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Research Article

Existence Theorems for Hybrid Fractional Differential Equations with ψ-Weighted Caputo–Fabrizio Derivatives

Mohammad Alshammari,[1] **Saleh Alshammari**,[1] **and Mohammed S. Abdo**[2,3]

[1]*Department of Mathematics, College of Science, University of Háil, Háil 2440, Saudi Arabia*
[2]*Department of Mathematics, Hodeidah University, P.O. Box 3114, Al-Hudaydah, Yemen*
[3]*Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia*

Correspondence should be addressed to Mohammed S. Abdo; msabdo@hoduniv.net.ye

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In this study, two classes of hybrid boundary value problems involving ψ-weighted Caputo–Fabrizio fractional derivatives are considered. Based on the properties of the given operator, we construct the hybrid fractional integral equations corresponding to the hybrid fractional differential equations. Then, we establish and extend the existence theory for given problems in the class of continuous functions by Dhage's fixed point theory. Furthermore, as special cases, we offer further analogous and comparable conclusions. Finally, we give two examples as applications to illustrate and validate the results.

1. Introduction

The theory of fractional calculus has attracted the attention of a remarkable number of researchers from various fields in recent years. The physical meaning of fractional orders is that the dynamical systems of fractional order can be represented by a fractional differential equation (FDE) with a noninteger derivative. These systems are referred described as having fractional dynamics. Undoubtedly, it has been demonstrated that the use of fractional derivatives (FDs) is very beneficial for modeling a wide range of problems and natural phenomena in engineering and applied sciences; for example, see renowned monographs by Osler [1], Samko et al. [2], Kilbas et al. [3], and Diethelm and Ford [4]. The literature contains a variety of FD concepts, including those presented by Riemann–Liouville and Caputo [3], which include the singular kernel $k(t,s) = (t-s)^{-\nu}/\Gamma(1-\nu)$, $0<\nu<1$.

These FDs play an important role in modeling numerous physical and biological phenomena. In any case, as was referenced in Caputo and Fabrizio [5, 6], certain peculiarities connected with material heterogeneities cannot be well modeled utilizing Riemann–Liouville or Caputo FDs. Because of this reality, the authors in [5] proposed another FD involving the nonsingular kernel $k(t,s) = \exp(-\nu\ (t-s)/1-\nu)$, $0<\nu<1$; then, Losada and Nieto [7] studied some of its properties. The existence and uniqueness of solutions are essential properties of mathematical models [8, 9] and are among the advantages of applied theory. The model must possess these properties in order to be reliable and useful. Existence refers to the fact that the model must describe a well-defined problem, which can be solved within a certain mathematical framework. In other words, the model should not have any ambiguity or inconsistency that would make it impossible to solve. In [10–18], the authors studied the existence of solutions for various types of FDEs involving the Caputo–Fabrizio FD and other fractional operators. For instance, Abbas et al. [11] handled some existence results for Caputo–Fabrizio type implicit FDEs in b-metric spaces. The existence and uniqueness of solutions for the following problem

$$\begin{cases} {}^{\mathrm{CF}}\mathbb{D}_{0^+}^{\nu} x(s) = f(s, x(s)), 0 < \nu < 1, \\ x(0) = x_0, \end{cases} \tag{1}$$

were demonstrated by Shaikh et al. [14]. It is possible to think of hybrid differential equations as quadratic perturbations of nonlinear differential equations. They are of great interest to scholars because they are particular instances of dynamical systems. Dhage and Lakshmikantham [19] provide details on various perturbations for nonlinear differential and integral equations. For additional updates on the availability of hybrid FDEs theory, we refer to [19–21]. For instance, the following hybrid classical

$$\begin{cases} \dfrac{d}{ds}\left(\dfrac{x(s)}{Z(s, x(s))}\right) = f(s, x(s)), 0 \le s < T, \\ \\ x(s_0) = x_0 \in \mathbb{R}, \end{cases} \tag{2}$$

has been studied by Dhage and Lakshmikantham [19]. Taking on the analogous approach of [19], Zhao et al. in [20] extended the investigation of hybrid (2) to the following Riemann–Liouville type hybrid FDE:

$$\begin{cases} {}^{\mathrm{RL}}\mathbb{D}_{0^+}^{\nu}\left(\dfrac{x(s)}{Z(s, x(s))}\right) = f(s, x(s)), 0 \le s < T, \\ \\ x(0) = 0,\ 0 < \nu < 1. \end{cases} \tag{3}$$

Herzallah and Baleanu [21] discussed the existence results of hybrid FDE (3) using the Caputo FD. Furthermore, various classes of hybrid FDEs subject to different conditions have additionally been concentrated on by many specialists, see [22–27]; e.g., Ali et al. [26] developed an existence analysis for nonlinear hybrid FDEs with ψ-Hilfer FD and hybrid boundary conditions. For a nonlocal hybrid of Caputo fractional integrodifferential equations, Ahmad et al. [22] presented the existence results.

Almeida [28] proposed a general operator so called ψ-Caputo FD when the kernel is $k(t, s) = \psi(t) - \psi(s)$ and the derivative is $((1/\psi'(t))(d/dt))$. Then, the authors in [29] expanded some of the properties of this operator to include the Laplace transform of it. Regarding this, Jarad et al. [30] developed the idea of weighted FDs with another function. Abdo et al. [31] proved the positive solutions of the following ψ-weighted Caputo problem:

$$\begin{cases} {}^{\mathrm{C}}\mathbb{D}_{0^+}^{\nu;\psi,w} x(s) = f(s, x(s)), 0 < \nu \le 1, \\ x(0) = x_0. \end{cases} \tag{4}$$

Recently, Al-Rafai and Jarrah [32] extended the concept of weighted FD to the ψ-weighted Caputo–Fabrizio FD, where ψ and w are monotone function and weight function, respectively.

Motivated by the abovementioned studies, we discuss the existence of solutions of the following weighted hybrid FDE:

$$\begin{cases} {}^{CF}\mathbb{D}_{0;w}^{\nu}\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))} = f(s, x(s)), s \in \mho := [0, T], \\ \\ a\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))}\Bigg|_{s=0} + b\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))}\Bigg|_{s=T} = c, \end{cases} \tag{5}$$

and the following ψ-weighted hybrid FDE

$$\begin{cases} {}^{CF}\mathbb{D}_{0;w}^{\nu;\psi}\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))} = f(s, x(s)), s \in \mho := [0, T], \\ \\ a\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))}\Bigg|_{s=0} + b\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))}\Bigg|_{s=T} = c, \end{cases} \tag{6}$$

where $0 < \nu < 1$ and $a, b, c \in \mathbb{R}$, with $a \ne 0$, $f, U \in C(\mho \times \mathbb{R}, \mathbb{R})$, $Z \in C(\mho \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, and ${}^{\mathrm{CF}}\mathbb{D}_{0;w}^{\nu;\psi}$ denote the weighted Caputo–Fabrizio and ψ-weighted Caputo–Fabrizio FDs, respectively, and $w, \psi \in C^1(\mho, \mathbb{R})$ with $w, w', \psi' > 0$ on $\mho$. Our first contribution focuses on several special cases of problems (5) and (6) connecting the weighted function to another monotone function. Then, using Dhage's fixed-point theory and some added conditions of functions $f(.,.)$, $U(.,.)$, and $Z(.,.)$, we show the existence of solutions to (5) and (6) under ψ-weighted Caputo–Fabrizio FDs or the diverse hybrid cases. Specifically, our results support, extend, and enhance those found in [31–33].

