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Research Article

Study of Nonlocal Multiorder Implicit Differential Equation Involving Hilfer Fractional Derivative on Unbounded Domains

Sabri T. M. Thabet [1] **and Imed Kedim** [2]

[1]*Department of Mathematics, University of Lahej, Lahej, Yemen*
[2]*Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia*

Correspondence should be addressed to Sabri T. M. Thabet; th.sabri@yahoo.com

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This paper aims to study the existence and uniqueness of the solution for nonlocal multiorder implicit differential equation involving Hilfer fractional derivative on unbounded domains (a, ∞), $a \geq 0$, in an applicable Banach space by utilizing the Banach contraction principle. Furthermore, we discuss various types of stability such as Ulam–Hyers–Rassias (UHR), Ulam–Hyers (UH), and semi-Ulam–Hyers–Rassias (sUHR) for nonlocal boundary value problem. Absolutely, our results cover that several outcomes have existed in the literature.

1. Introduction

The subject of fractional calculus is an extension of classical calculus, and this means extend an integer order to an arbitrary order. In the field of fractional calculus, there are many definitions of integrals and derivatives. Most of the general public fractional definitions are Caputo and Riemann–Liouville (R-L), which attracted many researcher to extend differential equations of integer orders to fractional such as [1]. Newly, Hilfer has connected the Caputo and R-L derivative by a general formula, and it is known by Hilfer or generalized R-L derivatives [2], which is attracted attention of many authors in the literature, such as [3] studied the Ulam stability and existence results of differential equations in the sense of fractional Hadamard and Hilfer derivatives. Bulavatsky [4] proved the closed solutions to several problems for the anomalous diffusion equation involving Hilfer fractional derivative. The author of this work [5] derived an equivalent form of the Hilfer derivative. Furthermore, the authors in [6] constructed operational calculus of generalized R-L derivatives and used it to solve linear equation of n–term with constant coefficients via Hilfer derivative. The authors in [7, 8] established existence and uniqueness of solutions for a nonlinear Hilfer fractional problem in weighted spaces by employing various fixed point theorems. Thabet et al. [9] studied an abstract Hilfer fractional integrodifferential equation via technique of measure of noncompactness and established continuous dependence of ϵ-approximate solutions by using generalized Gronwall's inequality. In 2020, Abdo et al. [10] investigated sufficient conditions UH stability and the existence and uniqueness of solutions for ψ-Hilfer fractional integrodifferential equations by Schaefer, Banach, and Schauder theorems with helping of generalized Gronwall's inequality.

The subject of fixed point theory represents a highly important part of various areas of mathematics and may be regarded a wealth in nonlinear analysis. The fixed point for an adequate map is equivalent to the existence and uniqueness of the solutions for many engineering problems and realistic phenomena. We indicate that some works have been concerned on fixed point theorems and its applications; for example, the authors in [11–13] proved UH, UHR, sUHR, and asymptotic stability as well as the existence and uniqueness of solutions for some fractional problems by topological degree theory and Banach contraction principle. The Banach, Schauder, and Monch fixed point theorems with helping of the measure of noncompactness used to study some qualitative properties and stability of fractional integrodifferential equations [14, 15]. Moreover, several fixed point theorems are employed to treat the qualitative properties of various class systems such as implicit problems [16, 17], hybrid multifractional problem [18], and thermistor fractional problem [19]. In the literature, engineering and physical problems have proved that the dealing with nonlocal initial conditions has a good effect more than the classical initial, and see some papers considered nonlocal conditions [20–23].

In 2018, the authors in [24] studied the existence, uniqueness, and stability of the solution for the following nonlocal implicit differential equation via Hilfer fractional derivative on the bounded interval $[0, T]$:

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta}\varphi(u) = h\left(u, \varphi(u), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta}\varphi(u)\right), & u \in J = [0, T], \\ \mathfrak{I}_{0^+}^{1-\gamma}\varphi(0) = \sum_{j=1}^{m}\delta_j\varphi\left(\epsilon_j\right), & \epsilon_j \in J, 0 < \alpha < 1, 0 \leq \beta \leq 1, \gamma = \alpha + \beta - \alpha\beta. \end{cases} \tag{1}$$

In 2021, the authors of these works [25, 26] established sufficient conditions of the existence and uniqueness solution and discussed UHR stability, UH stability, and sUHR stability for initial ψ–Hilfer fractional integrodifferential equations. Ali et al. [27] discussed the stability of pantograph-type implicit fractional differential equations with impulsive conditions. Also, the existence, uniqueness, and UH stabilities result for a coupled ψ–Hilfer fractional integrodifferential equation on bounded domains investigated by [28]. Almalahi et al. [29, 30] established qualitative theories for fractional functional differential equation with boundary condition and finite delay as well as a coupled system of hybrid fractional differential equations via ϕ–Hilfer fractional derivatives. Very recently, Xie et al. [31] investigated some qualitative properties of multiorder differential equations with initial condition involving R-L fractional derivatives of the following form:

$$\begin{cases} \mathscr{L}\left({}^{R}|D\right)\varphi(u) = h\left(u, \varphi(u), {}^{R}\mathfrak{D}^{\theta}\varphi(u)\right), & u \in J = [0, \infty), \\ u^{1-\alpha_n}\varphi(u)|_{u=0} = 0, \end{cases} \tag{2}$$

where $\mathscr{L}({}^{R}\mathfrak{D}) = {}^{R}\mathfrak{D}^{\alpha_n} - \sum_{j=1}^{n-1}c_j\,{}^{R}\mathfrak{D}^{\alpha_{n-j}}, n \in \mathbb{N}$.

Inspired by these works [24, 31], this work aims to study the existence and uniqueness results along with stability types of the UH, UHR, and sUHR for the following nonlocal multiorder implicit differential equation involving Hilfer fractional derivative on unbounded domains:

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha_n,\beta_n}\varphi(u) - \sum_{i=1}^{n-1}c_i\,{}^{H}\mathfrak{D}_{a^+}^{\alpha_n,\beta_n}\varphi(u) = h\left(u, \varphi(u), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right), \\ \mathfrak{I}_{a^+}^{1-\gamma_n}\varphi\left(a^+\right) = \sum_{j=1}^{m}\delta_j\varphi\left(\epsilon_j\right), \quad \epsilon_j, u \in I = (a, \infty), a \geq 0, \end{cases} \tag{3}$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, $\beta_1 < \beta_2 < \cdots < \beta_n, c_i$, $\delta_j \in \mathbb{R}$, $i = 1, 2, \ldots, n-1; j = 1, 2, \ldots, m), n, m \in \mathbb{N}$, ${}^{H}\mathfrak{D}_{a^+}^{\alpha_n,\beta_n}, {}^{H}\mathfrak{D}_{a^+}^{\alpha_i,\beta_i}, {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}$, are the Hilfer fractional derivatives of order $\alpha_n, \alpha_i, \theta \in (0, 1]$, and types $\beta_n, \beta_i, \vartheta \in [0, 1]$, respectively, such that $\theta < \alpha_i, \alpha_i + \theta \leq \alpha_n$, and $\mathfrak{I}_{a^+}^{1-\gamma_n}$ is R-L fractional integral of order $(1-\gamma_n), 0 \leq \gamma_n = \alpha_n + \beta_n - \alpha_n\beta_n < 1$, and a function $h: I \times \chi \times \chi \longrightarrow \chi$, where χ is the real Banach space. The novelty and contributions of this paper are to consider a more general problem containing n–term of generalized fractional derivative in applicable Banach space Ξ on unbounded domains, in order to include a lot of outcomes which are existing in the literature, for example, the results in [24, 31].

The organization of this work is as follows: in Section 2, we recall some notations, definitions, and preliminaries. Section 3 establishes the existence and uniqueness of the solution of the problem (3), in an applicable Banach space by utilizing the fixed point theorem. In Section 4, we prove various types of stability by employing the nonlinear analysis topics.

2. Preliminaries

Throughout this section, we present some interesting notations and preliminaries, in order to use it in achieving desired results.

