{N-p}{p-1}\right)^p \int_{\mathbb{R}^N} \frac{|x|^{p/p-1+k_1}}{\left(1+|x|^{p/p-1}\right)^N} dx,$$
$$C_{k_1,2}(a_{k_1}) = \left(\frac{N-p}{p-1}\right)^p a_{k_1}\omega_N,$$
$$C_{k_1,3}(a_{k_1}) = a_{k_1}\left(\frac{N-p}{p-1}\right)^p (\operatorname{diam}(\Omega))^{k_1+1-((N-p)/p)-1}, \tag{63}$$

where ω_N is the area of S^{N-1} and by the same way we define $C_{k_2,i}(\tilde{a}_{k_2})$.

Using equations (61) and (62), we distinct the following cases:

(1) If $N = p^2$, $k_1 > N - p/p - 1$, and $k_2 > N - p/p - 1$, then

$$Q^{q_1,q_2}_{\mu_1,\mu_2}(Au_\varepsilon, Bu_\varepsilon) \leq s^{p\beta+1/p^*} q_1(x_0)S_0^1 + s^{-p\alpha+1/p^*} q_2(x_0)S_0^1 + - \mu_1\eta\frac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right). \tag{64}$$

(2) If $N = p^2, k_1 = N - p/p - 1$, and $k_2 > N - p/p - 1$, we have the following equation:

$$Q^{q_1,q_2}_{\mu_1,\mu_2}(Au_\varepsilon, Bu_\varepsilon) \leq s^{p\beta+1/p^*} q_1(x_0)S_0^1 + s^{-p\alpha+1/p^*} q_2(x_0)S_0^1 + -\left(\mu_1 - \frac{s^{p\beta+1/p^*}}{\eta}\frac{C_{k_1,2}(a_{k_1})}{K_{3,2}}\right)\eta\frac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right). \tag{65}$$

(3) If $N = p^2, k_1 > N - p/p - 1$, and $k_2 = N - p/p - 1$, we have the following equation:

$$Q^{q_1,q_2}_{\mu_1,\mu_2}(Au_\varepsilon, Bu_\varepsilon) \leq s^{p\beta+1/p^*} q_1(x_0)S_0^1 + s^{-p(\alpha+1/p^*)} q_2(x_0)S_0^1 + -\left(\mu_1 - \frac{s^{-p\alpha+1/p^*}}{\eta}\frac{C_{k_2,2}(\tilde{a}_{k_2})}{K_{3,2}}\right)\eta\frac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right). \tag{66}$$

(4) If $N = p^2, k_1 = N - p/p - 1$, and $k_2 = N - p/p - 1$, we have the following equation:

$$Q^{q_1,q_2}_{\mu_1,\mu_2}(Au_\varepsilon, Bu_\varepsilon) \leq s^{p\beta+1/p^*} q_1(x_0)S_0^1 + s^{-p\alpha+1/p^*} q_2(x_0)S_0^1 + -\left(\mu_1 - \frac{1}{\eta}\left(s^{p\beta+1/p^*}\frac{C_{k_1,2}(a_{k_1})}{K_{3,2}} + s^{-p\beta+1/p^*}\frac{C_{k_2,2}(\tilde{a}_{k_2})}{K_{3,2}}\right)\eta\frac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)|\right) + + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right). \tag{67}$$

(5) If $N > p^2$, $k_1 > p$, and $k_2 > p$, we have the following equation:

$$Q^{q_1,q_2}_{\mu_1,\mu_2}\left(Au_\varepsilon, Bu_\varepsilon\right) \leq s^{p\beta+1/p^*} q_1\left(x_0\right)S_0^1 + s^{-p\alpha+1/p^*} q_2\left(x_0\right)S_0^1 - \eta \frac{K_{3,1}}{K_2}\mu_1 \varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right). \tag{68}$$

(6) If $N > p^2$, $k_1 > p$, and $k_2 = p$, we obtain the following equation:

$$\begin{aligned} Q^{q_1,q_2}_{\mu_1,\mu_2}\left(Au_\varepsilon, Bu_\varepsilon\right) &\leq s^{p\beta+1/p^*} q_1\left(x_0\right)S_0^1 + s^{-p\alpha+1/p^*} q_2\left(x_0\right)S_0^1 + \\ &-\left(\mu_1 - \frac{s^{-p\alpha+1/p^*}}{\eta}\frac{C_{p,1}\left(\tilde{a}_p\right)}{K_{3,1}}\right)\eta \frac{K_{3,1}}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right). \end{aligned} \tag{69}$$

