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Research Article

Solutions of the Diophantine Equations $B_r = J_s + J_t$ and $C_r = J_s + J_t$

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Let $\{B_r\}_{r\geq 0}$, $\{J_r\}_{r\geq 0}$, and $\{C_r\}_{r\geq 0}$ be the balancing, Jacobsthal, and Lucas balancing numbers, respectively. In this paper, the diophantine equations $B_r = J_s + J_t$ and $C_r = J_s + J_t$ are completely solved. The solutions rely basically on Matveev's theorem on linear forms in logarithms of algebraic numbers and a procedure of reducing the upper bound due to Dujella and Pethö.

1. Introduction

Balancing numbers are generated by the equation $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$ with $B_0 = 0$ and $B_1 = 1$. So, the initial terms are as follows:

$$0, 1, 6, 35, 204, 1189, 6930, \ldots. \quad (1)$$

Lucas balancing numbers are strongly related to balancing numbers and defined by $C_0 = 1, C_1 = 3$, and $C_{n+1} = 6C_n - C_{n-1}$ for ≥ 1. Its first terms are as follows:

$$1, 3, 17, 99, 591, 3543, 21255, \ldots. \quad (2)$$

The initial terms of the Jacobsthal sequence are $J_0 = 0$ and $J_1 = 1$, and it follows the equation $J_{n+1} = J_n + 2J_{n-1}$ for $n \geq 1$. So, we have the following equation:

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \ldots. \quad (3)$$

Balancing numbers and associated sequences have been considered in many papers concerning diophantine equations. In [1], Ray solved some diophantine equations that involve balancing and Lucas balancing numbers. In [2], Dey and Rout found the perfect powers in the sequences of balancing and Lucas balancing numbers and identified the Lucas balancing numbers which are products of a power of 3 and a perfect power. In addition, they proved that many diophantine equations that contains balancing and Lucas balancing numbers have no solutions. In [3], Rayaguru and Panda found all the repdigits that exist in the product of consecutive balancing or Lucas balancing numbers, and in [4], they explored the repdigits that are expressible as products of balancing and Lucas balancing numbers with their indices in arithmetic progressions. In [5], Erduvan and Keskin studied and determined Fibonacci numbers which are products of two balancing numbers. In [6], Rayaguru et al. found the factoriangular numbers in the sequences of balancing and Lucas balancing numbers. In [7], Ddamulira obtained all the repdigits that can be written as sums of three balancing numbers. In [8], Patra and Panda solved the equation $x^s - 8C_n xy + 16y^t = \pm 2^r$ for $(s,t) \in \{(2,2), (2,4), (4,2)\}$. In [9], Nansoko et al. solved completely the diophantine equation $B_n^x + B_{n+1}^x + \cdots + B_{n+k-1}^x = B_m$ in positive integers (m, n, k, x).

In this paper, our purpose is to solve the following equations:

$$B_r = J_s + J_t, \quad (4)$$

and

$$C_r = J_s + J_t. \quad (5)$$

All the solutions of the two equations are given as follows:

Theorem 1. *Let $s \geq t$ and let (r, s, t) be a non-negative solution of equation (4). Then, the solutions are as follows:*

$$(1,1,0),\ (2,3,3),\ (2,4,1),\ (2,4,2). \tag{6}$$

Theorem 2. *Let $s \geq t$ and let (r, s, t) be a non-negative solution of equation (5). Then, the solutions are as follows:*

$$(0,1,0),\ (0,2,0),\ (1,3,0). \tag{7}$$

The key idea is to use a variant of Baker' theory due to Matveev to find an upper bound for all the implied variables in terms of a single variable. The obtained upper bound is usually too large to be investigated by computer calculations. Therefore, we apply a reduction method of Dujella and Pethö to cut down the upper bound. Lastly, we use Sage to determine all the solutions.

2. Preliminary Results

This section gathers the relevant background material that will be used throughout the paper.