Let the Banach space of continuous functions denoted by $\mathscr{C}(I, \chi)$, with supremum norm $\|\varphi\|_{\chi} = \sup_{u\in I}\|\varphi(u)\|$. For reaching our goal, we define the following Banach space:

$$\Xi = \left\{ \varphi | \varphi(u) \in \mathscr{C}(I,\chi), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u) \in \mathscr{C}^1(I,\chi), \sup_{u\in I}\frac{\|\varphi(u)\|}{1+u^{\rho}} < \infty, \sup_{u\in I}\frac{\left\| {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right\|}{1+u^{\rho}} < \infty \right\}, \tag{4}$$

where $\rho > 1$, endowed with the norm

$$\|\varphi\|_{\Xi} = \max\left\{ \sup_{u\in I}\frac{\|\varphi(u)\|}{1+u^{\rho}}, \sup_{u\in I}\frac{\left\| {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right\|}{1+u^{\rho}} \right\}. \tag{5}$$

It is easy to prove that $(\Xi, \|\cdot\|_{\Xi})$ be a Banach space as a procedure in the works [32–34].

Definition 1 (see [35]). The R-L fractional integral of order $\mu > 0$ of a function g is given by

$$\left(\mathfrak{J}_{a^+}^{\mu} g\right)(u) = \frac{1}{\Gamma(\mu)}\int_a^u (u-v)^{\mu-1} g(v)\mathrm{d}v, u > a. \tag{6}$$

Definition 2 (see [35]). The R-L fractional derivative of order $\mu \in (0,1]$ of a function g is given by

$$\left({}^{R}\mathfrak{D}_{a^+}^{\mu} g\right)(u) = \left(\mathfrak{D}\mathfrak{J}_{a^+}^{(1-\mu)} g\right)(u) = \frac{1}{\Gamma(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}u}\int_a^u (u-v)^{-\mu} g(v)\mathrm{d}v, u > a, \tag{7}$$

where $\mathfrak{D} := \mathrm{d}/\mathrm{d}u$.

Definition 3 (see [35]). The Caputo fractional derivative of order $\mu \in (0,1]$ of a function g is given by

$$\left({}^{C}\mathfrak{D}_{a^+}^{\mu} g\right)(u) = \left(\mathfrak{J}_{a^+}^{(1-\mu)}\mathfrak{D} g\right)(u) = \frac{1}{\Gamma(1-\mu)}\int_a^u (u-v)^{-\mu} g''(v)\mathrm{d}v, u > a. \tag{8}$$

Definition 4 (see [2]). The Hilfer fractional derivative of order $0 < \mu \le 1$ and type $0 \le \nu \le 1$ of a function g is given by

$$\left({}^{H}\mathfrak{D}_{a^+}^{\mu,\nu} g\right)(u) = \left(\mathfrak{J}_{a^+}^{\nu(1-\mu)}\mathfrak{D}\left(\mathfrak{J}_{a^+}^{(1-\nu)(1-\mu)}\right) g\right)(u). \tag{9}$$

Remark 1. We observe that the Hilfer fractional derivative returns to the R-L derivative when $\nu = 0$, i.e., $({}^{H}\mathfrak{D}_{a^+}^{\mu,\nu} = \mathfrak{D}\mathfrak{J}_{a^+}^{(1-\mu)} = {}^{R}\mathfrak{D}_{a^+}^{\mu})$, and returns to the Caputo derivatives when $\nu = 1$, i.e., $({}^{H}\mathfrak{D}_{a^+}^{\mu,\nu} = \mathfrak{J}_{a^+}^{(1-\mu)}\mathfrak{D} = {}^{C}\mathfrak{D}_{a^+}^{\mu})$.

Lemma 1 (see [36]). *If* $0 < \mu \le 1, 0 \le \nu \le 1, \quad 0 \le \xi < 1$, $\xi = \mu + \nu - \mu\nu$, *and* $g \in L^1(I)$, $\mathfrak{J}_{a^+}^{1-\mu} g \in L^1(I)$, *then*

$$\mathfrak{J}_{a^+}^{\mu}\, {}^{H}\mathfrak{D}_{a^+}^{\mu,\nu} g(u) = \mathfrak{J}_{a^+}^{\xi}\, {}^{R}\mathfrak{D}_{a^+}^{\xi} g(u) = g(u) - \frac{\left(\mathfrak{J}_{a^+}^{1-\xi} g\right)(a^+)}{\Gamma(\xi)}(u-a)^{\xi-1}, \quad \forall u \in I. \tag{10}$$

Lemma 2 (see [36]). *Let* $\zeta > 0$ *and* $\mu > 0$. *Then,*

$$\begin{aligned} \left[\mathfrak{J}_{a^+}^{\mu}(u-a)^{\zeta-1}\right](t) &= \frac{\Gamma(\zeta)}{\Gamma(\zeta+\mu)}(t-a)^{\zeta+\mu-1}, \\ \left[{}^{R}\mathfrak{D}_{a^+}^{\mu}(u-a)^{\zeta-1}\right](t) &= \frac{\Gamma(\zeta)}{\Gamma(\zeta-\mu)}(t-a)^{\zeta-\mu-1}. \end{aligned} \tag{11}$$

Lemma 3. *Let* $\alpha > 0, 0 < \mu \le 1, 0 \le \nu \le 1$, *and* $g \in L^1(I)$. *Then,*

$$\left({}^{H}\mathfrak{D}_{a^+}^{\mu,\nu}\mathfrak{J}_{a^+}^{\alpha} g\right)(u) = \left(\mathfrak{J}_{a^+}^{\alpha-\mu} g\right)(u), \text{and}\left(\mathfrak{J}_{a^+}^{\alpha}\, {}^{H}\mathfrak{D}_{a^+}^{\mu,\nu} g\right)(u) = \left(\mathfrak{J}_{a^+}^{\alpha-\mu} g\right)(u). \tag{12}$$

Proof. The proof follows from Definition 4, with some calculations. □

Definition 5 (see [37]). Let Ξ be a nonempty set. Then, $\mathfrak{d}$: $\Xi \times \Xi \longrightarrow [0,\infty)$ is called a generalized metric on Ξ, if

(i) $\mathfrak{d}(u,v)=0$, iff $u=v$, $\forall u,v\in\Xi$
(ii) $\mathfrak{d}(u,v)=\mathfrak{d}(v,u)$, $\forall u,v\in\Xi$
(iii) $\mathfrak{d}(u,v)\leq\mathfrak{d}(u,t)+\mathfrak{d}(t,v)$, $\forall u,v,t\in\Xi$

Theorem 1 (see [37]). *Assume that the generalized complete metric space is denoted by* $(\Xi,\mathfrak{d})$*, and let the operator* $\Pi\colon\Xi\longrightarrow\Xi$ *is contractive with the Lipschitz constant* $\ell<1$*. If there is a positive integer r, where* $\mathfrak{d}(\Pi^{r+1}u,\Pi^r u)<\infty$*, for some* $u\in\Xi$*, then the following hold:*

(i) The sequence $\{\Pi^r\}$ *converges to a fixed point* $u_0\in\Pi$

(ii) u_0 *is the unique fixed point of* Π *in* $\Xi^*=\{v\in\Xi|\mathfrak{d}(\Pi^r u,v)<\infty\}$

(iii) if $v\in\Xi^*$*, then* $\mathfrak{d}(v,u_0)\leq 1/1-\ell\mathfrak{d}(\Pi v,v)$

3. Existence and Uniqueness of Solution

Let us start this section by deriving the integral solution of the problem (3), as in the following lemma.

Lemma 4. *Let a function* φ *be continuously differentiable. Then, the solution of the system (3) is equivalent to the following Volterra integral equation:*

$$\begin{aligned}\varphi(u)=&\frac{(u-a)^{\gamma_n-1}}{\Delta}\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}\delta_j\varphi(\epsilon_j)+\frac{(u-a)^{\gamma_n-1}}{\Delta}\sum_{j=1}^{m}\delta_j\mathfrak{I}_{a^+}^{\alpha_n}h\left(\epsilon_j,\varphi(\epsilon_j),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right)\\&+\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\varphi(u)+\mathfrak{I}_{a^+}^{\alpha_n}h\left(u,\varphi(u),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right),\end{aligned}\tag{13}$$

provided that $\Delta=\Gamma(\gamma_n)-\sum_{j=1}^{m}\delta_j(\epsilon_j-a)^{\gamma_n-1}\neq 0$.