(7) If $N > p^2$, $k_1 = p$, and $k_2 > p$, we have the following equation:

$$\begin{aligned} Q^{q_1,q_2}_{\mu_1,\mu_2}\left(Au_\varepsilon, Bu_\varepsilon\right) &\leq s^{p\left(\beta+1/p^*\right)} q_1\left(x_0\right)S_0^1 + s^{-p\left(\alpha+1/p^*\right)} q_2\left(x_0\right)S_0^1 + \\ &-\left(\mu_1 - \frac{s^{p\left(\beta+1/p^*\right)}}{\eta}\frac{C_{p,1}\left(a_p\right)}{K_{3,1}}\right)\eta \frac{K_{3,1}}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right). \end{aligned} \tag{70}$$

(8) If $N > p^2, k_1 = p$, and $k_2 = p$, it result in the following equation:

$$\begin{aligned} Q^{q_1,q_2}_{\mu_1,\mu_2}\left(Au_\varepsilon, Bu_\varepsilon\right) &\leq s^{p\left(\beta+1/p^*\right)} q_1\left(x_0\right)S_0^1 + s^{-p\left(\alpha+1/p^*\right)} q_2\left(x_0\right)S_0^1 + \\ &-\left(\mu_1 - \frac{1}{\eta}\left(s^{p\left(\beta+1/p^*\right)}\frac{C_{k_1,1}\left(a_{k_1}\right)}{K_{3,1}} + s^{-p\left(\alpha+1/p^*\right)}\frac{C_{k_2,1}\left(\tilde{a}_{k_2}\right)}{K_{3,1}}\right)\right)\eta \frac{K_{3,1}}{K_2}\varepsilon^{p-1} + \\ &+ o\left(\varepsilon^{p-1}\right). \end{aligned} \tag{71}$$

(9) If $N < p^2, k_1 > (N - p/p - 1)$, and $k_2 > (N - p/p- 1)$, we have the following equation:

$$\begin{aligned} Q^{q_1,q_2}_{\mu_1,\mu_2}\left(Au_\varepsilon, Bu_\varepsilon\right) &\leq s^{p\left(\beta+1/p^*\right)} q_1\left(x_0\right)S_0^1 + s^{-p\left(\alpha+1/p^*\right)} q_2\left(x_0\right)S_0^1 + \\ &-\left(\mu_1 - \frac{1}{\eta}\left(s^{p\left(\beta+1/p^*\right)}\frac{C_{k_1,3}\left(a_{k_1}\right)}{K_{3,3}} + s^{-p\left(\alpha+1/p^*\right)}\frac{C_{k_2,3}\left(\tilde{a}_{k_2}\right)}{K_{3,3}}\right)\right)\eta \frac{K_{3,3}}{K_2}\varepsilon^{N-p/p} + \\ &+ o\left(\varepsilon^{N-p/p}\right). \end{aligned} \tag{72}$$

Using the fact that,

$$\begin{aligned} q_1(x_0)S_0^1 &\le S_0^{q_1}, \\ q_2(x_0)S_0^1 &\le S_0^{q_2}, \end{aligned} \tag{73}$$

and from Lemma 3, we obtain the following equation:

$$s^{p(\beta+1/p^*)}(q_1(x_0))S_0^1 + s^{-p(\alpha+1/p^*)}q_2(x_0)S_0^1 \le K(\alpha,\beta)\left(S_0^{q_1}\right)^{(\beta+1/p^*)}\left(S_0^{q_2}\right)^{(\alpha+1/p^*)} \le S_{0,0}^{q_1,q_2}. \tag{74}$$