2.1. Balancing and Lucas Balancing Sequences. The sequences of balancing and Lucas balancing numbers are characterized by the following equation:

$$\Theta(\nu) := \nu^2 - 6\nu + 1 = 0. \tag{8}$$

Let $\rho = 3 + \sqrt{8}$ and $\delta = 3 - \sqrt{8}$ be the solutions. Their Binet formulas are as follows:

$$\begin{aligned} B_n &= \frac{\rho^n - \delta^n}{2\sqrt{8}} \quad \text{for all } n \geq 0, \\ C_n &= \frac{\rho^n + \delta^n}{2} \quad \text{for } n \geq 0. \end{aligned} \tag{9}$$

One can show that

$$\rho^{n-1} \leq B_n < \rho^n \quad \text{holds for all } n \geq 1, \tag{10}$$

and

$$\frac{\rho^n}{2} \leq C_n < \frac{\rho^{n+1}}{2} \quad \text{holds for all } k \geq 2. \tag{11}$$

For more details concerning balancing and Lucas balancing numbers, see [10, 11].

2.2. Jacobsthal Sequence. The Jacobsthal numbers obey the following Binet formula:

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{12}$$

For $n \geq 1$, one can show that

$$2^{n-2} \leq J_n \leq 2^{n-1}. \tag{13}$$

We refer to [12, 13] for more details. The next inequality of linear forms is fundamental. Bugeaud, Mignotte, and Siksek deduced it, see [14], from Matveev's theorem [15].

2.3. A Theorem of Matveev. Consider an algebraic number α. Suppose the minimal polynomial (over $\mathbb{Z}$) of α has degree m and let $\alpha^{(i)}$'s be the conjugates of α. Then, the minimal polynomial can be written as follows:

$$c_0 x^m + c_1 x^{m-1} + \cdots + c_m = c_0 \prod_{i=1}^{m} \left(x - \alpha^{(i)}\right), \tag{14}$$

where c_0 is positive integer. The logarithmic Weil height (over algebraic real field) of α is given by the following equation:

$$h(\alpha) := \frac{1}{m}\left(\log c_0 + \sum_{i=1}^{m} \log\left(\max\left\{\left|\alpha^{(i)}\right|, 1\right\}\right)\right). \tag{15}$$

The function of logarithmic height satisfies the following properties (see [16] for proofs):

$$\begin{aligned} h(\alpha_1 \pm \alpha_2) &\leq h(\alpha_1) + h(\alpha_2) + \log 2, \\ h\left(\alpha_1 \alpha_2^{\pm 1}\right) &\leq h(\alpha_1) + h(\alpha_1), \\ h(\alpha^s) &= |s| h(\alpha)\ (s \in \mathbb{Z}). \end{aligned} \tag{16}$$

Theorem 3 (Matveev). *Suppose that $\alpha_1, \ldots, \alpha_k$ are positive real algebraic numbers in a real algebraic number field $\mathbb{A}$ of degree $D_{\mathbb{A}}$ and that $t_1, \ldots, t_k$ are nonzero integers such that the quantity*

$$\Omega_1 := \alpha_1^{t_1} \alpha_2^{t_2} \ldots \alpha_k^{t_k} - 1 \neq 0. \tag{17}$$

Let $H_i \geq \max\{D_{\mathbb{A}} h(\alpha_i), |\log \alpha_i|, 0.16\}$, for $i = 1, \ldots, k$ and $\beta \geq \max\{|t_1|, \ldots, |t_k|\}$. Then,

$$\log|\Omega| > -1.4 \cdot 30^{k+3} \cdot k^{4.5} \cdot D_{\mathbb{A}}^2 \cdot (1 + \log D_{\mathbb{A}}) \cdot (1 + \log \beta) H_1 \ldots H_k. \tag{18}$$

2.4. Reduction Lemma. Let $\|\theta\|$ the distance between a real number θ and the closest integer. The subsequent result is due to Dujella and Pethö, see [17].

Lemma 4. *Let $\lambda, \theta, A > 0, B > 1$ be given real numbers. Let K be a positive integer. Assume p/q is a convergent of λ with $q > 6K$. If $\epsilon := \|\theta q\| - K\|\lambda q\| > 0$ and $s, t, \omega > 0$ satisfy*

$$0<|s\lambda - t + \theta| < \frac{A}{B^{\omega}}, \tag{19}$$

with $s \leq K$, *then*

$$\omega < \frac{\log (Aq/\epsilon)}{\log B}. \tag{20}$$

2.5. Legendre Theorem. Legendre proved the following essential result in his book [18]. We will use this theorem in some cases of our investigation of balancing numbers. The interested reader can see [19] for more details.