Proof. By applying $\mathfrak{I}_{a^+}^{\alpha_n}$ on both sides of (3), then using Lemmas 1 and 3, we get

$$\begin{aligned}\varphi(u)&=\frac{\mathfrak{I}_{a^+}^{1-\gamma_n}\varphi(a^+)}{\Gamma(\gamma_n)}(u-a)^{\gamma_n-1}+\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n}\,{}^H\mathfrak{D}_{a^+}^{\alpha_i,\beta_i}\varphi(u)+\mathfrak{I}_{a^+}^{\alpha_n}h\left(u,\varphi(u),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right)\\&=\frac{\mathfrak{I}_{a^+}^{1-\gamma_n}\varphi(a^+)}{\Gamma(\gamma_n)}(u-a)^{\gamma_n-1}+\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\varphi(u)+\mathfrak{I}_{a^+}^{\alpha_n}h\left(u,\varphi(u),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right).\end{aligned}\tag{14}$$

Now, by putting $u=\epsilon_j$ into equation (14) and multiplying by δ_j, we get

$$\delta_j\varphi(\epsilon_j)=\frac{\delta_j\mathfrak{I}_{a^+}^{1-\gamma_n}\varphi(a^+)}{\Gamma(\gamma_n)}(\epsilon_j-a)^{\gamma_n-1}+\delta_j\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\varphi(\epsilon_j)+\delta_j\mathfrak{I}_{a^+}^{\alpha_n}h\left(\epsilon_j,\varphi(\epsilon_j),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right).\tag{15}$$

Then,

$$\begin{aligned}\mathfrak{I}_{a^+}^{1-\gamma_n}\varphi(a^+)&=\sum_{j=1}^{m}\delta_j\varphi(\epsilon_j)\\&=\frac{\mathfrak{I}_{a^+}^{1-\gamma_n}\varphi(a^+)}{\Gamma(\gamma_n)}\sum_{j=1}^{m}\delta_j(\epsilon_j-a)^{\gamma_n-1}+\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}\delta_j\varphi(\epsilon_j)\\&\quad+\sum_{j=1}^{m}\delta_j\mathfrak{I}_{a^+}^{\alpha_n}h\left(\epsilon_j,\varphi(\epsilon_j),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right),\end{aligned}\tag{16}$$

which yields that

$$\mathfrak{J}_{a^+}^{1-\gamma_n}\varphi(a^+) = \frac{\Gamma(\gamma_n)}{\Delta}\sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}\delta_j\varphi(\epsilon_j) + \frac{\Gamma(\gamma_n)}{\Delta}\sum_{j=1}^{m}\delta_j\mathfrak{J}_{a^+}^{\alpha_n}h\left(\epsilon_j, \varphi(\epsilon_j), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right), \tag{17}$$

where $\Delta = \Gamma(\gamma_n) - \sum_{j=1}^{m}\delta_j(\epsilon_j - a)^{\gamma_n - 1} \neq 0$. Hence, by substituting equation (17) into equation (14), the proof is finished. □

Next, we need to present the following assumptions which are required to investigate the existence and uniqueness of the solution of the problem (3).

(A_1) Suppose that $f_1(\cdot), f_2(\cdot) \geq 0$ are continuous functions and the continuously differentiable function $h\colon I \times \Xi \times \Xi \longrightarrow \Xi$ such that

$$\begin{aligned}&\left\| h(u, (1+u^\rho)\varphi_1(u), (1+u^\rho)\overline{\varphi}_1(u)) - h(u, (1+u^\rho)\varphi_2(u), (1+u^\rho)\overline{\varphi}_2(u))\right\| \leq f_1(u)\left\|\varphi_1(u) - \varphi_2(u)\right\| \\ &\quad + f_2(u)\left\|\overline{\varphi}_1(u) - \overline{\varphi}_2(u)\right\|,\end{aligned} \tag{18}$$

far all $\varphi_1, \varphi_2, \overline{\varphi}_1, \overline{\varphi}_2 \in \Xi$ and $u \in I$.

(A_2) There exist the constants $P, Q, L, \Omega > 0$, such that $0 < P + Q < 1$, which are verifying the following requirements:

$$\begin{aligned}&\sup_{u\in I}\left\{\frac{\overline{\Omega}(u-a)^{\gamma_n-1}}{|\Delta|(1+u^\rho)}\left(\sum_{i=1}^{n-1}|c_i|\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|\delta_j|\left(1+\varepsilon_j^\rho\right) + \sum_{j=1}^{m}|\delta_j|\mathfrak{J}_{a^+}^{\alpha_n}\left[f_1(\varepsilon_j) + f_2(\varepsilon_j)\right]\right)\right\} \leq P, \\ &\sup_{u\in I}\left\{\frac{1}{(1+u^\rho)}\sum_{i=1}^{n-1}\mathfrak{J}_{a^+}^{\eta-\alpha_i}|c_i|(1+u^\rho) + \frac{1}{(1+u^\rho)}\mathfrak{J}_{a^+}^{\eta}\left[f_1(u) + f_2(u)\right]\right\} \leq Q, \\ &\sup_{u\in I}\left\{\frac{\overline{\Omega}(u-a)^{\gamma_n-1}}{|\Delta|(1+u^\rho)}\sum_{j=1}^{m}|\delta_j|\mathfrak{J}_{a^+}^{\alpha_n}\left\|h(\varepsilon_j,0,0)\right\| + \frac{1}{(1+u^\rho)}\mathfrak{J}_{a^+}^{\eta}\|h(u,0,0)\|\right\} \leq L < \infty, \\ &\Omega = \sup_{u\in I}\left\{\frac{\Gamma(\gamma_n)(u-a)^{-\theta}}{\Gamma(\gamma_n-\theta)}\right\} < \infty,\end{aligned} \tag{19}$$

where $\overline{\Omega} = \Omega$ or 1, and $\eta = \alpha_n$ or $\alpha_n - \theta$. □

Theorem 2. *Suppose that (A_1) and (A_2) are fulfilled. Then, the problem (3) has an one and only one solution on unbounded interval I.*

Proof. In the light of Lemma 4, we consider the operator $\Pi\colon \Xi \longrightarrow \Xi$ defined as follows:

$$\begin{aligned}(\Pi\varphi)(u) &= \frac{(u-a)^{\gamma_n-1}}{\Delta}\sum_{i=1}^{n-1}c_i\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}\delta_j\varphi(\epsilon_j) + \frac{(u-a)^{\gamma_n-1}}{\Delta}\sum_{j=1}^{m}\delta_j\mathfrak{J}_{a^+}^{\alpha_n}h\left(\epsilon_j, \varphi(\epsilon_j), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right) \\ &\quad + \sum_{i=1}^{n-1}c_i\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\varphi(u) + \mathfrak{J}_{a^+}^{\alpha_n}h\left(u, \varphi(u), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right).\end{aligned} \tag{20}$$