Let $(\nu_i)_{1\le i\le 5}$ such that

$$\begin{aligned} \nu_1 &:= \frac{C_{k_1,2}\left(a_{k_1}\right)}{K_{3,2}}\left(\frac{(\beta+1)S_0^{q_1}}{(\alpha+1)S_0^{q_2}}+1\right)^{-1}, \nu_2 := \frac{C_{k_2,2}\left(\tilde{a}_{k_2}\right)}{K_{3,2}}\left(\frac{(\alpha+1)S_0^{q_2}}{(\beta+1)S_0^{q_1}}+1\right)^{-1}, \\ \nu_3 &:= \frac{C_{p,1}\left(\tilde{a}_p\right)}{K_{3,1}}\left(\frac{(\alpha+1)S_0^{q_2}}{(\beta+1)S_0^{q_1}}+1\right)^{-1}, \nu_4 := \frac{C_{p,1}\left(a_p\right)}{K_{3,1}}\left(\frac{(\beta+1)S_0^{q_1}}{(\alpha+1)S_0^{q_2}}+1\right)^{-1}, \\ \nu_5 &:= \left(\frac{(\beta+1)S_0^{q_1}}{(\alpha+1)S_0^{q_2}}+1\right)^{-1}\frac{C_{k_1,3}\left(a_{k_1}\right)}{K_{3,3}} + \left(\frac{(\alpha+1)S_0^{q_2}}{(\beta+1)S_0^{q_1}}+1\right)^{-1}\frac{C_{k_2,3}\left(\tilde{a}_{k_2}\right)}{K_{3,3}}, \end{aligned} \tag{75}$$

then, we have $Q_{\mu_1,\mu_2}^{q_1,q_2}(Au_\varepsilon, Bu_\varepsilon) \le$

$$\begin{cases}
S_{0,0}^{q_1,q_2} - \mu_1\eta\dfrac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right), & \text{if } \begin{cases} N = p^2, \\ k_1 > \dfrac{N-p}{p-1}, k_2 > \dfrac{N-p}{p-1}, \end{cases} \\
S_{0,0}^{q_1,q_2} - (\mu_1-\nu_1)\eta\dfrac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right), & \text{if } \begin{cases} N = p^2, \\ k_1 = \dfrac{N-p}{p-1}, k_2 > \dfrac{N-p}{p-1}, \end{cases} \\
S_{0,0}^{q_1,q_2} - (\mu_1-\nu_2)\eta\dfrac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right), & \text{if } \begin{cases} N = p^2, \\ k_1 > \dfrac{N-p}{p-1}, k_2 = \dfrac{N-p}{p-1}, \end{cases} \\
S_{0,0}^{q_1,q_2} - (\mu_1-(\nu_1+\nu_2))\eta\dfrac{K_{3,2}}{K_2}\varepsilon^{p-1}|\log(\varepsilon)| + o\left(\varepsilon^{p-1}|\log(\varepsilon)|\right), & \text{if } \begin{cases} N = p^2, \\ k_1 = k_2 = \dfrac{N-p}{p-1}, \end{cases} \\
S_{0,0}^{q_1,q_2} - \eta\dfrac{K_{3,1}}{K_2}\mu_1\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right), & \text{if } \begin{cases} N > p^2, \\ k_1 > p, k_2 > p, \end{cases} \\
S_{0,0}^{q_1,q_2} - (\mu_1-\nu_3)\eta\dfrac{K_{3,1}}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right), & \text{if } \begin{cases} N > p^2, \\ k_1 > p, k_2 = p, \end{cases} \\
S_{0,0}^{q_1,q_2} - (\mu_1-\nu_4)\eta\dfrac{K_{3,1}}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right), & \text{if } \begin{cases} N > p^2, \\ k_1 = p, k_2 > p, \end{cases} \\
S_{0,0}^{q_1,q_2} - (\mu_1-(\nu_3+\nu_4))\eta\dfrac{K_{3,1}}{K_2}\varepsilon^{p-1} + o\left(\varepsilon^{p-1}\right), & \text{if } \begin{cases} N > p^2, \\ k_1 = k_2 = p, \end{cases} \\
S_{0,0}^{q_1,q_2} - (\mu_1-\nu_5)\eta\dfrac{K_{3,3}}{K_2}\varepsilon^{N-p/p} + o\left(\varepsilon^{N-p/p}\right), & \text{if } \begin{cases} N < p^2, \\ k_1 > \dfrac{N-p}{p-1}, k_2 > \dfrac{N-p}{p-1}. \end{cases}
\end{cases} \tag{76}$$

The conclusion follows from the previous inequalities. □

3. Nonexistence Result

The main goal of this section is the nonexistence result. So we use Pohozaev identity to prove Theorem 1.