Remark 1. This work can be a generalization of some of the studied problems in the literature, for example,

(i) In problem (5), if we choose $w = 1$, $U(s, x(s)) \equiv 0$, and $Z(s, x(s)) \equiv 1$, then we obtain the following problem:

$$\begin{cases} {}^{CF}\mathbb{D}_{0}^{\nu} x(s) = f(s, x(s)), 0 < \nu < 1, \\ \mathrm{ax}(s)|_{s=0} + \mathrm{bx}(s)|_{s=T} = c, \end{cases} \tag{7}$$

which has been studied by Salim et al. [33].

(ii) If we choose $a = 1$, $b = 0$, $U(s, x(s)) \equiv 0$, and $Z(s, x(s)) \equiv 1$, then our problem (6) reduces to problem (4), which was considered by Abdo et al. [31].

Remark 2

(1) If $\psi(s) = s$, then problem (6) reduces to problem (5)

(2) If $\psi(s) = s$, $w = 1$, $U \equiv 0$, and $Z \equiv 1$, then problem (6) reduces to problem (7), see [33]

(3) If $\psi(s) = s$, $w = 1$, $a = 1$, $b = 0$, $U \equiv 0$, and $Z \equiv 1$, then problem (6) reduces to problem (1), see [14]

(4) Many problems with less general operators with various values of w and ψ, such as the one proposed by Caputo and Fabrizio in [5] are part of our current problems

The rest of this work is arranged as follows. Section 2 gives some basic results about generalized Caputo–Fabrizio FD and functional spaces. Our main results of problems (5) and (6) are discussed in Section 3. Two examples that confirm the validity of the main results are provided in Section 4. Finally, we include the conclusions in Section 5.

2. Primitive Results

We start off this section by providing some definitions and fundamental results. Let $\mho := [0,b]$, $0<b<\infty$. $\mathbb{R}$ will constantly represent real space. Define the supremum norm $\|\cdot\|$ in $\mathcal{X} := \mathscr{C}(\mho,\mathbb{R})$ by $\|x\|_{\mathcal{X}} = \sup_{s\in\mho}|x(s)|$, and the multiplication in $\mathcal{X}$ by $(xy)(s) = x(s)y(s)$. Obviously, $\mathcal{X}$ is a Banach algebra with the norm and multiplication in it. The weight function and the monotone function, respectively, are represented by $\psi(s)$ and $w(s)$ with $w,\psi\in\mathcal{X}^1$ and $w, w', \psi' > 0$ on $\mho$.

Definition 3 (see [32]). Let $0<\nu<1$, and $x\in\mathcal{X}$. Then, the ψ-weighted Caputo–Fabrizio FD of x is given by the following equation:

$$ {}^{CF}\mathbb{D}_{0;w}^{\nu;\psi}x(s) = \frac{\aleph(\nu)}{1-\nu}\frac{1}{w(s)}\int_0^s e^{-\lambda_\nu(\psi(s)-\psi(\xi))}\frac{d}{d\xi}(\mathrm{wx})(\xi)d\xi, \tag{8} $$

where $\lambda_\nu = \nu/1-\nu$, and $\aleph(\nu)$ is a normalization function satisfying $\aleph(0)=\aleph(1)=1$.

The above-given operator can be written as follows:

$$ {}^{CF}\mathbb{D}_{0;w}^{\nu;\psi}x(s) = \frac{\aleph(\nu)}{1-\nu}\frac{e^{-\lambda_\nu\psi(s)}}{w(s)}\int_0^s e^{\lambda_\nu\psi(\xi)}\frac{d}{d\xi}(\mathrm{wx})(\xi)d\xi. \tag{9} $$

Definition 4 (see [32]). Let $0<\nu<1$, and $x\in\mathcal{X}$. Then, the ψ-weighted Caputo–Fabrizio fractional integral is defined as follows:

$$ {}^{CF}\mathbb{I}_{0;w}^{\nu;\psi}x(s) = \frac{1-\nu}{\aleph(\nu)}x(s) + \frac{\nu}{\aleph(\nu)}\frac{1}{w(s)}\int_0^s \psi'(\xi)w(\xi)x(\xi)d\xi. \tag{10} $$

Lemma 5 (see [32]). *Let $0<\nu<1$, and $x\in\mathcal{X}$. Then,*

$$ \begin{aligned} {}^{CF}\mathbb{D}_{0;w}^{\nu;\psi}\,{}^{CF}\mathbb{I}_{0;w}^{\nu;\psi}\,x(s) &= x(s),\\ {}^{CF}\mathbb{I}_{0;w}^{\nu;\psi}\,{}^{CF}\mathbb{D}_{0;w}^{\nu;\psi}\,x(s) &= x(s) - \frac{w(0)x(0)}{w(s)}. \end{aligned} \tag{11} $$

In particular, if $x(0)=0$, then ${}^{CF}\mathbb{I}_{0;w}^{\nu;\psi}\,{}^{CF}\mathbb{D}_{0;w}^{\nu;\psi}\,x(s) = x(s)$.

Lemma 6 (see [32]). *Let $0<\nu<1$, and $x\in\mathcal{X}$ with $f(0)=0$. Then, the following FDE*

$$ \begin{aligned} {}^{CF}\mathbb{D}_{0;w}^{\nu;\psi}x(s) &= f(s),\\ x(0) &= c, \end{aligned} \tag{12} $$

has the unique solution

$$ x(s) = \frac{w(0)}{w(s)}c + \frac{1-\nu}{\aleph(\nu)}f(s) + \frac{\nu}{\aleph(\nu)}\frac{1}{w(s)}\int_0^s \psi'(\xi)w(\xi)f(\xi)d\xi, \quad s\in\mho. \tag{13} $$

Theorem 7 (Dhage's fixed point theorem [34]). *Let $\mathcal{D}$ be a nonempty, convex, closed subset of a Banach algebra $\mathcal{X}$. Let the operators $\mathcal{O}_1, \mathcal{O}_3\colon \mathcal{X}\longrightarrow\mathcal{X}$ and $\mathcal{O}_2\colon\mathcal{D}\longrightarrow\mathcal{X}$ such that (i) $\mathcal{O}_1$ and $\mathcal{O}_3$ are Lipschitzian with a Lipschitz constants κ_1 and κ_2, respectively; (ii) $\mathcal{O}_2$ is continuous and compact; (iii) $x=\mathcal{O}_1x\mathcal{O}_2y+\mathcal{O}_3x \Longrightarrow x\in\mathcal{X}$ for each $y\in\mathcal{D}$, (iv) $\kappa_1\mathcal{N}+\kappa_2<1$, where $\mathcal{N}=\|\mathcal{O}_2(\mathcal{D})\|$. Then, there exists $x\in\mathcal{D}$ such that $\mathcal{O}_1x\mathcal{O}_2x+\mathcal{O}_3x = x$.*

3. Main Results

Here, we provide some qualitative analyses of two types of Caputo–Fabrizio hybrid problems that are (5) and (6).