Clearly, Π is itself mapping. Since for any $\varphi \in \Xi$, by using (A_1) and (A_2), we have

$$\begin{aligned}
\frac{\|(\Pi\varphi)(u)\|}{1+u^{\rho}} &\leq \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|\delta_j|\left\|\varphi(\epsilon_j)\right\| + \frac{1}{(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\|\varphi(u)\| \\
&\quad + \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\mathfrak{J}_{a^+}^{\alpha_n}\left\|h\left(\epsilon_j,\varphi(\epsilon_j),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right)\right\| + \frac{1}{(1+u^{\rho})}\mathfrak{J}_{a^+}^{\alpha_n}\left\|h\left(u,\varphi(u),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right)\right\| \\
&\leq \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\,\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|\delta_j|\left\|\varphi(\epsilon_j)\right\| + \frac{1}{(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\,\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\|\varphi(u)\| \\
&\quad + \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\,\mathfrak{J}_{a^+}^{\alpha_n}\left\|h\left(\epsilon_j,\varphi(\epsilon_j),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right) - h(\epsilon_j,0,0)\right\| \\
&\quad + \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\,\mathfrak{J}_{a^+}^{\alpha_n}\left\|h(\epsilon_j,0,0)\right\| \\
&\quad + \frac{1}{(1+u^{\rho})}\mathfrak{J}_{a^+}^{\alpha_n}\left\|h\left(u,\varphi(u),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right) - h(u,0,0)\right\| + \frac{1}{(1+u^{\rho})}\mathfrak{J}_{a^+}^{\alpha_n}\|h(u,0,0)\| \\
&\leq \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|\delta_j|\left\|\varphi(\epsilon_j)\right\| + \frac{1}{(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\|\varphi(u)\| \\
&\quad + \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\,\mathfrak{J}_{a^+}^{\alpha_n}\left[f_1(\epsilon_j)\frac{\left\|\varphi(\epsilon_j)\right\|}{\left(1+\epsilon_j^{\rho}\right)} + f_2(\epsilon_j)\frac{\left\|{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right\|}{\left(1+\epsilon_j^{\rho}\right)}\right] \\
&\quad + \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\mathfrak{J}_{a^+}^{\alpha_n}\left\|h(\epsilon_j,0,0)\right\| \\
&\quad + \frac{1}{(1+u^{\rho})}\,\mathfrak{J}_{a^+}^{\alpha_n}\left[f_1(u)\frac{\|\varphi(u)\|}{(1+u^{\rho})} + f_2(u)\frac{\left\|{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right\|}{(1+u^{\rho})}\right] + \frac{1}{(1+u^{\rho})}\,\mathfrak{J}_{a^+}^{\alpha_n}\|h(u,0,0)\| \\
&\leq \frac{(u-a)^{\gamma_n-1}\|\varphi\|_{\Xi}}{|\Delta|(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\,\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|\delta_j|\left(1+\epsilon_j^{\rho}\right) + \frac{\|\varphi\|_{\Xi}}{(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\,\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}(1+u^{\rho}) \\
&\quad + \frac{(u-a)^{\gamma_n-1}\|\varphi\|_{\Xi}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\,\mathfrak{J}_{a^+}^{\alpha_n}\left[f_1(\epsilon_j)+f_2(\epsilon_j)\right] + \frac{\|\varphi\|_{\Xi}}{(1+u^{\rho})}\,\mathfrak{J}_{a^+}^{\alpha_n}\left[f_1(u)+f_2(u)\right] \\
&\quad + \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\,\mathfrak{J}_{a^+}^{\alpha_n}\left\|h(\epsilon_j,0,0)\right\| + \frac{1}{(1+u^{\rho})}\,\mathfrak{J}_{a^+}^{\alpha_n}\|h(u,0,0)\| \\
&\leq (P+Q)\|\varphi\|_{\Xi} + L < \infty.
\end{aligned} \tag{21}$$

Also, by using Lemma 3, we find

$$\begin{aligned}
\frac{\left\| {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\Pi\varphi(u)\right\|}{1+u^{\rho}} &\le \frac{\Gamma(\gamma_n)(u-a)^{\gamma_n-\theta-1}}{\Gamma(\gamma_n-\theta)|\Delta|(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|\delta_j|\left\|\varphi(\epsilon_j)\right\| \\
&\quad + \frac{1}{(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i-\theta}\|\varphi(u)\| + \frac{\Gamma(\gamma_n)(u-a)^{\gamma_n-\theta-1}}{\Gamma(\gamma_n-\theta)|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\mathfrak{I}_{a^+}^{\alpha_n}\left\|h\left(\epsilon_j,\varphi(\epsilon_j),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right)\right\| \\
&\quad + \frac{1}{(1+u^{\rho})}\mathfrak{I}_{a^+}^{\alpha_n-\theta}\left\|h\left(u,\varphi(u),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right)\right\| \\
&\le \frac{\Omega(u-a)^{\gamma_n-1}\|\varphi\|_{\Xi}}{|\Delta|(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|c_i|\left(1+\epsilon_j^{\rho}\right) + \frac{\|\varphi\|_{\Xi}}{(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i-\theta}(1+u^{\rho}) \\
&\quad + \frac{\Omega(u-a)^{\gamma_n-1}\|\varphi\|_{\Xi}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\mathfrak{I}_{a^+}^{\alpha_n}\left[f_1(\epsilon_j)+f_2(\epsilon_j)\right] + \frac{\|\varphi\|_{\Xi}}{(1+u^{\rho})}\mathfrak{I}_{a^+}^{\alpha_n-\theta}\left[f_1(u)+f_2(u)\right] \\
&\quad + \frac{\Omega(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\mathfrak{I}_{a^+}^{\alpha_n}\left\|h(\epsilon_j,0,0)\right\| + \frac{1}{(1+u^{\rho})}\mathfrak{I}_{a^+}^{\alpha_n-\theta}\|h(u,0,0)\| \\
&\le (P+Q)\|\varphi\|_{\Xi} + L < \infty.
\end{aligned} \tag{22}$$

In the following, we establish that Π is contractive mapping on Ξ. For any $\varphi,\overline{\varphi}\in\Xi$, and by using (A_1) and (A_2), we get

$$\begin{aligned}
\frac{\|(\Pi\varphi)(u)-(\Pi\overline{\varphi})(u)\|}{1+u^{\rho}} &\le \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|\delta_j|\left\|\varphi(\epsilon_j)-\overline{\varphi}(\epsilon_j)\right\| \\
&\quad + \frac{1}{(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\|\varphi(u)-\overline{\varphi}(u)\| \\
&\quad + \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\mathfrak{I}_{a^+}^{\alpha_n}\left\|h\left(\epsilon_j,\varphi(\epsilon_j),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right)-h\left(\epsilon_j,\overline{\varphi}(\epsilon_j),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\overline{\varphi}(\epsilon_j)\right)\right\| \\
&\quad + \frac{1}{(1+u^{\rho})}\mathfrak{I}_{a^+}^{\alpha_n}\left\|h\left(u,\varphi(u),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right)-h\left(u,\overline{\varphi}(u),{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\overline{\varphi}(u)\right)\right\| \\
&\le \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|\delta_j|\left\|\varphi(\epsilon_j)\right\| + \frac{1}{(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\|\varphi(u)\| \\
&\quad + \frac{(u-a)^{\gamma_n-1}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\mathfrak{I}_{a^+}^{\alpha_n}\left[f_1(\epsilon_j)\frac{\left\|\varphi(\epsilon_j)-\overline{\varphi}(\epsilon_j)\right\|}{\left(1+\epsilon_j^{\rho}\right)} + f_2(\epsilon_j)\frac{\left\|{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)-{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\overline{\varphi}(\epsilon_j)\right\|}{\left(1+\epsilon_j^{\rho}\right)}\right] \\
&\quad + \frac{1}{(1+u^{\rho})}\mathfrak{I}_{a^+}^{\alpha_n}\left[f_1(u)\frac{\|\varphi(u)-\overline{\varphi}(u)\|}{(1+u^{\rho})} + f_2(u)\frac{\left\|{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)-{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\overline{\varphi}(u)\right\|}{(1+u^{\rho})}\right] \\
&\le \frac{(u-a)^{\gamma_n-1}\|\varphi-\overline{\varphi}\|_{\Xi}}{|\Delta|(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}|\delta_j|\left(1+\epsilon_j^{\rho}\right) + \frac{\|\varphi-\overline{\varphi}\|_{\Xi}}{(1+u^{\rho})}\sum_{i=1}^{n-1}|c_i|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}(1+u^{\rho}) \\
&\quad + \frac{(u-a)^{\gamma_n-1}\|\varphi-\overline{\varphi}\|_{\Xi}}{|\Delta|(1+u^{\rho})}\sum_{j=1}^{m}|\delta_j|\mathfrak{I}_{a^+}^{\alpha_n}\left[f_1(\epsilon_j)+f_2(\epsilon_j)\right] + \frac{\|\varphi-\overline{\varphi}\|_{\Xi}}{(1+u^{\rho})}\mathfrak{I}_{a^+}^{\alpha_n}\left[f_1(u)+f_2(u)\right] \\
&\le (P+Q)\|\varphi-\overline{\varphi}\|_{\Xi}.
\end{aligned} \tag{23}$$