Proof of Theorem 1. Let $(u, v) \in E$ be the solution of $(\mathcal{S}_{\mu_1,\mu_2}^{q_1,q_2})$. Multiplying the first equation in the system $(\mathcal{S}_{\mu_1,\mu_2}^{q_1,q_2})$ by $\nabla u.(x - x_0)$ on both sides and integrating by parts, we obtain the following equation:

$$\begin{aligned}&\frac{p-N}{p}\int_{\Omega} q_1(x)|\nabla u(x)|^p \mathrm{d}x - \frac{1}{p}\int_{\Omega}\tilde{q}_1(x)|\nabla u(x)|^p \mathrm{d}x - \left(1-\frac{1}{p}\right)\int_{\partial\Omega} q_1(x)(x-x_0).\nu\left|\frac{\partial u}{\partial \nu}\right|^p \mathrm{d}\sigma\\ &= -\frac{N}{p}\mu_1\int_{\Omega}|u|^p \mathrm{d}x + \int_{\Omega}\nabla\left(|u|^{\alpha+1}\right).(x-x_0)|v|^{\beta+1}\mathrm{d}x.\end{aligned} \tag{77}$$

where ν denotes the outward normal to $\partial\Omega$.

Similarly, we obtain for the second equation of $(\mathcal{S}_{\mu_1,\mu_2}^{q_1,q_2})$ as follows:

$$\begin{aligned}&\frac{p-N}{p}\int_{\Omega} q_2(x)|\nabla v(x)|^p \mathrm{d}x - \frac{1}{p}\int_{\Omega}\tilde{q}_2(x)|\nabla v(x)|^p \mathrm{d}x - \left(1-\frac{1}{p}\right)\int_{\partial\Omega} q_2(x)(x-x_0).\nu\left|\frac{\partial v}{\partial \nu}\right|^p \mathrm{d}\sigma\\ &= -\frac{N}{p}\mu_2\int_{\Omega}|v|^p \mathrm{d}x + \int_{\Omega}|u|^{\alpha+1}\nabla\left(|v|^{\beta+1}\right).(x-x_0)\mathrm{d}x,\end{aligned} \tag{78}$$

Combining equations (77) and (78), we write the following equation:

$$\begin{aligned}&\frac{p-N}{p}\int_{\Omega}\left(q_1(x)|\nabla u(x)|^p + q_2(x)|\nabla v(x)|^p\right)\mathrm{d}x +\\ &-\frac{1}{p}\int_{\Omega}\left(\tilde{q}_1(x)|\nabla u(x)|^p + \tilde{q}_2(x)|\nabla v(x)|^p\right)\mathrm{d}x +\\ &-\left(1-\frac{1}{p}\right)\int_{\partial\Omega}\left(q_1(x)(x-x_0).\nu\left|\frac{\partial u}{\partial \nu}\right|^p + q_2(x)(x-x_0).\nu\left|\frac{\partial v}{\partial \nu}\right|^p\right)\mathrm{d}\sigma\\ &= -\frac{N}{p}\int_{\Omega}\left(\mu_1|u|^p + \mu_2|v|^p\right)\mathrm{d}x - N\int_{\Omega}|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x.\end{aligned} \tag{79}$$

On the other hand, multiplying the two equations in the system $(\mathcal{S}_{\mu_1,\mu_2}^{q_1,q_2})$ by $(N - p/p)u$ and $(N - p/p)v$, respectively, integrating by parts, and by summing the obtained results, we obtain the following equation:

$$\begin{aligned}&\frac{N-p}{p}\int_{\Omega}\left(q_1(x)|\nabla u(x)|^p + q_2(x)|\nabla v(x)|^p\right)\mathrm{d}x\\ &= \frac{N-p}{p}\int_{\Omega}\left(\mu_1|u|^p + \mu_2|v|^p\right)\mathrm{d}x + N\int_{\Omega}|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x.\end{aligned} \tag{80}$$