Lemma 8. *Let g be continuous function on $\mho$ with $g(0)=0$ and assume that $x\longrightarrow x-U(s,x)/Z(s,x)$ is increasing in $\mathbb{R}$, a.e., for each $s\in\mho$. Then, the solution of the ψ-weighted hybrid FDE*

$$ \begin{cases} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{\nu;\psi}\left(\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\right) = g(s), s\in\mho, 0<\nu<1,\\[2ex] a\left(\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\right)_{s=0} + b\left(\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\right)_{s=T} = c, \end{cases} \tag{14} $$

satisfies the following equation:

$$
\begin{aligned}
x(s) = {} & Z(s, x(s)) \\
& \cdot \left[\frac{\beta_w}{w(s)} \left(\frac{c}{a} - \frac{b\mu_\nu}{a} g(T) - \frac{b\eta_\nu}{aw(T)} \int_0^T \psi'(\xi) w(\xi) g(\xi) \mathrm{d}\xi \right) \right. \\
& \left. + \mu_\nu g(s) + \frac{\eta_\nu}{w(s)} \int_0^s \psi'(\xi) w(\xi) g(\xi) \mathrm{d}\xi \right] \\
& + U(s, x(s)), \quad s \in \mho,
\end{aligned}
\tag{15}
$$

where $\mu_\nu := 1 - \nu/\aleph(\nu)$, $\eta_\nu := \nu/\aleph(\nu)$, *and* $\beta_w := aw(0)w(T)/aw(T) + bw(0)$ *with* $aw(T) + bw(0) \neq 0$.

Proof. Applying the operator ${}^{\mathrm{CF}}\mathbb{I}_{0;w}^{\nu;\psi}$ of the first equation of (14), we have the following equation:

$$
{}^{\mathrm{CF}}\mathbb{I}_{0;w}^{\nu;\psi} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{\nu;\psi} \left(\frac{x(s) - U(s, x(s))}{Z(s, x(s))} \right) = {}^{\mathrm{CF}}\mathbb{I}_{0;w}^{\nu;\psi} g(s). \tag{16}
$$

Using Lemma 5, we have the following equation:

$$
{}^{\mathrm{CF}}\mathbb{I}_{0;w}^{\nu;\psi} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{\nu;\psi} \left(\frac{x(s) - U(s, x(s))}{Z(s, x(s))} \right) = \left(\frac{x(s) - U(s, x(s))}{Z(s, x(s))} \right) - \frac{w(0)}{w(s)} \left(\frac{x(0) - U(0, x(0))}{Z(0, x(0))} \right). \tag{17}
$$

Comparing (16) and (17), we obtain the following equation:

$$
\left(\frac{x(s) - U(s, x(s))}{Z(s, x(s))} \right) - \frac{w(0)}{w(s)} \left(\frac{x(0) - U(0, x(0))}{Z(0, x(0))} \right) = {}^{\mathrm{CF}}\mathbb{I}_{0;w}^{\nu;\psi} g(s), \tag{18}
$$

which implies

$$
\begin{aligned}
x(s) = {} & Z(s, x(s)) \left[\frac{w(0)}{w(s)} \left(\frac{x(s) - U(s, x(s))}{Z(s, x(s))} \right)_{s=0} + \mu_\nu g(s) + \frac{\eta_\nu}{w(s)} \int_0^s \psi'(\xi) w(\xi) g(\xi) \mathrm{d}\xi \right] \\
& + U(s, x(s)).
\end{aligned}
\tag{19}
$$

Taking $s \longrightarrow T$ to both sides of (19), we have the following equation:

$$
\left(\frac{x(s) - U(s, x(s))}{Z(s, x(s))} \right)_{s=T} = \left[\frac{w(0)}{w(T)} \left(\frac{x(s) - U(s, x(s))}{Z(s, x(s))} \right)_{s=0} + \mu_\nu g(T) + \frac{\eta_\nu}{w(T)} \int_0^s \psi'(\xi) w(\xi) g(\xi) \mathrm{d}\xi \right]. \tag{20}
$$

Applying the boundary condition of (14) and using (20), we obtain the following equation:

$$\begin{aligned}\left(\frac{x(s)-U(s,x(s))}{Z(s,x(s))}\right)_{s=0} &= \frac{c}{a}-\frac{b}{a}\left(\frac{x(s)-U(s,x(s))}{Z(s,x(s))}\right)_{s=T}\\ &= \frac{c}{a}-\frac{b}{a}\left[\frac{w(0)}{w(T)}\left(\frac{x(s)-U(s,x(s))}{Z(s,x(s))}\right)_{s=0}+\mu_\nu g(T)+\frac{\eta_\nu}{w(T)}\int_0^T \psi'(\xi)w(\xi)g(\xi)\mathrm{d}\xi\right].\end{aligned}\tag{21}$$

Hence,

$$\left(\frac{x(s)-U(s,x(s))}{Z(s,x(s))}\right)_{s=0} = \frac{\beta_w}{w(0)}\left(\frac{c}{a}-\frac{b\mu_\nu}{a}g(T)-\frac{b\eta_\nu}{\mathrm{a}w(T)}\int_0^T \psi'(\xi)w(\xi)g(\xi)\mathrm{d}\xi\right).\tag{22}$$

Substituting (22) into (19), we obtain (15). $\square$

Due to Lemma 8, we can infer the following result:

Corollary 9. *Let $Z \in C(\mho,\mathbb{R}\backslash\{0\})$ and $f,U \in C(\mho,\mathbb{R})$ with $f(0,x(0))=0$. Then, the solution of (6) satisfies the following equation:*

$$\begin{aligned}x(s) &= Z(s,x(s))\\ &\cdot\left[\frac{\beta_w}{w(s)}\left(\frac{c}{a}-\frac{b\mu_\nu}{a}f(T,x(T))-\frac{b\eta_\nu}{\mathrm{a}w(T)}\int_0^T \psi'(\xi)w(\xi)f(\xi,x(\xi))d\xi\right)+\mu_\nu f(s,x(s))+\frac{\eta_\nu}{w(s)}\int_0^s \psi'(\xi)w(\xi)f(\xi,x(\xi))d\xi\right]\\ &+U(s,x(s)),\quad s\in\mho,\end{aligned}\tag{23}$$

where η_ν,μ_ν, and β_w as in Lemma 8.