Theorem 5. *Let* $p, q \in \mathbb{Z}$. *Assume* r *is a real number and* $r = [a_0, a_1, \ldots]$. *If*

$$\left|\frac{p}{q} - r\right| < \frac{1}{2q^2}, \tag{21}$$

then p/q *is a convergent of the continued fraction of* r. *Let* S *and* t *be integers (non-negative) such that* $q_t > S$ *and let* $b = \max_{0\leq i\leq t}\{a_i\}$. *Then,*

$$\frac{1}{(b+2)q^2} < \left|\frac{p}{q} - r\right|. \tag{22}$$

We assume, by symmetry of equations (4) and (5), that $s \geq t$.

3. Solving the Equation $B_r = J_s + J_t$

3.1. Bounding the Variables. Applying the inequalities (10) and (13), we get the following inequalities:

$$\begin{aligned} \rho^{r-1} &\leq B_r \leq 2^s, \\ 2^{s-2} &\leq B_r \leq \rho^r. \end{aligned} \tag{23}$$

These imply that

$$(s-2)\frac{\log 2}{\log \rho} \leq r \leq s\frac{\log 2}{\log \rho} + 1. \tag{24}$$

The value of $(\log 2/\log \rho)$ is approximately 0.39, so we can take $r < 2s$. Binet formulas can be inserted into equation (4) to give the following equation:

$$\frac{\rho^r - \delta^r}{2\sqrt{8}} = \frac{2^s - (-1)^s}{3} + \frac{2^t - (-1)^t}{3}. \tag{25}$$

Then,

$$\left|\frac{\rho^r}{2\sqrt{8}} - \frac{2^s}{3}\right| = \left|\frac{2^t}{3} - \frac{\left((-1)^s + (-1)^t\right)}{3} + \frac{\delta^r}{2\sqrt{8}}\right|. \tag{26}$$

This implies that

$$\left|\frac{\rho^r}{2\sqrt{8}} - \frac{2^s}{3}\right| < \frac{4\cdot 2^t}{3}. \tag{27}$$

Thus,

$$\left|\frac{3\rho^r 2^{-s}}{2\sqrt{8}} - 1\right| < \frac{4}{2^{s-t}}. \tag{28}$$

Let

$$\Omega_1 = \frac{3\rho^r 2^{-s}}{2\sqrt{2}} - 1, k = 3, \alpha_1 = \frac{3}{2\sqrt{8}}, \alpha_2 = \rho, \alpha_3 = 2, t_1 = 1, t_2 = r, t_3 = -s. \tag{29}$$

If $\Omega_1 = 0$, then $3\rho^k = 2^{s+2}.\sqrt{2}$. Let σ be the automorphism given by $\sigma(\rho) = \delta$. Therefore, $|3\delta^r| = 2^{s+2}.\sqrt{2}$. Indeed, $|3\delta^r| < 3$, a contradiction. So, $\Omega_1 \neq 0$.

Take $\mathbb{A} = \mathbb{Q}(\rho)$. Then, $D_{\mathbb{A}} = 2$. The logarithmic heights are as follows:

$$\begin{aligned} h(\alpha_1) &\leq h(2\sqrt{8}) + h(3) \leq \frac{5}{2}\log 2 + \log 3, \\ h(\alpha_2) &= \frac{1}{2}\log \rho, \\ h(\alpha_3) &= \log 2. \end{aligned} \tag{30}$$

Setting

$$\begin{aligned} &H_1 = 5\log 2 + 2\log 3, H_2 = \log \rho, H_3 = 2\log 2, \\ &\beta = 2s, \end{aligned} \tag{31}$$

and using the theorem of Matveev, we obtain the following inequality:

$$\log |\Omega_1| > -c_1(1 + \log 2s), \tag{32}$$

where $c_1 = 1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1 + \log 2)(2\log 3 + 5\log 2)(2\log 2 \log \rho)$. So,

$$\log |\Omega_1| > -2 \times 10^{13}(1 + \log 2s). \tag{33}$$

On the other hand, by inequality (28), one gets the following inequality:

$$\log |\Omega_1| < (t-s)\log 2 + \log 4. \tag{34}$$

A combination of inequalities (33) and (34) gives the following inequality:

$$-\log 4 + (s-t)\log 2 < 2 \times 10^{13}(1 + \log 2s). \tag{35}$$

Hence,

$$t\log 2 > s\log 2 - 2\times 10^{13}(1 + \log 2s) - \log 4. \tag{36}$$

From equation (25),

$$\frac{\rho^r}{2\sqrt{8}} - \frac{2^s\left(1+2^{t-s}\right)}{3} = \frac{\delta^r}{2\sqrt{8}} - \frac{(-1)^s-(-1)^t}{3}. \tag{37}$$