Similarly,

$$\begin{aligned}
&\frac{\left\| {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\,\Pi\varphi(u) - {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\,\Pi\overline{\varphi}(u)\right\|}{1+u^{\rho}} \\
&\le \frac{\Omega(u-a)^{\gamma_n-1}\|\varphi-\overline{\varphi}\|_{\Xi}}{|\Delta|\left(1+u^{\rho}\right)}\sum_{i=1}^{n-1}\left|c_i\right|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}\left|\delta_j\right|\left(1+\epsilon_j^{\rho}\right) \\
&\quad+\frac{\|\varphi-\overline{\varphi}\|_{\Xi}}{\left(1+u^{\rho}\right)}\sum_{i=1}^{n-1}\left|c_i\right|\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i-\theta}\frac{\left(1+u^{\rho}\right)}{(u-a)^{1-\gamma_n}} \\
&\quad+\frac{\Omega(u-a)^{\gamma_n-1}\|\varphi-\overline{\varphi}\|_{\Xi}}{|\Delta|\left(1+u^{\rho}\right)}\sum_{j=1}^{m}\left|\delta_j\right|\mathfrak{I}_{a^+}^{\alpha_n}\left[f_1\left(\epsilon_j\right)+f_2\left(\epsilon_j\right)\right] \\
&\quad+\frac{\|\varphi-\overline{\varphi}\|_{\Xi}}{\left(1+u^{\rho}\right)}\mathfrak{I}_{a^+}^{\alpha_n-\theta}\left[f_1(u)+f_2(u)\right] \\
&\le (P+Q)\|\varphi-\overline{\varphi}\|_{\Xi}.
\end{aligned} \tag{24}$$

Therefore, we deduce that $\|\Pi\varphi-\Pi\overline{\varphi}\|_{\Xi}\le(P+Q)\|\varphi-\overline{\varphi}\|_{\Xi}$, which yields that Π is a contractive operator, since $P+Q\in(0,1)$. In view of the Banach contraction principle, Π has an one and only one fixed point φ_0 in Ξ, which is verifying $\Pi\varphi_0=\varphi_0$. Hence, problem (3) has an one and only one solution on unbounded interval (a,∞). □

4. Stability of Solution

In this situation, in order to discuss UHR stability, UH stability, and sUHR stability, we need to introduce applicable metrics $\mathfrak{d}_1(\cdot)$ and $\mathfrak{d}_2(\cdot)$ on Banach space Ξ. The metric $\mathfrak{d}_1(\cdot)$ is defined by

$$\mathfrak{d}_1(\varphi,\overline{\varphi})=\inf_{u\in I}\left\{M\in I\,|\,\frac{\|\varphi(u)-\overline{\varphi}(u)\|}{1+u^{\rho}}\le M\psi(u),\frac{\left\|{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)-{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\overline{\varphi}(u)\right\|}{1+u^{\rho}}\le M\psi(u)\right\}, \tag{25}$$

where $\psi(u)$ is a nonnegative increasing continuous function on unbounded interval I. Also, the metric $\mathfrak{d}_2(\cdot)$ is defined by

$$\mathfrak{d}_2(\varphi,\overline{\varphi})=\sup_{u\in I}\left\{M\in I\,|\,\frac{\|\varphi(u)-\overline{\varphi}(u)\|}{\psi(u)\left(1+u^{\rho}\right)}\le M,\frac{\left\|{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)-{}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\overline{\varphi}(u)\right\|}{\psi(u)\left(1+u^{\rho}\right)}\le M\right\}, \tag{26}$$

where $\psi(u)$ is a nonnegative decreasing continuous function on unbounded interval I. We can guarantee that $\mathfrak{d}_1(\cdot)$ and $\mathfrak{d}_2(\cdot)$ are metrics on Banach space Ξ, as given in the work [38] and references therein.

In the following, we will give the definitions of UHR, UH, and sUHR stability, and then state and prove their theorems.

Definition 6 (see [12]). The solution of the problem (3) is UHR stable if for every continuously differentiable function $\varphi: I = (a, \infty) \longrightarrow \Xi$ verifying

$$\begin{aligned}
&\Bigg\| \varphi(u) - \frac{(u-a)^{\gamma_n - 1}}{\Delta} \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i} \sum_{j=1}^{m} \delta_j \varphi(\epsilon_j) - \frac{(u-a)^{\gamma_n - 1}}{\Delta} \sum_{j=1}^{m} \delta_j \mathfrak{J}_{a^+}^{\alpha_n} h\Big(\epsilon_j, \varphi(\epsilon_j), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(\epsilon_j)\Big) \\
&- \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i} \varphi(u) - \mathfrak{J}_{a^+}^{\alpha_n} h\Big(u, \varphi(u), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(u)\Big) \Bigg\| \\
&\leq \mathfrak{J}_{a^+}^{\alpha_n} \psi(u), u \in I, \\
&\Bigg\| {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(u) - \frac{\Gamma(\gamma_n)(u-a)^{\gamma_n - \theta - 1}}{\Gamma(\gamma_n - \theta)\Delta} \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i} \sum_{j=1}^{m} \delta_j \varphi(\epsilon_j) - \frac{\Gamma(\gamma_n)(u-a)^{\gamma_n - \theta - 1}}{\Gamma(\gamma_n - \theta)\Delta} \sum_{j=1}^{m} \delta_j \mathfrak{J}_{a^+}^{\alpha_n} h\Big(\epsilon_j, \varphi(\epsilon_j), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(\epsilon_j)\Big) \\
&- \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i - \theta} \varphi(u) - \mathfrak{J}_{a^+}^{\alpha_n - \theta} h\Big(u, \varphi(u), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(u)\Big) \Bigg\| \\
&\leq \mathfrak{J}_{a^+}^{\alpha_n - \theta} \psi(u), u \in I,
\end{aligned} \tag{27}$$

where $\psi(u)$ is a nonnegative nondecreasing continuous function on unbounded interval I, and there is a unique solution φ_0 of the problem (3), and a constant $M > 0$ independent of φ, φ_0, where

$$\begin{aligned}
&\frac{\|\varphi(u) - \varphi_0(u)\|}{1 + u^{\rho}} \leq M\psi(u), \forall u \in I, \\
&\frac{\left\| {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(u) - {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi_0(u) \right\|}{1 + u^{\rho}} \leq M\psi(u), \forall u \in I.
\end{aligned} \tag{28}$$

Moreover, if we replace $\psi(u)$ by $\omega \geq 0$, then the solution of the problem (3) is UH stable.