Now, we need the following assumptions on U,Z, and f.

(As1) $Z\colon \mho\times\mathbb{R}\longrightarrow\mathbb{R}\times\{0\}$ and $U,f\colon \mho\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous

(As2) There exist positive functions ϑ_U and ϑ_Z with bounds $\|\vartheta_U\|$ and $\|\vartheta_Z\|$, respectively, such that

$$\begin{aligned}&|U(s,x)-U(s,\overline{x})|\le\vartheta_U(s)|x-\overline{x}|, s\in\mho, x,\overline{x}\in\mathbb{R}\text{ and}\\ &|Z(s,x)-Z(s,\overline{x})|\le\vartheta_Z(s)|x-\overline{x}|, s\in\mho, x,\overline{x}\in\mathbb{R}.\end{aligned}\tag{24}$$

(As3) There exist two functions $\delta_f\in\mathcal{X}$ and $\Upsilon\colon \mathbb{R}^+\longrightarrow\mathbb{R}^+$ be nondecreasing continuous such that

$$|f(s,x)|\le\delta_f(s)\Upsilon(|x|), s\in\mho, x\in\mathbb{R}.\tag{25}$$

(As4) There exists $r>0$ such that

$$r\ge\frac{Z_0\varpi+U_0}{\left(1-\|\vartheta_Z\|\varpi-\|\vartheta_U\|\right)},\tag{26}$$

and $\|\vartheta_U\|\varpi+\|\vartheta_Z\|<1$, where $U_0=\sup_{s\in\mho}|U(s,0)|$, $Z_0=\sup_{s\in\mho}|Z(s,0)|$, and

$$\begin{aligned}\varpi &:= \frac{|\beta_w||c|}{w(0)|a|}+\|\delta_f\|\Upsilon(r)\\ &\times\left[\mu_\nu+\frac{|\beta_w||b|\mu_\nu}{w(0)|a|}+\left(\frac{|b|\eta_\nu}{|a|w(T)}+\frac{\eta_\nu}{w(0)}\right)w(T)[\psi(T)-\psi(0)]\right].\end{aligned}\tag{27}$$

Theorem 10. *Suppose that (As1)–(As4) hold, then the ψ-weighted problem (6) has at least one solution defined on $\mho$.*

Proof. Define the set $\mathscr{D} = \{x \in \mathscr{X}: \|x\|_{\mathscr{X}} \leq r\}$, where r satisfies (As4). Certainly, $\mathscr{D}$ is a convex, closed, and bounded subset of $\mathscr{X}$. By Corollary 9, we define three operators $\mathscr{O}_1, \mathscr{O}_3: \mathscr{X} \longrightarrow \mathscr{X}$ and $\mathscr{O}_2: \mathscr{D} \longrightarrow \mathscr{X}$ by

$$\begin{aligned} &\mathscr{O}_1 x(s) = Z(s, x(s)), \quad s \in \mho, \\ &\mathscr{O}_2 x(s) = \frac{\beta_w}{w(s)}\left(\frac{c}{a} - \frac{b\mu_\nu}{a} f(T, x(T)) - \frac{b\eta_\nu}{aw(T)} \int_0^T \psi'(\xi) w(\xi) f(\xi, x(\xi)) \mathrm{d}\xi\right) \\ &\qquad + \mu_\nu f(s, x(s)) + \frac{\eta_\nu}{w(s)} \int_0^s \psi'(\xi) w(\xi) f(\xi, x(\xi)) \mathrm{d}\xi, s \in \mho \text{ and} \\ &\mathscr{O}_3 x(s) = U(s, x(s)), \quad s \in \mho. \end{aligned} \tag{28}$$

So, we can write the formula (23) in the operator form as follows:

$$x(s) = \mathscr{O}_1 x(s) \mathscr{O}_2 x(s) + \mathscr{O}_3 x(s), \quad s \in \mho. \tag{29}$$

Now, we show that $\mathscr{O}_1$, $\mathscr{O}_2$, and $\mathscr{O}_3$ fulfill all the assumptions of Theorem 7, through the following claims: □

Step 11. $\mathscr{O}_1$ and $\mathscr{O}_3$ are Lipschitzian on $\mathscr{X}$.

For $s \in \mho$ and $x, \overline{x} \in \mathscr{X}$, we have from (As2) that

$$\begin{aligned} &\left|\mathscr{O}_1 x(s) - \mathscr{O}_1 \overline{x}(s)\right| = |Z(s, x(s)) - Z(s, \overline{x}(s))| \leq \vartheta_Z(s)|x(s) - \overline{x}(s)| \text{and} \\ &\left|\mathscr{O}_3 x(s) - \mathscr{O}_3 \overline{x}(s)\right| = |U(s, x(s)) - U(s, \overline{x}(s))| \leq \vartheta_U(s)|x(s) - \overline{x}(s)|, \end{aligned} \tag{30}$$

which implies

$$\begin{aligned} &\left\|\mathscr{O}_1 x - \mathscr{O}_1 \overline{x}\right\| \leq \left\|\vartheta_Z\right\| \|x - \overline{x}\| \text{and} \\ &\left\|\mathscr{O}_3 x - \mathscr{O}_3 \overline{x}\right\| \leq \left\|\vartheta_U\right\| \|x - \overline{x}\|. \end{aligned} \tag{31}$$

Thus, $\mathscr{O}_1, \mathscr{O}_3: \mathscr{X} \to \mathscr{X}$ are Lipschitzian on $\mathscr{X}$ with Lipschitz constants $\|\vartheta_Z\|$ and $\|\vartheta_U\|$, respectively.

Step 12. $\mathscr{O}_2: \mathscr{D} \longrightarrow \mathscr{X}$ is a completely continuous.