Therefore,

$$\left|\frac{3\rho^r 2^{-s}}{2\sqrt{8}\left(1+2^{t-s}\right)} - 1\right| = \left|\frac{3\cdot 2^{-s}}{2\sqrt{8}\left(1+2^{t-s}\right)}\left(\frac{\delta^r}{2\sqrt{8}} - \frac{(-1)^s-(-1)^t}{3}\right)\right|. \tag{38}$$

Hence,

$$\left|\frac{3\rho^r 2^{-s}}{2\sqrt{8}\left(1+2^{t-s}\right)} - 1\right| < \frac{5}{2^t}. \tag{39}$$

Let $\Omega_2 = (3\rho^r 2^{-s})/(2\sqrt{8}\,(1+2^{t-s})) - 1$. Then,

$$\log\left|\Omega_2\right| < \log 5 - t\log 2. \tag{40}$$

Let

$$\alpha_1 = \frac{3}{2\sqrt{8}\left(1+2^{t-s}\right)}, \alpha_2 = \rho, \alpha_3 = 2, k = 3, t_1 = 1, t_2 = r, t_3 = -s, B = 2s. \tag{41}$$

First, we show that $\Omega_2 \neq 0$. If $\Omega_2 = 0$, then $3\rho^r = 2\sqrt{8}\,(2^s+2^t)$. Again, let $\sigma(\rho) = \delta$. This gives $|3\delta^r| = 2\sqrt{8}\,(2^s+2^t)$. Since $|3\delta^r| < 3$, a contradiction, then we take $\mathbb{A} = \mathbb{Q}(\rho)$. Immediately, we get the following equation:

$$\begin{aligned} &h(\alpha_1) \le h(3) + h(2\sqrt{8}) + h\left(1+2^{t-s}\right) \le \log 3 + \frac{7}{2}\log 2 + (s-t)\log 2,\\ &h(\alpha_2) = \frac{1}{2}\log\rho,\\ &h(\alpha_3) = \log 2. \end{aligned} \tag{42}$$

Set

$$\begin{aligned} &H_1 = 2\log 3 + 2(s-t)\log 2 + 7\log 2, H_2 = \log\rho,\\ &H_3 = 2\log 2. \end{aligned} \tag{43}$$

By Matveev's Theorem, we get the following inequality:

$$\log\left|\Omega_2\right| > -c_2(1+\log 2s)(2\log 3 + 2(s-t)\log 2 + 7\log 2), \tag{44}$$

where $c_2 = 1.4\times 30^6\times 3^{4.5}\times 4\times(1+\log 2)(2\log 2\log\rho)$.

Using equations (35), (36), and (40) with simple calculations give the following inequality:

$$s\log 2 < 6\times 10^{13}(1+\log 2s) + 1.2\times 10^{26}(1+\log 2s)^2 + 3. \tag{45}$$

By a simple Mathematica calculation, we find the following estimation:

$$s < 3\times 10^{29}. \tag{46}$$

3.2. Reducing the Upper Bound. It is known that $|a| < |e^a - 1|$ for $a \in (-1/2, 1/2)$. Now, we aim to cut down the bound on n. Let

$$\Delta_1 = \log\left(\frac{3}{2\sqrt{8}}\right) + r\log\rho - s\log 2. \tag{47}$$

Equation (28) gives, for $t - s \ge 5$,

$$\left|\Omega_1\right| = \left|e^{\Delta_1} - 1\right| < \frac{4}{2^{t-s}} < \frac{1}{4}, \tag{48}$$

which implies that

$$\left|\Delta_1\right| < \frac{1}{2}. \tag{49}$$

Then, $|\Delta_1| < 2|e^{\Delta_1} - 1|$. Therefore, we get

$$\left|\Delta_1\right| < \frac{8}{2^{s-t}}. \tag{50}$$

We observe that $\Delta_1 \neq 0$ since $\Omega_1 \neq 0$. Then,

$$0<\left|\frac{\log(3/2\sqrt{8})}{\log 2}-s+r\left(\frac{\log\rho}{\log 2}\right)\right|<\frac{8}{2^{s-t}\log 2}<\frac{12}{2^{s-t}}. \quad (51)$$