Definition 7 (see [12]). The solution of the problem (3) is sUHR stable, if for every continuously differentiable function $\varphi: I = (a, \infty) \longrightarrow \Xi$ verifying

$$\begin{aligned}
&\Bigg\| \varphi(u) - \frac{(u-a)^{\gamma_n - 1}}{\Delta} \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i} \sum_{j=1}^{m} \delta_j \varphi(\epsilon_j) - \frac{(u-a)^{\gamma_n - 1}}{\Delta} \sum_{j=1}^{m} \delta_j \mathfrak{J}_{a^+}^{\alpha_n} h\Big(\epsilon_j, \varphi(\epsilon_j), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(\epsilon_j)\Big) \\
&- \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i} \varphi(u) - \mathfrak{J}_{a^+}^{\alpha_n} h\Big(u, \varphi(u), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(u)\Big) \Bigg\| \\
&\leq \mathfrak{J}_{a^+}^{\alpha_n} \omega, u \in I, \\
&\Bigg\| {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(u) - \frac{\Gamma(\gamma_n)(u-a)^{\gamma_n - \theta - 1}}{\Gamma(\gamma_n - \theta)\Delta} \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i} \sum_{j=1}^{m} \delta_j \varphi(\epsilon_j) - \frac{\Gamma(\gamma_n)(u-a)^{\gamma_n - \theta - 1}}{\Gamma(\gamma_n - \theta)\Delta} \sum_{j=1}^{m} \delta_j \mathfrak{J}_{a^+}^{\alpha_n} h\Big(\epsilon_j, \varphi(\epsilon_j), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(\epsilon_j)\Big) \\
&- \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i - \theta} \varphi(u) - \mathfrak{J}_{a^+}^{\alpha_n - \theta} h\Big(u, \varphi(u), {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta} \varphi(u)\Big) \Bigg\| \\
&\leq \mathfrak{J}_{a^+}^{\alpha_n - \theta} \omega, \quad u \in I,
\end{aligned} \tag{29}$$

where $\omega \geq 0$, and there is a unique solution φ_0 of the problem (3), and a constant $M > 0$ independent of φ, φ_0 for some positive decreasing continuous function $\psi(u)$ on unbounded interval I, where

$$\begin{aligned} \frac{\left\|\varphi(u) - \varphi_0(u)\right\|}{1 + u^\rho} &\leq M\psi(u), \quad \forall u \in I, \\ \frac{\left\|{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u) - {}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi_0(u)\right\|}{1 + u^\rho} &\leq M\psi(u), \quad \forall u \in I. \end{aligned} \tag{30}$$

Theorem 3. *Suppose that* (A_1) *and* (A_2) *are fulfilled,* $\psi(u)$ *be a positive continuous increasing function on unbounded interval* I, *and* φ: $I = (a, \infty) \longrightarrow \Xi$ *is continuously differentiable function verifying*

$$\begin{aligned} &\Bigg\|\varphi(u) - \frac{(u-a)^{\gamma_n - 1}}{\Delta}\sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i}\sum_{j=1}^{m}\delta_j\varphi(\epsilon_j) - \frac{(u-a)^{\gamma_n - 1}}{\Delta}\sum_{j=1}^{m}\delta_j \mathfrak{J}_{a^+}^{\alpha_n} h\Big(\epsilon_j, \varphi(\epsilon_j), {}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\Big) - \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i}\varphi(u) \\ &- \mathfrak{J}_{a^+}^{\alpha_n} h\Big(u, \varphi(u), {}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\Big)\Bigg\| \leq \mathfrak{J}_{a^+}^{\alpha_n}\psi(u), \quad u \in I, \end{aligned} \tag{31}$$

$$\begin{aligned} &\Bigg\|{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u) - \frac{\Gamma(\gamma_n)(u-a)^{\gamma_n - \theta - 1}}{\Gamma(\gamma_n - \theta)\Delta}\sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i}\sum_{j=1}^{m}\delta_j\varphi(\epsilon_j) - \frac{\Gamma(\gamma_n)(u-a)^{\gamma_n - \theta - 1}}{\Gamma(\gamma_n - \theta)\Delta}\sum_{j=1}^{m}\delta_j \mathfrak{J}_{a^+}^{\alpha_n} h\Big(\epsilon_j, \varphi(\epsilon_j), {}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\Big) \\ &- \sum_{i=1}^{n-1} c_i \mathfrak{J}_{a^+}^{\alpha_n - \alpha_i - \theta}\varphi(u) - \mathfrak{J}_{a^+}^{\alpha_n - \theta} h\Big(u, \varphi(u), {}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\Big)\Bigg\| \leq \mathfrak{J}_{a^+}^{\alpha_n - \theta}\psi(u), \quad u \in I. \end{aligned} \tag{32}$$

Then, there is one and only one solution $\varphi_0 \in \Xi$ such that

$$\begin{aligned} \frac{\left\|\varphi(u) - \varphi_0(u)\right\|}{1 + u^\rho} &\leq \frac{K}{1 - (P + Q)}\psi(u), \quad \forall u \in I, 0 < (P + Q) < 1, \\ \frac{\left\|{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u) - {}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi_0(u)\right\|}{1 + u^\rho} &\leq \frac{K}{1 - (P + Q)}\psi(u), \quad \forall u \in I, 0 < (P + Q) < 1, \end{aligned} \tag{33}$$

where $\sup_u \in I(u-a)^\eta/\Gamma(\eta + 1)(1 + u^\rho) \leq K < \infty$, for $\eta = \alpha_n$ or $\alpha_n - \theta$, which yields that the solution of problem (3) is UHR stable and consequently is UH stable.

Proof. Let Π: $\Xi \longrightarrow \Xi$ be the contractive operator as given in (20).

Now, for $\varphi, \overline{\varphi} \in \Xi$, we present from metric $\mathfrak{d}_1(\cdot)$ and the assumptions (A_1), (A_2) that

$$\begin{aligned} \frac{\|(\Pi\varphi)(u) - (\Pi\overline{\varphi})(u)\|}{1 + u^\rho} \leq &\frac{M\psi(u)(u-a)^{\gamma_n - 1}}{|\Delta|(1 + u^\rho)}\sum_{i=1}^{n-1}|c_i|\mathfrak{J}_{a^+}^{\alpha_n - \alpha_i}\sum_{j=1}^{m}|\delta_j|\big(1 + \epsilon_j^\rho\big) \\ &+ \frac{M\psi(u)}{(1 + u^\rho)}\sum_{i=1}^{n-1}|c_i|\mathfrak{J}_{a^+}^{\alpha_n - \alpha_i}(1 + u^\rho) \\ &+ \frac{M\psi(u)(u-a)^{\gamma_n - 1}}{|\Delta|(1 + u^\rho)}\sum_{j=1}^{m}|\delta_j|\mathfrak{J}_{a^+}^{\alpha_n}\big[f_1(\epsilon_j) + f_2(\epsilon_j)\big] \\ &+ \frac{M\psi(u)}{(1 + u^\rho)}\mathfrak{J}_{a^+}^{\alpha_n}\big[f_1(u) + f_2(u)\big] \end{aligned}$$

$$\leq (P+Q)M\psi(u), \quad \forall u \in I, 0<(P+Q)<1,$$

$$\begin{aligned}\frac{\left\| {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\Pi\varphi(u) - {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\Pi\overline{\varphi}(u)\right\|}{1+u^{\rho}} &\leq \frac{M\Omega\psi(u)(u-a)^{\gamma_n-1}}{|\Delta|\left(1+u^{\rho}\right)}\sum_{i=1}^{n-1}\left|c_i\right|\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}\left|\delta_j\right|\left(1+\epsilon_j^{\rho}\right) \\ &+ \frac{M\psi(u)}{\left(1+u^{\rho}\right)}\sum_{i=1}^{n-1}\left|c_i\right|\mathfrak{J}_{a^+}^{\alpha_n-\alpha_i-\theta}\left(1+u^{\rho}\right) \\ &+ \frac{M\Omega\psi(u)(u-a)^{\gamma_n-1}}{|\Delta|\left(1+u^{\rho}\right)}\sum_{j=1}^{m}\left|\delta_j\right|\mathfrak{J}_{a^+}^{\alpha_n}\left[f_1\left(\epsilon_j\right)+f_2\left(\epsilon_j\right)\right] \\ &+ \frac{M\psi(u)}{\left(1+u^{\rho}\right)}\mathfrak{J}_{a^+}^{\alpha_n-\theta}\left[f_1(u)+f_2(u)\right] \\ &\leq (P+Q)M\psi(u), \quad \forall u \in I, 0<(P+Q)<1. \end{aligned} \tag{34}$$

Hence, we obtain

$$\mathfrak{d}_1\left(\Pi\varphi,\Pi\overline{\varphi}\right) \leq (P+Q)M = (P+Q)\mathfrak{d}_1\left(\varphi,\overline{\varphi}\right), 0<(P+Q)<1. \tag{35}$$