In the beginning, we show that $\mathscr{O}_2$ is continuous on $\mathscr{D}$. Let $\{x_n\}_{n\geq 1}$ be a sequence in $\mathscr{D}$ with $x_n \longrightarrow x \in \mathscr{D}$. Then, from Lebesgue's convergence theorem [35], we obtain the following equation:

$$\begin{aligned} \lim_{n\longrightarrow\infty} \mathscr{O}_2 x_n(s) &= \frac{\beta_w}{w(s)}\left(\frac{c}{a} - \frac{b\mu_\nu}{a} \lim_{n\longrightarrow\infty} f(T, x_n(T)) - \frac{b\eta_\nu}{aw(T)} \lim_{n\longrightarrow\infty} \int_0^T \psi'(\xi) w(\xi) f(\xi, x_n(\xi)) d\xi\right) \\ &\quad + \mu_\nu \lim_{n\longrightarrow\infty} f(s, x_n(s)) + \frac{\eta_\nu}{w(s)} \lim_{n\longrightarrow\infty} \int_0^s \psi'(\xi) w(\xi) f(\xi, x_n(\xi)) \mathrm{d}\xi \\ &= \frac{\beta_w}{w(s)}\left(\frac{c}{a} - \frac{b\mu_\nu}{a} \lim_{n\longrightarrow\infty} f(T, x_n(T)) - \frac{b\eta_\nu}{aw(T)} \int_0^T \psi'(\xi) w(\xi) \lim_{n\longrightarrow\infty} f(\xi, x_n(\xi)) \mathrm{d}\xi\right) \end{aligned}$$

$$\begin{aligned}
&+\mu_\nu \lim_{n\longrightarrow\infty} f(s,x_n(s)) + \frac{\eta_\nu}{w(s)}\int_0^s \psi'(\xi)w(\xi)\lim_{n\longrightarrow\infty} f(\xi,x_n(\xi))\mathrm{d}\xi\\
&= \frac{\beta_w}{w(s)}\left(\frac{c}{a}-\frac{b\mu_\nu}{a}f(T,x(T)) - \frac{b\eta_\nu}{\mathrm{a}w(T)}\int_0^T \psi'(\xi)w(\xi)f(\xi,x(\xi))\mathrm{d}\xi\right)\\
&+\mu_\nu f(s,x(s)) + \frac{\eta_\nu}{w(s)}\int_0^s \psi'(\xi)w(\xi)f(\xi,x(\xi))\mathrm{d}\xi.
\end{aligned} \tag{32}$$

Hence, $\lim_{n\longrightarrow\infty}\mathcal{O}_2 x_n(s) = \mathcal{O}_2 x(s)$, for all $s\in \mho$. Thus, $\mathcal{O}_2$ is a continuous on $\mathcal{D}$.

Next, let $x\in\mathcal{D}$. Then, by (As3), we have the following equation:

$$\begin{aligned}
\left|\mathcal{O}_2 x(s)\right| &= \frac{|\beta_w|}{w(s)}\left(\frac{|c|}{|a|}+\frac{|b|\mu_\nu}{|a|}|f(T,x(T))| + \frac{|b|\eta_\nu}{|a|w(T)}\int_0^T \psi'(\xi)w(\xi)|f(\xi,x(\xi))|d\xi\right)\\
&+\mu_\nu|f(s,x(s))| + \frac{\eta_\nu}{w(s)}\int_0^s \psi'(\xi)w(\xi)|f(\xi,x(\xi))|\mathrm{d}\xi\\
&\le \frac{|\beta_w|}{w(s)}\left(\frac{|c|}{|a|}+\frac{|b|\mu_\nu}{|a|}\delta_f(T)\Upsilon(|x(T)|) + \frac{|b|\eta_\nu}{|a|w(T)}\int_0^T \psi'(\xi)w(\xi)\delta_f(\xi)\Upsilon(|x(\xi)|)d\xi\right)\\
&+\mu_\nu\delta_f(s)\Upsilon(|x|) + \frac{\eta_\nu}{w(s)}\int_0^s \psi'(\xi)w(\xi)\delta_f(\xi)\Upsilon(|x(\xi)|)\mathrm{d}\xi\\
&\le \frac{|\beta_w|}{w(s)}\left(\frac{|c|}{|a|}+\frac{|b|\mu_\nu}{|a|}\left\|\delta_f\right\|\Upsilon(r) + \frac{|b|\eta_\nu}{|a|w(T)}\int_0^T \psi'(\xi)w(\xi)\left\|\delta_f\right\|\Upsilon(r)\mathrm{d}\xi\right)\\
&+\mu_\nu\left\|\delta_f\right\|\Upsilon(r) + \frac{\eta_\nu}{w(s)}\int_0^s \psi'(\xi)w(\xi)\left\|\delta_f\right\|\Upsilon(r)d\xi\\
&\le \frac{|\beta_w||c|}{w(0)|a|} + \left\|\delta_f\right\|\Upsilon(r)\left(\frac{|\beta_w||b|\mu_\nu}{w(0)|a|} + \frac{|b|\eta_\nu}{|a|w(T)}\int_0^T \psi'(\xi)w(\xi)\mathrm{d}\xi + \mu_\nu + \frac{\eta_\nu}{w(0)}\int_0^s \psi'(\xi)w(\xi)\mathrm{d}\xi\right).
\end{aligned} \tag{33}$$

Since $\psi', w > 0$ and applying the mean value theorem for integral, we obtain the following equation:

$$\int_0^\theta \psi'(\xi)w(\xi)d\xi = w(\kappa)\int_0^\theta \psi'(\xi)\mathrm{d}\xi = w(\kappa)[\psi(\theta)-\psi(0)], \tag{34}$$

for some $0<\kappa<T$. It follows that

$$\int_0^\theta \psi'(\xi)w(\xi)d\xi \le w(T)[\psi(T)-\psi(0)]. \tag{35}$$

Hence, (33) becomes

$$\left|\mathcal{O}_2 x(s)\right| \leq \frac{\left|\beta_w\right||c|}{w(0)|a|} + \left\|\delta_f\right\|\Upsilon(r) \times\left[\mu_\nu + \frac{\left|\beta_w\right||b|\mu_\nu}{w(0)|a|} + \left(\frac{|b|\eta_\nu}{|a|w(T)} + \frac{\eta_\nu}{w(0)}\right)w(T)[\psi(T)-\psi(0)]\right]. \tag{36}$$

Therefore, $\|\mathcal{O}_2 x\| \leq \varpi$, for all $x \in \mathcal{D}$, where ϖ given by (As4). This consequence proves that $\mathcal{O}_2(\mathcal{D})$ is uniformly bounded set on $\mathcal{D}$. Finally, we show that the set $\mathcal{O}_2(\mathcal{D})$ is an equicontinuous in $\mathcal{X}$.