Using Lemma 4 with $K=6\times 10^{29}$ $(K>2s>r)$, $\lambda=(\log\rho/\log 2)$, $\theta=(\log(3/2\sqrt{8})/\log 2)$, $A=12$, and $B=2$, let $\lambda=[a_0,a_1,\ldots]$, we find that $q_{61}=6332847229674209482244367144203>6K$. Compute

$$\epsilon=\|\theta q_{61}\|-K\|\lambda q_{61}\|>0.4. \quad (52)$$

It follows, by Lemma 4, that $s-t<108$. Set

$$\Delta_2=\log\left(\frac{3}{2\sqrt{8}\left(1+2^{t-s}\right)}\right)+r\log\rho-s\log 2. \quad (53)$$

Let $m\geq 5$. By equation (39), we have the following inequality:

$$\left|\Omega_2\right|=\left|e^{\Delta_2}-1\right|<\frac{5}{2^t}<\frac{1}{4}. \quad (54)$$

This implies that

$$\left|\Delta_2\right|<\frac{1}{2}. \quad (55)$$

Then, $|\Delta_2|<2|e^{\Delta_2}-1|$. Therefore,

$$\left|\Delta_2\right|<\frac{10}{2^t}. \quad (56)$$

Then,

$$0<\left|\frac{\log\left(3/\left(2\sqrt{8}\left(1+2^{t-s}\right)\right)\right)}{\log 2}-s+r\left(\frac{\log\rho}{\log 2}\right)\right|<\frac{15}{2^t}. \quad (57)$$

Applying Lemma 4 with $K=6\times 10^{29}$ $(K>2s>r)$, $\lambda=(\log\rho/\log 2)$, $\theta=(\log(3/2\sqrt{8}\,(1+2^{t-s})))/(\log 2)$, $A=15$, and $B=2$, we have $q_{65}>6K$. Consider ϵ in the following cases:

Case I: $s-t<108$ and $s-t\neq 1$

$$\epsilon=\|\theta q_{65}\|-K\|\lambda q_{65}\|>0.0008. \quad (58)$$

By Lemma 4, we get $t<124$, so $s<232$ and <464.

Case II: $s-t=1$. In this case the value of ϵ will be always negative and equation (4) becomes as follows:

$$B_r=2^t. \quad (59)$$

Then, $r<2t$, and from equation (89), we get $t<3\times 10^{29}$. As before, we can prove that

$$\gamma^r 2^{-(t+3)}-1<\frac{1}{2^t}, \quad (60)$$

This gives, for $t\geq 3$, that

$$\left|r\frac{\log\gamma}{\log 2}-(t+3)\right|<\frac{4}{2^t}<\frac{1}{4}. \quad (61)$$

Since $16t<2^t$ for $t\geq 7$, we get $4/2^t<1/2r^2$. Then, $|(\log\rho/\log 2)-t+3/r|<1/2r^2$. So, $t+3/r$ is a convergent of $(\log\rho/\log 2)$. Using $r<K$ and some computations we find that

$$q_{58}<K<q_{59}, \qquad b=\max_{0\leq i\leq 59}\{a_i\}=200. \quad (62)$$

Therefore,

$$\frac{1}{(200+2)r}<\frac{4}{2^t}. \quad (63)$$

Thus,

$$2^t<5\cdot 10^{32}. \quad (64)$$

Then, $t\leq 108$. Solving, using Sage, equation (102) for $t<108$ gives no solutions and for $t<124$, $s<232$, and $r<464$ gives the triples in Theorem 1.

4. Solving the Equation $C_r=J_s+J_t$

4.1. Bounding the Variables. By equations (11) and (13), we have the following equation:

$$\frac{\rho^r}{2}\leq C_r\leq 2^s, \qquad 2^{s-2}\leq C_r\leq\frac{\rho^{r+1}}{2}. \quad (65)$$

These imply that

$$(s-1)\frac{\log 2}{\log\rho}-1\leq r\leq(s+1)\frac{\log 2}{\log\rho}. \quad (66)$$

We take $r<2s$. Using Binet formulas, equation (5) can be written as follows:

$$\frac{\rho^r+\delta^r}{2}=\frac{2^s-(-1)^s}{3}+\frac{2^t-(-1)^t}{3}. \quad (67)$$