In view of the inequalities (31) and (32), we have

$$\frac{\|(\varphi)(u)-(\Pi\varphi)(u)\|}{1+u^{\rho}} \leq \sup_{u\in I}\frac{(u-a)^{\alpha_n}}{\Gamma\left(\alpha_n+1\right)\left(1+u^{\rho}\right)}\psi(u) = K\psi(u), \quad u \in I, \tag{36}$$

$$\frac{\left\| {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u) - {}^{H}\mathfrak{D}_{a^+}^{\theta,\vartheta}\Pi\varphi(u)\right\|}{1+u^{\rho}} \leq \sup_{u\in I}\frac{(u-a)^{\alpha_n-\theta}}{\Gamma\left(\alpha_n-\theta+1\right)\left(1+u^{\rho}\right)}\psi(u) = K\psi(u), \quad u \in I. \tag{37}$$

Due to the inequalities (36) and (37), we get

$$\mathfrak{d}_1(\varphi,\Pi\varphi) \leq K < \infty. \tag{38}$$

Based on (i) and (ii) of Theorem 1, there is one and only one fixed point φ_0 such that $\Pi\varphi_0 = \varphi_0$. As a consequence of (iii) of Theorem 1, we can conclude that

$$\mathfrak{d}_1\left(\varphi,\varphi_0\right) \leq \frac{1}{1-(P+Q)}\mathfrak{d}_1(\Pi\varphi,\varphi) \leq \frac{K}{1-(P+Q)}, 0<(P+Q)<1. \tag{39}$$

According to the above conclusions, the solution of problem (3) is UHR stable. Along with this, if $\psi(u)=1$, then the solution of problem (3) is UH stable. □

Theorem 4. *Suppose that* (A_1) *and* (A_2) *are fulfilled,* $\psi(u)$ *be a positive decreasing continuous function on unbounded interval* I*, and* φ: $I=(a,\infty) \longrightarrow \Xi$ *is continuously differentiable function verifying*

$$\left\|\varphi(u)-\frac{(u-a)^{\gamma_n-1}}{\Delta}\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}\delta_j\varphi(\epsilon_j)-\frac{(u-a)^{\gamma_n-1}}{\Delta}\sum_{j=1}^{m}\delta_j\mathfrak{I}_{a^+}^{\alpha_n}h\left(\epsilon_j,\varphi(\epsilon_j),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right)-\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\varphi(u)\right.$$
$$\left.-\mathfrak{I}_{a^+}^{\alpha_n}h\left(u,\varphi(u),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right)\right\|\leq\mathfrak{I}_{a^+}^{\alpha_n}\omega,\quad u\in I, \tag{40}$$

$$\left\|{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)-\frac{\Gamma(\gamma_n)(u-a)^{\gamma_n-\theta-1}}{\Gamma(\gamma_n-\theta)\Delta}\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i}\sum_{j=1}^{m}\delta_j\varphi(\epsilon_j)-\frac{\Gamma(\gamma_n)(u-a)^{\gamma_n-\theta-1}}{\Gamma(\gamma_n-\theta)\Delta}\sum_{j=1}^{m}\delta_j\mathfrak{I}_{a^+}^{\alpha_n}h\left(\epsilon_j,\varphi(\epsilon_j),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(\epsilon_j)\right)\right.$$
$$\left.-\sum_{i=1}^{n-1}c_i\mathfrak{I}_{a^+}^{\alpha_n-\alpha_i-\theta}\varphi(u)-\mathfrak{I}_{a^+}^{\alpha_n-\theta}h\left(u,\varphi(u),{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)\right)\right\|\leq\mathfrak{I}_{a^+}^{\alpha_n-\theta}\omega,\quad u\in I, \tag{41}$$

where $\omega>0$. Then, there is one and only one solution $\varphi_0\in\Xi$, and a constant $\Psi>0$ such that

$$\frac{\|\varphi(u)-\varphi_0(u)\|}{1+u^\rho}\leq\frac{\omega K\Psi}{1-(P+Q)}\psi(u),\quad\forall u\in I, 0<(P+Q)<1,$$
$$\frac{\left\|{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)-{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi_0(u)\right\|}{1+u^\rho}\leq\frac{\omega K\Psi}{1-(P+Q)}\psi(u),\quad\forall u\in I, 0<(P+Q)<1, \tag{42}$$

where $\sup_u\in I(u-a)^\eta/\Gamma(\eta+1)(1+u^\rho)\leq K<\infty$, for $\eta=\alpha_n$ or $\alpha_n-\theta$, which yields that the solution of problem (3) is sUHR stable.

Proof. Similar to Theorem 3, we take the contractive operator $\Pi:\Xi\longrightarrow\Xi$ as given in (20). From metric $\mathfrak{d}_2(\cdot)$ and assumptions (A_1), (A_2), we find that

$$\frac{\|(\Pi\varphi)(u)-(\Pi\overline{\varphi})(u)\|}{\psi(u)(1+u^\rho)}\leq(P+Q)M,\quad\forall u\in I, 0<(P+Q)<1,$$
$$\frac{\left\|{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\Pi\varphi(u)-{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\Pi\overline{\varphi}(u)\right\|}{\psi(u)(1+u^\rho)}\leq(P+Q)M,\quad\forall u\in I, 0<(P+Q)<1. \tag{43}$$

Then, we get

$$\mathfrak{d}_2(\Pi\varphi,\Pi\overline{\varphi})\leq(P+Q)M=(P+Q)\mathfrak{d}_2(\varphi,\overline{\varphi}), 0<(P+Q)<1. \tag{44}$$

Due to the positiveness, continuity of a decreasing function $\psi(u),\forall u\in I$, we find

$$\frac{1}{\psi(u)}\leq\Psi,\quad\forall u\in I, 0<\Psi. \tag{45}$$

Based on the inequalities (40) and (41), we obtain

$$\frac{\|(\varphi)(u)-(\Pi\varphi)(u)\|}{\psi(u)(1+u^\rho)}\leq\sup_{u\in I}\frac{\omega(u-a)^{\alpha_n}}{\psi(u)\Gamma(\alpha_n+1)(1+u^\rho)}=K\Psi\omega,\quad u\in I, \tag{46}$$

$$\frac{\left\|{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\varphi(u)-{}^H\mathfrak{D}_{a^+}^{\theta,\vartheta}\Pi\varphi(u)\right\|}{\psi(u)(1+u^\rho)}\leq\sup_{u\in I}\frac{\omega(u-a)^{\alpha_n-\theta}}{\psi(u)\Gamma(\alpha_n-\theta+1)(1+u^\rho)}=K\Psi\omega,\quad u\in I. \tag{47}$$

From the inequalities (46) and (47), we have

$$\mathfrak{d}_2(\varphi, \Pi\varphi) \le K\Psi\omega < \infty. \tag{48}$$

In the light of (i) and (ii) of Theorem 1, there is one and only one fixed point φ_0 such that $\Pi\varphi_0 = \varphi_0$. As consequence from (iii) of Theorem 1, we can deduce that

$$\mathfrak{d}_2(\varphi, \varphi_0) \le \frac{1}{1-(P+Q)}\mathfrak{d}_2(\Pi\varphi, \varphi) \le \frac{K\Psi\omega}{1-(P+Q)}, 0 < (P+Q) < 1. \tag{49}$$

According to the above conclusions, the solution of the problem (3) is sUHR stable, and the proof is finished. □

5. Conclusion

This paper declared that, by convenience Hilfer fractional derivative and Banach contraction principle in an applicable Banach space, the nonlocal multiorder implicit differential equation (3), on unbounded domains $(a, \infty), a \ge 0$, provides existence and uniqueness of the solution as well as the stability of UHR, UH, and sUHR. Our problem is more general and returns to the sense of the Caputo fractional problem when $\beta_i, \vartheta = 1, (i = 1, 2, \ldots, n)$ and returns to the sense of the R-L fractional problem when $\beta_i, \vartheta = 0, (i = 1, 2, \ldots, n)$. Moreover, these results were generalized to include many outcomes are existed in the literature, for example, the results in [24, 31]. As a future target, the studied problem with this approach would be more exciting if it was discussed under integral value conditions via ψ−Hilfer fractional operators.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have equal contributions. All authors read and approved the final manuscript.