Let $s_1, s_2 \in \mho$ with $s_1 \leq s_2$, and $x \in \mathcal{D}$. Then,

$$\begin{aligned}
&\left|\mathcal{O}_2 x(s_2) - \mathcal{O}_2 x(s_1)\right| \\
&\leq \frac{\left|w(s_2)-w(s_1)\right|}{w(s_2)w(s_1)}\beta_w\left(\frac{c}{a} - \frac{b\mu_\nu}{a} f(T,x(T)) - \frac{b\eta_\nu}{\mathrm{aw}(T)}\int_0^T \psi'(\xi)w(\xi)f(\xi,x(\xi))\mathrm{d}\xi\right) \\
&\quad + \mu_\nu\left|f(s_2,x(s_2)) - f(s_1,x(s_1))\right| + \eta_\nu\left|w(s_2)-w(s_1)\right|\left|\int_0^{s_2}\psi'(\xi)w(\xi)|f(\xi,x(\xi))|d\xi\right. \\
&\quad + \frac{\eta_\nu}{w(s_1)}\int_{s_1}^{s_2}\psi'(\xi)w(\xi)|f(\xi,x(\xi))|\mathrm{d}\xi \\
&\leq \frac{\left|w(s_2)-w(s_1)\right|}{w(s_2)w(s_1)}\beta_w\left(\frac{c}{a} - \frac{b\mu_\nu}{a} f(T,x(T)) - \frac{b\eta_\nu}{\mathrm{aw}(T)}\int_0^T \psi'(\xi)w(\xi)f(\xi,x(\xi))\mathrm{d}\xi\right) \\
&\quad + \mu_\nu\left|f(s_2,x(s_2)) - f(s_1,x(s_1))\right| + \eta_\nu\left|w(s_2)-w(s_1)\right|\left\|\delta_f\right\|\Upsilon(r)w(s_2)[\psi(s_2)-\psi(0)] \\
&\quad + \frac{w(s_2)}{w(s_1)}\eta_\nu\left\|\delta_f\right\|\Upsilon(r)[\psi(s_2)-\psi(s_1)].
\end{aligned} \tag{37}$$

As $s_2 \longrightarrow s_1$, the continuity of f, ψ and w imply that

$$\left|\mathcal{O}_2 x(s_2) - \mathcal{O}_2 x(s_1)\right| \longrightarrow 0, \text{as } s_2 \longrightarrow s_1. \tag{38}$$

Thus, $\mathcal{O}_2$ is equicontinuous on $\mathcal{D}$. As a result of the Ascoli–Arzelà theorem [4], $\mathcal{O}_2\colon \mathcal{D} \longrightarrow \mathcal{X}$ is a completely continuous.

Step 13. Assumption (iii) of Theorem 7 is satisfied.

Let $x \in \mathcal{X}$ and $y \in \mathcal{D}$ such that $x = \mathcal{O}_1 x\mathcal{O}_2 y + \mathcal{O}_3 x$. Then,

$$\begin{aligned}
|x(s)| &\leq \left|\mathcal{O}_1 x(s)\right|\left|\mathcal{O}_2 y(s)\right| + \left|\mathcal{O}_3 x(s)\right| \\
&\leq |Z(s,x(s))|\left\{\frac{\left|\beta_w\right|}{w(s)}\left(\frac{|c|}{|a|} + \frac{|b|\mu_\nu}{|a|}|f(T,y(T))| + \frac{|b|\eta_\nu}{|a|w(T)}\int_0^T \psi'(\xi)w(\xi)|f(\xi,y(\xi))|d\xi\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\mu_\nu|f(s,y(s))|+\frac{\eta_\nu}{w(s)}\int_0^s\psi'(\xi)w(\xi)|f(\xi,y(\xi))|d\xi\Bigg\}+|U(s,x(s))|\\
&\le[|Z(s,x(s))-Z(s,0)|+|Z(s,0)|]\left\{\frac{|\beta_w|}{w(s)}\left(\frac{|c|}{|a|}+\frac{|b|\mu_\nu}{|a|}|f(T,y(T))|\right.\right.\\
&\left.+\frac{|b|\eta_\nu}{|a|w(T)}\int_0^T\psi'(\xi)w(\xi)|f(\xi,y(\xi))|\mathrm{d}\xi\right)+\mu_\nu|f(s,y(s))|\\
&\left.+\frac{\eta_\nu}{w(s)}\int_0^s\psi'(\xi)w(\xi)|f(\xi,y(\xi))|\mathrm{d}\xi\right\}+[|U(s,x(s))-U(s,0)|+|U(s,0)|]\\
&\le\left[\|\vartheta_Z\||x(s)|+Z_0\right]\left\{\frac{|\beta_w|}{w(0)}\left(\frac{|c|}{|a|}+\frac{|b|\mu_\nu}{|a|}\|\delta_f\|\Upsilon(r)\right.\right.\\
&\left.\left.+\left(\frac{|b|\eta_\nu}{|a|w(T)}+\frac{\eta_\nu}{w(0)}\right)\|\delta_f\|\Upsilon(r)w(T)[\psi(T)-\psi(0)]\right)+\mu_\nu\|\delta_f\|\Upsilon(r)\right\}\\
&+\|\vartheta_U\||x(s)|+U_0\\
&=\left[\|\vartheta_Z\|\varpi+\|\vartheta_U\|\right]|x(s)|+Z_0\varpi+U_0,
\end{aligned}
\tag{39}
$$

which implies

$$
|x(s)|\le\frac{Z_0\varpi+U_0}{1-\|\vartheta_Z\|\varpi-\|\vartheta_U\|}\le r.\tag{40}
$$

Step 14. Assumption (iv) of Theorem 7 is satisfied, i.e., $\kappa_1\mathcal{N}+\kappa_2<1$, where $\mathcal{N}=\|\mathcal{O}_2(\mathcal{D})\|$.

Since

$$
\mathcal{N}=\|\mathcal{O}_2(\mathcal{D})\|=\sup_{x\in\mathcal{D}}\left\{\sup_{x\in\mathcal{D}}\mathcal{O}_2x(s)\right\}\le\varpi,\tag{41}
$$

we have the following equation:

$$
\kappa_1\mathcal{N}+\kappa_2:=\|\vartheta_Z\|\mathcal{N}+\|\vartheta_U\|<\|\vartheta_Z\|\varpi+\|\vartheta_U\|<1.\tag{42}
$$

Thus, all the assumptions of Theorem 7 are satisfied, and hence, the equation $x=\mathcal{O}_1x\mathcal{O}_2x+\mathcal{O}_3x$ has a solution in $\mathcal{D}$. As a result, ψ-weighted hybrid problem (6) has a solution on $\mho$.

3.1. Special Results. In this subsection, we discuss some special cases of problem (6).

Consider $\psi(s)=s$ in problem (6), we obtain the following weighted hybrid FDE:

$$
\begin{cases}
{}^{\mathrm{CF}}\mathbb{D}^{\nu}_{0;w}\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}=f(s,x(s)),s\in\mho:=[0,T],\\[2ex]
a\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\Bigg|_{s=0}+b\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\Bigg|_{s=T}=c,
\end{cases}
\tag{43}
$$

where all constants and symbols correspond to those in problem (6). Since the next Lemma is a duplication of Lemma 8 with $\psi(s)=s$, we shall omit its proof.