Then,

$$\left|\frac{\rho^r}{2}-\frac{2^s}{3}\right|=\left|\frac{2^t}{3}-\frac{\left((-1)^s+(-1)^t\right)}{3}-\delta^r\right|. \quad (68)$$

Therefore,

$$\left|\frac{\rho^r}{2}-\frac{2^s}{3}\right|\leq\frac{2^t}{3}+\frac{2}{3}+\left|\delta^r\right|. \quad (69)$$

Then,

$$\left|\frac{\rho^r}{2}-\frac{2^s}{3}\right|<\frac{4\cdot 2^t}{3}. \quad (70)$$

Thus,

$$\left|\frac{3\rho^r}{2^{s+1}}-1\right|<\frac{4}{2^{s-t}}. \quad (71)$$

Consider the following:

$$\Omega_3 = 3\rho^r 2^{-(s+1)} - 1, k = 3, \alpha_1 = 3, \alpha_2 = \rho, \alpha_3 = 2, t_1 = 1, t_2 = r, t_3 = -(s+1). \quad (72)$$

As before, we can prove that $\Omega_3 \neq 0$. Let $\mathbb{A} = \mathbb{Q}(\rho)$. Then,

$$h(\alpha_1) = \log 3,\quad h(\alpha_2) = \frac{1}{2}\log\rho,\quad h(\alpha_3) = \log 2. \quad (73)$$

Setting

$$H_1 = 2\log 3, H_2 = \log\rho, H_3 = 2\log 2. \quad (74)$$

Taking $\beta = 2s$, it follows that

$$\log|\Omega_3| > -c_3(1+\log 2s), \quad (75)$$

where $c_3 = -1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1+\log 2)(2\log 3)(2\log 2\log\rho)$.

Consequently,

$$\log|\Omega_3| > -3\times 10^{12}(1+\log 2s). \quad (76)$$

In addition, it follows from inequality (71) that

$$\log|\Omega_3| < \log 4 + (t-s)\log 2. \quad (77)$$

Comparing inequalities (76) and (77) entails that

$$(s-t)\log 2 - \log 6 < 3\times 10^{12}(1+\log 2s). \quad (78)$$

Hence,

$$t\log 2 > s\log 2 - 3\times 10^{12}(1+\log 2s) - \log 4. \quad (79)$$

Equation (67) is equivalent to

$$\frac{\rho^r}{2} - \frac{2^s(1+2^{t-s})}{3} = -\frac{\delta^r}{2} - \frac{(-1)^s - (-1)^t}{3}. \quad (80)$$

So,

$$\left|\frac{3\rho^r 2^{-(s+1)}}{1+2^{t-s}} - 1\right| = \left|\frac{3\cdot 2^{-s}}{1+2^{t-s}}\left(-\frac{\delta^r}{2} - \frac{(-1)^s-(-1)^t}{3}\right)\right|. \quad (81)$$

Then,

$$\left|\frac{3\rho^r 2^{-(s+1)}}{1+2^{t-s}} - 1\right| < \frac{5}{2^t}. \quad (82)$$

Let $\Omega_4 = 3/1 + 2^{t-s}\rho^k 2^{-(s+1)} - 1$. Then,

$$\log|\Omega_4| < \log 5 - t\log 2. \quad (83)$$

Set

$$\alpha_1 = \frac{3}{1+2^{t-s}}, \alpha_2 = \rho, \alpha_3 = 2, k = 3, t_1 = 1, t_2 = r, t_3 = -(s+1). \quad (84)$$

Again, $\Omega_4 \neq 0$. Let $\mathbb{A} = \mathbb{Q}(\rho)$. Then,

$$h(\alpha_1) \le \log 3 + (s-t)\log 2 + \log 2,\quad h(\alpha_2) = \frac{1}{2}\log\rho,\quad h(\alpha_3) = \log 2. \quad (85)$$

Let

$$H_1 = 2\log 3 + 2(s-t)\log 2 + 2\log 2, H_2 = \log\gamma, H_3 = 2\log 2,\quad \beta = 2s. \quad (86)$$