Combining equations (79) and (80), we obtain the following equation:

$$\int_\Omega (\mu_1|u|^p + \mu_2|v|^p)\mathrm{d}x - \frac{1}{p}\int_\Omega (\tilde{q}_1(x)|\nabla u(x)|^p + \tilde{q}_2(x)|\nabla v(x)|^p)\mathrm{d}x + -\left(1-\frac{1}{p}\right)\int_{\partial\Omega}\left(q_1(x)(x-x_0).\nu\left|\frac{\partial u}{\partial \nu}\right|^p + q_2(x)(x-x_0).\nu\left|\frac{\partial v}{\partial \nu}\right|^p\right)\mathrm{d}\sigma = 0. \tag{81}$$

If Ω is a star-shaped domain about x_0, we have the following equation:

$$\int_\Omega (\mu_1|u|^p + \mu_2|v|^p)\mathrm{d}x > \frac{1}{p}\int_\Omega (\tilde{q}_1(x)|\nabla u(x)|^2 + \tilde{q}_2(x)|\nabla v(x)|^2)\mathrm{d}x. \tag{82}$$

By $(\mathscr{H}_0)$, we reach that

$$\mu_2 > \frac{1}{p}\frac{\int_\Omega (\tilde{q}_1(x)|\nabla u(x)|^p + \tilde{q}_2(x)|\nabla v(x)|^p)\mathrm{d}x}{\int_\Omega (|u|^p + |v|^p)\mathrm{d}x} \geq \gamma(q_1, q_2), \tag{83}$$

which is a contradiction. □

4. Existence Results

We first verify that J satisfies the geometric conditions of the mountain pass theorem.

Lemma 5. *Assume that* $(\mathscr{H}_0)$ *is satisfied, then*

(i) *There exist* $\rho > 0$ *and* $R > 0$ *such that* $J(u,v) \geq \rho$ *for all* $(u,v) \in E$ *with* $\|(u,v)\| = R$.

(ii) *There exists* $(u_0, v_0) \in E$, *with* $\|(u_0, v_0)\| > R$ *such that* $J(u_0, v_0) \leq 0$.

Proof

(i) From Hölder's inequality, Sobolev embedding and $(\mathscr{H}_0)$ it follows that

$$\begin{aligned} J(u,v) &= \frac{1}{p}\|(u,v)\|^p - \frac{1}{p}\int_\Omega (\mu_1|u|^p + \mu_2|v|^p)\mathrm{d}x - \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x \\ &\geq \frac{1}{p}\|(u,v)\|^p - \frac{\mu_2}{p}\int_\Omega (|u|^p + |v|^p)\mathrm{d}x - C\|(u,v)\|^{p^*} \\ &\geq \frac{1}{p}\min\left\{\left(1-\frac{\mu_2}{\lambda_1^{q_1}}\right),\left(1-\frac{\mu_2}{\lambda_1^{q_2}}\right)\right\}\|(u,v)\|^p - C\|(u,v)\|^{p^*}, \end{aligned} \tag{84}$$

where C is a positive constant. Then, there exists $(u,v) \in E$ such that $J(u,v) \geq \rho > 0$, for $\|(u,v)\| = R$ small enough.

(ii) We have $J(tu, tv) \longrightarrow -\infty$ as $t \longrightarrow \infty$, for any $(u,v) \in E/\{(0,0)\}$; thus, there exists (u_0, v_0) with $\|(u_0, v_0)\| > R$ such that $J(u_0, v_0) < 0$.

Next, we prove an important lemma which ensures the local compactness of a Palais–Smale sequence for J. □

Lemma 6. *If* $c < c^* := p/N - p(1/p^*)^{N/p}(S_{0,0}^{q_1,q_2})^{N/p}$, *then* J *satisfies* $(P-S)_c$ *condition.*