Lemma 15. *Let g be continuous function on $\mho$ with $g(0)=0$ and assume that $x\longrightarrow x-U(s,x)/Z(s,x)$ is increasing in $\mathbb{R}$, a.e., for each $s\in\mho$. Then, the solution to the following weighted hybrid FDE*

$$
\begin{cases}
{}^{\mathrm{CF}}\mathbb{D}^{\nu}_{0;w}\left(\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\right)=g(s),s\in\mho,0<\nu<1,\\[2ex]
a\left(\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\right)_{s=0}+b\left(\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\right)_{s=T}=c,
\end{cases}
\tag{44}
$$

satisfies the equation

$$
\begin{aligned}
x(s)&=Z(s,x(s))\\
&\cdot\left[\frac{\beta_w}{w(s)}\left(\frac{c}{a}-\frac{b\mu_\nu}{a}g(T)-\frac{b\eta_\nu}{\mathrm{a}w(T)}\int_0^T w(\xi)g(\xi)\mathrm{d}\xi\right)+\mu_\nu g(s)+\frac{\eta_\nu}{w(s)}\int_0^s w(\xi)g(\xi)\mathrm{d}\xi\right]\\
&+U(s,x(s)),\quad s\in\mho.
\end{aligned}
\tag{45}
$$

where η_ν,μ_ν, and β_w are as in Lemma 8.

According to Lemma 15, we can define three operators $\mathcal{O}_1, \mathcal{O}_3\colon \mathcal{X} \longrightarrow \mathcal{X}$, and $\mathcal{O}_2\colon \mathcal{D} \longrightarrow \mathcal{X}$ by

$$\begin{aligned} \mathcal{O}_1 x(s) &= Z(s, x(s)), \quad s \in \mho, \\ \mathcal{O}_2 x(s) &= \frac{\beta_w}{w(s)}\left(\frac{c}{a} - \frac{b\mu_\nu}{a} f(T, x(T)) - \frac{b\eta_\nu}{\mathrm{aw}(T)} \int_0^T w(\xi) f(\xi, x(\xi))\mathrm{d}\xi\right) \\ &\quad + \mu_\nu f(s, x(s)) + \frac{\eta_\nu}{w(s)} \int_0^s w(\xi) f(\xi, x(\xi))\mathrm{d}\xi, \quad s \in \mho, \\ \mathcal{O}_3 x(s) &= U(s, x(s)), \quad s \in \mho. \end{aligned} \tag{46}$$

In addition, we must provide some constants as follows:

$$\begin{aligned} \varpi &:= \frac{|\beta_w||c|}{w(0)|a|} + \|\delta_f\|\Upsilon(r) \\ &\quad \times \left[\mu_\nu + \frac{|\beta_w||b|\mu_\nu}{w(0)|a|} + \left(\frac{|b|\eta_\nu}{|a|w(T)} + \frac{\eta_\nu}{w(0)}\right) w(T)T\right], \end{aligned} \tag{47}$$

and

$$\int_0^\theta w(\xi)d\xi \le Tw(T), \text{for some } 0 < \theta < T. \tag{48}$$

The following existence theorem can be stated without proof.

Theorem 16. *Suppose that (As1)–(As4) hold with (47) and (48). Then, the weighted hybrid problem (5) has at least one solution defined on* $\mho$.

Remark 17. Our results for problem (6) are applied for the following special cases:

Case 1: if $a = 1$ and $b = 0$, then, we have the initial value problem of hybrid FDE:

$$\begin{cases} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{\nu;\psi} \dfrac{x(s) - U(s, x(s))}{Z(s, x(s))} = f(s, x(s)), 0 < \nu < 1, \\ \left.\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))}\right|_{s=0} = c. \end{cases} \tag{49}$$

Case 2: if $a = 0$ and $b = 1$, then, we have the terminal value problem of hybrid FDE:

$$\begin{cases} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{\nu;\psi} \dfrac{x(s) - U(s, x(s))}{Z(s, x(s))} = f(s, x(s)), 0 < \nu < 1, \\ \left.\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))}\right|_{s=T} = c. \end{cases} \tag{50}$$

Case 3: if $a = b = 1$ and $c = 0$, then, we have the antiperiodic the problem of hybrid FDE:

$$\begin{cases} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{\nu;\psi} \dfrac{x(s) - U(s, x(s))}{Z(s, x(s))} = f(s, x(s)), 0 < \nu < 1, \\ \left.\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))}\right|_{s=0} + \left.\dfrac{x(s) - U(s, x(s))}{Z(s, x(s))}\right|_{s=T} = 0. \end{cases} \tag{51}$$

Case 4: if we choose $U(s, x(s)) \equiv 0$, and $Z(s, x(s)) \equiv 1$, then our problems (5) and (6) reduce to the following problems:

$$\begin{aligned} &\begin{cases} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{\nu} x(s) = f(s, x(s)), 0 < \nu < 1, \\ \mathrm{ax}(s)|_{s=0} + \mathrm{bx}(s)|_{s=T} = c, \end{cases} \\ &\text{and} \\ &\begin{cases} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{\nu;\psi} x(s) = f(s, x(s)), 0 < \nu < 1, \\ \mathrm{ax}(s)|_{s=0} + \mathrm{bx}(s)|_{s=T} = c. \end{cases} \end{aligned} \tag{52}$$

4. Examples

Here, we give two examples to demonstrate the attained results.

Example 1. Consider the following ψ-weighted hybrid problem:

$$\begin{cases} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{1/4;\psi}\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))} = f(s,x(s)), s\in[0,1],\\ 2\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\bigg|_{s=0} + 3\dfrac{x(s)-U(s,x(s))}{Z(s,x(s))}\bigg|_{s=1} = 1. \end{cases} \tag{53}$$

Clearly, $= 1/4, \psi(s) = s/6, w(s) = e^s$, $[0,T] = [0,1]$, $a = 2$, $b = 3$, and $c = 1$. Set

$$\begin{aligned} U(s,x(s)) &= \frac{s^2}{16}\cos|x(s)| + \frac{s}{2},\\ Z(s,x(s)) &= 1 + \frac{1}{12}\sin\left(\frac{\pi s}{3}\right)|x(s)|,\\ f(s,x(s)) &= \frac{e^s-1}{10+s^2}\sin x(s). \end{aligned} \tag{54}$$

Note that $f(0,x(0)) = 0$. For $s\in[0,1]$ and $x, x^\star \in [0,\infty)$. Then,

$$\begin{aligned} \left|U(s,x) - U(s,x^\star)\right| &\le \frac{s^2}{16}\left|x - x^\star\right|,\\ \left|Z(s,x) - Z(s,x^\star)\right| &\le \frac{1}{12}\sin\left(\frac{\pi s}{3}\right)\left|x - x^\star\right|,\\ |f(s,x)| &\le e^s|x|. \end{aligned} \tag{55}$$