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References

[1] V. S. Erturk, A. Ali, K. Shah, P. Kumar, and T. Abdeljawad, "Existence and stability results for nonlocal boundary value problems of fractional order," *Boundary Value Problems*, vol. 2022, no. 25, p. 15, 2022.
[2] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co., Inc, Singapore, 2000.
[3] S. Abbas, M. Benchohra, J. E. Lazreg, and Y. Zhou, "A survey on Hadamard and Hilfer fractional differential equations: analysis and stability," *Chaos, Solitons & Fractals*, vol. 102, pp. 47–71, 2017.
[4] V. M. Bulavatsky, "Closed form of the solutions of some boundary-value problems for anomalous diffusion equation with Hilfer's generalized derivative," *Cybernetics and Systems Analysis*, vol. 50, no. 4, pp. 570–577, 2014.
[5] R. Kamocki, "A new representation formula for the Hilfer fractional derivative and its application," *Journal of Computational and Applied Mathematics*, vol. 308, pp. 39–45, 2016.
[6] R. Hilfer, Y. Luchko, and Z. Tomovski, "Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives," *Fract. Calc. Appl. Anal.*, vol. 12, pp. 299–318, 2009.
[7] K. M. Furati, M. D. Kassim, and N. E. Tatar, "Existence and uniqueness for a problem involving Hilfer fractional derivative," *Computers & Mathematics with Applications*, vol. 64, no. 6, pp. 1616–1626, 2012.
[8] R. Kamocki and C. Obczynski, "On fractional Cauchy-type problems containing Hilfer's derivative," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 50, pp. 1–12, 2016.
[9] S. T. M. Thabet, B. Ahmad, and R. P. Agarwal, "On abstract Hilfer fractional integrodifferential equations with boundary conditions," *Arab Journal of Mathematical Sciences*, vol. 26, no. 1/2, pp. 107–125, 2020.
[10] M. S. Abdo, S. T. M. Thabet, and B. Ahmad, "The existence and Ulam-Hyers stability results for ψ-Hilfer fractional integro-differential equations," *Journal of Pseudo-differential Operators and Applications*, vol. 11, pp. 1757–1780, 2020.
[11] S. Rezapour, S. T. M. Thabet, M. M. Matar, J. Alzabut, and S. Etemad, "Some existence and stability criteria to a generalized FBVP having fractional composite p-Laplacian operator," *Journal of Function Spaces*, vol. 2021, Article ID 9554076, 10 pages, 2021.
[12] E. C. De Oliveira and J. V. C. Sousa, "Ulam–Hyers–Rassias stability for a class of fractional integro-differential equations," *Results in Mathematics*, vol. 73, no. 3, p. 111, 2018.
[13] S. Abbas, M. Benchohra, and A. M. Darwish, "Asymptotic stability for implicit Hilfer fractional differential equations," *Panam. Math. J., vo.*, vol. 27, no. 3, pp. 40–52, 2017.
[14] M. I. Ayari and S. T. M. Thabet, "Qualitative properties and approximate solutions of thermostat fractional dynamics system via a nonsingular kernel operator," *Arab Journal of Mathematical Sciences*, vol. 21, p. 2023, 2023.
[15] S. T. M. Thabet and M. B. Dhakne, "On nonlinear fractional integro-differential equations with two boundary conditions," *Adv. Stud. Contemp. Math.*vol. 26, pp. 513–526, 2016.
[16] S. Abbas, M. Benchohra, and M. Bohner, "Weak solutions for implicit differential equations with Hilfer–Hadamard fractional derivative," *Advances in Dynamical Systems and Applications*, vol. 12, p. 16, 2017.

[17] A. Ndiaye, "A nonlinear implicit fractional equation with Caputo derivative," *Journal of Mathematics*, vol. 2021, Article ID 5547003, 9 pages, 2021.

[18] F. Fredj and H. Hammouche, "On existence results for hybrid ψ-Caputo multi-fractional differential equations with hybrid conditions," CUBO," *The Mathematica Journal*, vol. 24, no. 2, p. 273—289, 2022.

[19] D. Vivek, K. Kanagarajan, and S. Sivasundaram, "Dynamics and stability results for Hilfer fractional type thermistor problem," *Fractal Fract*, vol. 1, no. 1, p. 5, 2017.

[20] S. Andras and J. J. Kolumban, "On the Ulam–Hyers stability of first order differential systems with nonlocal initial conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 82, p. 11, 2013.

[21] M. I. Abbas, M. Ghaderi, S. Rezapour, and S. T. M. . Thabet, "On a coupled system of fractional differential equations via the generalized proportional fractional derivatives," *Journal of Function Spaces*, vol. 2022, Article ID 4779213, 10 pages, 2022.

[22] S. T. M. Thabet, M. B. Dhakne, M. A. Salman, and R. Gubran, "Generalized fractional Sturm-Liouville and Langevin equations involving Caputo derivative with nonlocal conditions," *Progress in Fractional Differentiation and Applications*, vol. 6, no. 3, pp. 225–237, 2020.

[23] M. Benchohra and M. Said, "Souid, L^1-solutions for implicit fractional order differential equations with nonlocal conditions," *Filomat*, vol. 30, no. 6, pp. 1485–1492, 2016.

[24] D. Vivek, K. Kanagarajan, and E. M. Elsayed, "Some existence and stability results for Hilfer-fractional implicit differential equations with nonlocal conditions," *Mediterranean Journal of Mathematics*, vol. 15, no. 1, p. 15, 2018.

[25] J. Zhou, S. Zhang, and Y. He, "Existence and stability of solution for nonlinear differential equations with ψ-Hilfer fractional derivative," *Applied Mathematics Letters*, vol. 121, Article ID 107457, 2021.

[26] J. L. Zhou, S. Q. Zhang, and Y. B. He, "Existence and stability of solution for a nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 498, no. 1, Article ID 124921, 2021.

[27] A. Ali, I. Mahariq, K. Shah, T. Abdeljawad, and B. Al-Sheikh, "Stability analysis of initial value problem of pantograph-type implicit fractional differential equations with impulsive conditions," *Advances in Difference Equations*, vol. 2021, no. 1, p. 55, 2021.

[28] J. L. Zhou, Y. B. He, S. Q. Zhang, H. Y. Deng, and X. Y. Lin, "Existence and stability results for nonlinear fractional integrodifferential coupled systems," *Boundary Value Problems*, vol. 2023, no. 1, p. 10, 2023.

[29] M. A. Almalahi, O. Bazighifan, S. K. Panchal, S. S. Askar, and G. I. Oros, "Analytical study of two nonlinear coupled hybrid systems involving generalized Hilfer fractional operators," *Fractal Fract*, vol. 5, no. 4, p. 178, 2021.

[30] M. A. Almalahi, M. S. Abdo, and S. K. Panchal, "ψ-Hilfer fractional functional differential equation by Picard operator method," *Journal of Applied Nonlinear Dynamics*, vol. 9, no. 4, pp. 685–702, 2020.

[31] L. Xie, J. Zhou, H. Deng, and Y. He, "Existence and stability of solution for multi-order nonlinear fractional differential equations," *AIMS Mathematics*, vol. 7, no. 9, Article ID 16440, 2022.

[32] X. Su and S. Zhang, "Unbounded solutions to a boundary value problem of fractional order on the halfline," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1079–1087, 2011.

[33] X. Su, "Solutions to boundary value problem of fractional order on unbounded domains in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 8, pp. 2844–2852, 2011.

[34] C. Kou, H. Zhou, and Y. Yan, "Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 17, pp. 5975–5986, 2011.

[35] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B.V, Amsterdam, Netherland, 2006.

[36] K. Diethelm, "The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type," *Lecture Notes in Mathematics*, Vol. 2004, Springer-Verlag, Berlin, Germany, 2010.

[37] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, no. 2, pp. 305–309, 1968.

[38] L. Cădariu, L. Găvruta, and P. Găvruta, "Weighted space method for the stability of some nonlinear equations," *Applicable Analysis and Discrete Mathematics*, vol. 6, pp. 126–139, 2012.