Proof. Suppose that $\{(u_n, v_n)\} \subset E$ satisfies

$$\begin{aligned} &\frac{1}{p}\|(u_n, v_n)\|^p - \frac{1}{p}\int_\Omega (\mu_1|u_n|^p + \mu_2|v_n|^p)\mathrm{d}x - \int_\Omega |u_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x = c + ^\circ(1), \\ &\|(u_n, v_n)\|^p - \int_\Omega (\mu_1|u_n|^p + \mu_2|v_n|^p)\mathrm{d}x - p^*\int_\Omega |u_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x = \langle \varepsilon_n, (u_n, v_n)\rangle, \end{aligned} \tag{85}$$

with $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$, then $\{(u_n, v_n)\}$ is bounded in E. Going if necessary to a subsequence, we can assume that as $n \longrightarrow \infty$

$$\begin{aligned} &(u_n, v_n) \rightharpoonup (u, v) \text{ weakly in } E,\\ &(u_n, v_n) \longrightarrow (u, v) \text{ strongly in } L^{p_1} \times L^{p_2} \text{ for all } p \le p_1, p_2 < p^*,\\ &(u_n, v_n) \longrightarrow (u, v)\, a.\text{e. in } \Omega. \end{aligned} \tag{86}$$

It follows that (u, v) is a weak solution of the system, i.e.,

$$\langle J'(u, v), (\varphi, \psi)\rangle = 0, \quad \text{for all } (\varphi, \psi) \in E. \tag{87}$$

We set

$$\begin{aligned} \widetilde{u}_n &= u_n - u,\\ \widetilde{v}_n &= v_n - v. \end{aligned} \tag{88}$$

Applying the following relations as in Brézis-Lieb Lemma [19]:

$$\begin{aligned} \|\nabla u_n\|_p^p &= \|\nabla u\|_p^p + \|\nabla \widetilde{u}_n\|_p^p + ^\circ(1),\\ \|\nabla v_n\|_p^p &= \|\nabla v\|_p^p + \|\nabla \widetilde{v}_n\|_p^p + ^\circ(1),\\ \int_\Omega |u_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x &= \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x + \int_\Omega |\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\mathrm{d}x + ^\circ(1), \end{aligned} \tag{89}$$

we obtain the following equations:

$$J(u, v) + \frac{1}{p}\|(\widetilde{u}_n, \widetilde{v}_n)\|^p - \int_\Omega |\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\mathrm{d}x = c + ^\circ(1), \tag{90}$$

and

$$\|(u, v)\|^p + \|(\widetilde{u}_n, \widetilde{v}_n)\|^p = \int_\Omega (\mu_1|u|^p + \mu_2|v|^p)\mathrm{d}x + p^*\int_\Omega \left(|u|^{\alpha+1}|v|^{\beta+1} + |\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\right)\mathrm{d}x + ^\circ(1). \tag{91}$$

Since $\langle J'(u, v), (u, v)\rangle = 0$, then

$$\|(\widetilde{u}_n, \widetilde{v}_n)\|^p = p^*\int_\Omega |\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\mathrm{d}x + ^\circ(1). \tag{92}$$

We may therefore assume that

$$\begin{aligned} &\|(\widetilde{u}_n, \widetilde{v}_n)\|^p \longrightarrow L\\ &\mathrm{p}^*\int_\Omega |\widetilde{\mathrm{u}}_\mathrm{n}|^{\alpha+1}|\widetilde{\mathrm{v}}_\mathrm{n}|^{\beta+1}\mathrm{d}x \longrightarrow L \text{ as } n \longrightarrow +\infty. \end{aligned} \tag{93}$$

From the definition of $S_{0,0}^{q_1,q_2}$, we obtain the following equation:

$$\|(\widetilde{u}_n, \widetilde{v}_n)\|^p \ge S_{0,0}^{q_1,q_2}\left(\int_\Omega |\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\mathrm{d}x\right)^{p/p^*}. \tag{94}$$

Thus, $L \ge S_{0,0}^{q_1,q_2}(L/p^*)^{p/p^*}$. Assume that $L > 0$, then $L \ge (p^*)^{1-(N/p)}(S_{0,0}^{q_1,q_2})^{N/p}$.

Passing to the limit in equation (90), we obtain the following equation:

$$J(u, v) + \frac{L}{N} = c < \frac{p}{N-p}\left(\frac{S_{0,0}^{q_1,q_2}}{p^*}\right)^{N/p}, \tag{95}$$

and hence, $J(u, v) < 0$, for $(u, v) \in E$.