Thus, (As_1), (As_2), and (As_3) hold with $\vartheta_U(s) = s^2/16$, $\vartheta_Z(s) = 1/12\sin(\pi s/3)$, $\delta_f(s) = e^s$ and $\Upsilon(|x|) = |x|$. Then, $\|\vartheta_U\| = 1/16$, $\|\vartheta_Z\| = 1/8\sqrt{3}$, $\|\delta_f\| = e, U_0 = 9/16$, and $Z_0 = 1$. In addition, the condition $\|\vartheta_U\|\varpi + \|\vartheta_Z\| < 1$ holds. Indeed, we have $\aleph(\nu) = 1$, $\mu_\nu = 3/4$, $\eta_\nu = 1/4$, and $\beta_w = 2e/2e+3$, where $\mathrm{aw}(T) + \mathrm{bw}(0) = 2e + 3 \neq 0$, and

$$\varpi = \frac{\left(117e + 192e^2 + 4e^3\right)r + 48}{48(3+2e)}. \tag{56}$$

When

$$\left\|\vartheta_Z\right\|\varpi + \left\|\vartheta_U\right\| < 1, \tag{57}$$

we have the following equation:

$$\left\|\vartheta_Z\right\|\varpi + \left\|\vartheta_U\right\| = \frac{1}{8\sqrt{3}}\varpi + \frac{1}{16} < 1 \Longrightarrow \varpi < \frac{15\sqrt{3}}{2}. \tag{58}$$

From (56) and (58), we obtain the following equation:

$$\frac{\left(117e + 192e^2 + 4e^3\right)r + 48}{48(3+2e)} < \frac{15\sqrt{3}}{2}. \tag{59}$$

Hence, $r < 2.1634$. Thus, there exists $r > 0$ such that (57) holds. On the other hand, we have the following equation:

$$r \ge \frac{Z_0\varpi + U_0}{\left(1 - \left\|\vartheta_Z\right\|\varpi - \left\|\vartheta_U\right\|\right)} = \frac{48\varpi + 27}{45 - 2\sqrt{3}\varpi}. \tag{60}$$

Using the MATLAB program, we find that the constant r satisfies the inequality $r > 2.86862$. Thus, Theorem 10 shows that (53) has a solution on [0, 1].

Example 2. Let $\psi(s) = s$ and consider the following weighted hybrid problem:

$$\begin{cases} {}^{\mathrm{CF}}\mathbb{D}_{0;w}^{1/4}\dfrac{x(s)-U(s,x(s))}{x(s,x(s))} = \dfrac{se^s}{10e^s + e^{2s}}\dfrac{|x(s)|}{1+|x(s)|}, s\in\left[0,\dfrac{1}{2}\right],\\ 2\dfrac{x(s)-U(s,x(s))}{x(s,x(s))}\bigg|_{s=0} + 3\dfrac{x(s)-U(s,x(s))}{x(s,x(s))}\bigg|_{s=1/2} = 1. \end{cases} \tag{61}$$

Clearly, $\nu = 1/4, w(s) = e^s$, $T = 1/2$, $a = 2$, $b = 3$, $c = 1$, where

$$\begin{aligned} U(s,x(s)) &= \frac{s}{2}\left(\frac{1}{19}\cos|x(s)| + \frac{x^2(s) + |x(s)|}{1+|x(s)|}\right) + \frac{s}{8},\\ x(s,x(s)) &= \frac{e^{1-s}}{40}\cos|x(s)|,\\ f(s,x(s)) &= \frac{se^s}{10e^s + e^{2s}}\frac{|x(s)|}{1+|x(s)|}. \end{aligned} \tag{62}$$

Observe that $f(0,x(0)) = 0$. For $s\in[0,1/2]$, and $x, x^\star \in 0,\infty)$. Then,

$$\begin{aligned} \left|U(s,x) - U(s,x^\star)\right| &\le \frac{s}{2}\left|x - x^\star\right|,\\ \left|x(s,x) - x(s,x^\star)\right| &\le \frac{e^{1-s}}{40}\left|x - x^\star\right|,\\ |f(s,x)| &\le \frac{s}{10+e^s}|x|. \end{aligned} \tag{63}$$

Thus, (As_1), (As_2), and (As_3) hold with $\vartheta_U(s) = s/2$, $\vartheta_x(s) = e^{1-s}/40$, $\delta_f(s) = s/10 + e^s$ and $\Upsilon(|x|) = |x|$. Then, $\|\vartheta_U\| = 1/38$, $\|\vartheta_x\| = \sqrt{e}/40$, $\|\delta_f\| = 1/20 + 2\sqrt{e}, U_0 = 23/304$, and $x_0 = \sqrt{e}/40$. In addition, the condition $\|\vartheta_U\|\varpi + \|\vartheta_x\| < 1$ holds. Indeed, we have $\aleph(\nu) = 1$, $\mu_\nu \coloneqq 3/4$, $\eta_\nu \coloneqq 1/4$, and $\beta_w = 2e^{0.5}/2e^{0.5} + 3$ with $\mathrm{aw}(T) + \mathrm{bw}(0) = 6.29744 \neq 0$ and

$$\varpi = \frac{2.1448}{2e^{1/2} + 20} r + 0.26181. \tag{64}$$

Then, we have the following equation:

$$\|\vartheta_x\|\varpi + \|\vartheta_U\| = \frac{\sqrt{e}}{40}\varpi + \frac{1}{38} < 1. \tag{65}$$

Using the MATLAB program, we find that the constant r satisfies the inequality $0 < r < 267.63$. Thus, Theorem 16 shows that (61) has a solution on $[0, 1/2]$.

5. Conclusions

The existence and uniqueness of solutions are among the qualitative properties of mathematical models. These properties are important because they help ensure that the model provides reliable and accurate results and that the results are applicable to a wide range of situations. Without these qualitative properties, a model may not accurately reflect the real-world phenomena it is meant to describe, which can lead to incorrect conclusions and unreliable predictions. In this work, we have successfully analyzed the nonlinear hybrid differential equations by the application of fractional calculus. Specifically, problems (5) and (6) have been considered using the ψ-weighted Caputo–Fabrizio FDs, which incorporate a nonsingular kernel. First, we have provided several special results and various observations for our proposed problems in the frame of ψ-weighted Caputo–Fabrizio FDs, which made our results more generalizable and studyable to a wide range of previously studied and research-worthy problems. Then, through the utilization of Dhage's fixed point theory for sums of three operators, we have established the existence of solutions to the proposed hybrid problems. Finally, in order to support the theoretical results, we have offered two practical examples. In the future, it will be interesting if the current systems are studied in the frame of ψ-weighted Atangana–Baleanu–Caputo, recently introduced in [36, 37].

Data Availability

No data are used in this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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