Then,

$$\log|\Omega_4| > -c_4(1+\log 2s)(2\log 3 + 2\log 2 + 2(s-t)\log 2), \quad (87)$$

where $c_4 = 1.4\times 30^6 \times 3^{4.5}\times 4\times(1+\log 2)(2\log 2\log\rho)$. Using equations (78), (79), and (83) and simple manipulations, it follows that

$$s\log 2 < 1.1\times 10^{13}(1+\log 2s) + 1.2\times 10^{25}(1+\log 2s)^2 + 3. \quad (88)$$

Solution by Mathematica gives the following equation:

$$s < 3\times 10^{28}. \quad (89)$$

4.2. Reducing the Upper Bound. For $s - t \ge 5$, let

$$\Delta_3 = \log(3) + r\log\rho - (s+1)\log 2. \quad (90)$$

From equation (71), we have the following equation:

$$|\Omega_3| = |e^{\Delta_3} - 1| < \frac{4}{2^{s-t}} < \frac{1}{4}. \quad (91)$$

So,

$$|\Delta_3| < \frac{1}{2}. \quad (92)$$

Then, $|\Delta_3| < 2|e^{\Delta_3} - 1|$. Therefore, we have the following equation:

$$|\Delta_3| < \frac{8}{2^{s-t}}. \quad (93)$$

Since $\Delta_3 \neq 0$, then

$$0 < \left| \frac{\log 3}{\log 2} - (s+1) + r\left(\frac{\log \rho}{\log 2}\right) \right| < \frac{8}{2^{s-t}\log 2} < \frac{12}{2^{s-t}}. \quad (94)$$

Applying Lemma 4 with $K = 6 \times 10^{28}$ $(K > 2s > r)$, $\lambda = (\log \rho / \log 2)$, $\theta = (\log 3/ \log 2)$, $A = 12$, and $B = 2$, the expansion of λ entails that $q_{65} > 6K$. Computing

$$\epsilon = \|\theta q_{65}\| - K\|\lambda q_{65}\| > 0.3. \quad (95)$$

Thus, by Lemma 4, we get $s - t < 117$. Set

$$\Delta_4 = \log\left(\frac{3}{1 + 2^{t-s}}\right) + r\log\rho - (s+1)\log 2, \quad (96)$$

and assume that $t > 5$. From equation (82), we deduce that

$$|\Omega_4| = |e^{\Delta_4} - 1| < \frac{5}{2^t} < \frac{1}{4}. \quad (97)$$

We conclude that

$$|\Delta_4| < \frac{1}{2}. \quad (98)$$

Then, $|\Delta_4| < 2|e^{\Delta_4} - 1|$. Therefore, we get

$$|\Delta_4| < \frac{10}{2^m}. \quad (99)$$

We have $\Delta_4 \neq 0$. So,

$$0 < \left| \frac{\log\left(3/\left(1 + 2^{t-s}\right)\right)}{\log 2} - (s+1) + r\left(\frac{\log \rho}{\log 2}\right) \right| < \frac{15}{2^t}. \quad (100)$$

Applying Lemma 4 again with $K = 6 \times 10^{28}$, $\lambda = (\log \rho / \log 2)$, $\theta = (\log(3/1 + 2^{m-n}))/(\log 2)$, $A = 15$, and $B = 2$, we get $q_{65} > 6K$. Consider ϵ in two cases.

Case I: If $s - t < 117$ and $s - t \neq 1$

$$\epsilon = \|\theta q_{65}\| - K\|\lambda q_{65}\| > 0.01. \quad (101)$$

Hence, $t < 122$, $s < 239$, and $r < 478$.

Case II: If $s - t = 1$, we get ϵ always negative. Solving equation (5) for $s - t = 1$. In this case equation (5) can be reduced to

$$C_r = 2^t. \quad (102)$$

By induction, we can prove that all the Lucas balancing numbers are odd. Therefore, we have no solutions in this case.

Solving equation (5) for $t < 122$, $s < 239$, and $r < 478$ yields the solutions which appear in Theorem 2.

5. Conclusion

We determined all the balancing and Lucas balancing numbers that are sums of Jacobsthal numbers. We mainly used Matveev's theorem. Also, we use a reduction lemma due to Dujella and Pethö to reduce the obtained upper bound. We revealed that there are four balancing numbers and three Lucas balancing numbers expressible as sums of two Jacobsthal numbers. In future, this work may be extended to investigate cobalancing and k-balancing numbers that are expressible as sums of Jacobsthal numbers.

Data Availability

The data used in this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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