On the other hand, we have the following equation:

$$J(u, v) = \left(\frac{p^*}{p} - 1\right)\int_\Omega |\widetilde{u}_n|^{\alpha+1}|\widetilde{v}_n|^{\beta+1}\mathrm{d}x \ge 0, \tag{96}$$

which yields a contradiction. Thus, $(u_n, v_n) \longrightarrow (u, v)$ strongly in E. ☐

Lemma 7. *Let A, B satisfy equation (59) and u_ε as in Lemma 4 with $\int_\Omega |u_\varepsilon|^{p^*}dx = 1$. Then,*

$$\sup_{t\ge 0} J(tAu_\varepsilon, tBu_\varepsilon) < c^*. \tag{97}$$

Proof. We have the following equation:

$$J\left(tAu_{\varepsilon}, tBu_{\varepsilon}\right) \leq \frac{1}{p}t^{p}\left(\left\|\left(Au_{\varepsilon}, Bu_{\varepsilon}\right)\right\|^{p} - \mu_{1}\left(A^{p} + B^{p}\right)\left\|u_{\varepsilon}\right\|_{p}^{p}\right) - t^{p^{*}} A^{\alpha+1} B^{\beta+1}. \tag{98}$$

Denoting by $r(tu_{\varepsilon})$ the function in the right-hand side of the last inequality. A forward computation assures that

$$t_0 := \left[\frac{\left\|\left(Au_{\varepsilon}, Bu_{\varepsilon}\right)\right\|^{p} - \mu_{1}\left(A^{p} + B^{p}\right)\left\|u_{\varepsilon}\right\|_{p}^{p}}{p^{*} A^{\alpha+1} B^{\beta+1}}\right]^{1/p^{*}-p}, \tag{99}$$

is the maximum point for r. So,

$$J\left(tAu_{\varepsilon}, tBu_{\varepsilon}\right) \leq \left(\frac{p}{N-p}\right)\left(\frac{1}{p^{*}}\right)^{N/p}\left[s^{p\left((\beta+1)/p^{*}\right)} Q_{\mu_1}^{q_1}\left(u_{\varepsilon}\right) + s^{-p\left((\alpha+1)/p^{*}\right)} Q_{\mu_1}^{q_2}\left(u_{\varepsilon}\right)\right]^{N/p}. \tag{100}$$

By Lemmas 3 and 4 and for small ε, we obtain the following equation:

$$\sup_{t\geq 0} J\left(tAu_{\varepsilon}, tBu_{\varepsilon}\right) < c^{*}, \tag{101}$$

and thus, equation (97) holds.

Now, we can prove Theorems 2 and 3. □

Proofs of Theorem 2 and 3. By Lemma 5, there exists a Palais–Smale sequence a $(\mathrm{P}-\mathrm{S})_c$ sequence in E with

$$\begin{aligned} &c = \inf_{\phi\in\Gamma} \max\nolimits_{t\in[0,1]} J(\phi(t)),\\ &\Gamma = \{\phi \in C([0,1], E)\colon \phi(0) = 0, J(\phi(1)) < 0\}. \end{aligned} \tag{102}$$

Lemmas 6 and 7 imply that J verifies the condition $(\mathrm{P}-\mathrm{S})_c$. Using the mountain pass theorem in [20] whenever $c > 0$ and the Ghoussoub–Preiss version in [21] whenever $c = 0$, respectively, we obtain a nontrivial critical point (u, v) of J.

Consider

$$J^{+}(u, v) = \frac{1}{p}\|(u, v)\|^{p} - \frac{1}{p}\int_{\Omega}\left(\mu_1\left|u^{+}\right|^{p} + \mu_2\left|v^{+}\right|^{p}\right)\mathrm{d}x - \int_{\Omega}\left|u^{+}\right|^{\alpha+1}\left|v^{+}\right|^{\beta+1}\mathrm{d}x. \tag{103}$$

Repeating the above process for J^{+}, we obtain a nonnegative solution (u, v) to the problem $(\mathcal{S}_{\mu_1,\mu_2}^{q_1,q_2})$. From $(\mathscr{H}_0)$ and by using the maximum principle, we conclude that $u > 0$ and $v > 0$.

Data Availability

The functional analysis data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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