Hindawi
Journal of Mathematics
Volume 2023, Article ID 8865992, 14 pages
https://doi.org/10.1155/2023/8865992

Research Article

Some Novel Estimates of Integral Inequalities for a Generalized Class of Harmonical Convex Mappings by Means of Center-Radius Order Relation

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Received 28 November 2022; Revised 7 January 2023; Accepted 27 May 2023; Published 22 June 2023

Academic Editor: Ching-Feng Wen

In interval analysis, integral inequalities are determined based on different types of order relations, including pseudo, fuzzy, inclusion, and various other partial order relations. By developing a link between center-radius (CR) order relations, it seeks to develop a theory of inequalities with novel estimates. A (CR)-order relation relationship differs from traditional interval-order relationships in that it is calculated as follows: $q = \langle q_c, q_r\rangle = \langle \overline{q} + \underline{q}/2, \overline{q} - \underline{q}/2\rangle$. There are several advantages to using this ordered relationship, including the fact that the inequality terms deduced from it yield much more precise results than any other partial-order relation defined in the literature. This study introduces the concept of harmonical (h_1, h_2)-convex functions associated with the center-radius order relations, which is very novel in literature. Applied to uncertainty, the center-radius order relation is an effective tool for studying inequalities. Our first step was to establish the Hermite–Hadamard $(\mathscr{H}.\mathscr{H})$ inequality and then to establish Jensen inequality using these notions. We discuss a few exceptional cases that could have practical applications. Moreover, examples are provided to verify the applicability of the theory developed in the present study.

1. Introduction

Uncertainty problems can be distorted by using specific numbers. It is therefore crucial to avoid such errors and obtain effective results. Furthermore, Moore [1] proposed and investigated interval analysis for the first time in 1966. It is a discipline in which an uncertain variable is represented by an interval of real numbers. Through this analysis, the accuracy level of problems is improved. A variety of fields have been affected by it over the past 50 years, including differential equations with intervals [2], neural networks [3], aeroelasticity [4], and error analysis [5]. As a result, interval analysis has yielded many excellent results, and readers interested in reading more can consult reference [6].

The concept of convexity is becoming increasingly important to both the pure and applied sciences. Research on the concept of convexity with integral problems is an exciting area. Integral inequalities are useful for evaluating convexity and nonconvexity qualitatively and quantitatively. This area of research has grown in popularity due to its diverse applications in different fields. Convexity is an integral part of optimization concepts and is widely used in operation research, economics, control theory, decision-making, and management. There is a great deal of experience among mathematicians who deal with inequalities, such as those related to Ostrowski, Opial, Simpson, Jensen, and Hermite–Hadamard. In an attempt to promote convexity subjectively, we apply several fundamental integral inequalities. This has resulted in many inequalities as an application of convex functions and generalized convex functions, see reference [7]. Later, various partial order relations, as well as different integral operators, were used to establish a strong interrelationship between inequalities and interval-valued functions ($\mathscr{IVFS}$). Khan et al. [8] established Hermite–Hadamard type inequalities for left right interval-valued functions. Nwaeze et al. [9] developed a fractional version of these inequalities for polynomial convex interval-valued functions. Several practical applications have been developed based on these concepts see reference [10]. For $\mathscr{IVFS}$, initially, Breckner introduces the concept of continuity, see reference [11]. Chalco-Cano et al. [12] established the Ostrowski-type inequality, Costa et al. [13], Flores-Franulic and Román-Flores [14], and Costa et al. [13] established the Opial-type inequality for $\mathscr{IVFS}$. A famous double inequality is defined as follows:

$$\frac{\Psi(q)+\Psi(r)}{2} \geq \frac{qr}{r-q}\int_q^r \Psi(\omega)\mathrm{d}\omega \geq \Psi\left(\frac{2qr}{q+r}\right). \quad (1)$$

An approximation of the mean value of a continuous function is provided by the function. Despite its simplicity, it is well known because of its definition of convex mappings, the first geometrical interpretation in elementary mathematics. In 2007, Varoşanec [15] introduced the idea of h-convexity. Inspired by this idea, Zhao et al. [16], introduced the notion of h-convex$\mathscr{IVFS}$, and utilizing these notions, Jensen and $\mathscr{H}.\mathscr{H}$ inequality were established. There are a variety of interval-valued generalizations of these inequalities. Zhang et al. [17] developed Hermite–Hadamard and Jensen-type inequalities for the generalized class of Godunova–Levin functions. Afzal et al. [18] introduced the notion of Harmonical Godunova–Levin interval-valued functions and developed these inequalities see reference [19]. Initially, Awan et al. introduced the notion of (h_1, h_2)-convex functions and developed the following results [20]. Later, different authors used the notion of (h_1, h_2)-convexity and developed the following inequalities using related classes of convexity see references [21–23]. The results developed using partial order relations, including inclusion relations, pseudo-order relations, and fuzzy order relations are not as accurate as the results developed using the center-radius method. Therefore, center-radius order is an ideal tool for studying inequalities, and it was developed by the following author [24]. Based on harmonical CR-h-convex functions, this result can be proved by Liu et al. [25].

Theorem 1 (See [25]). *Consider* $\Psi\colon [q,r] \longrightarrow \mathbf{R}_I^+$. *Define* $h\colon (0,1) \longrightarrow \mathbf{R}^+$ *and* $h(1/2) \neq 0$. *If* $\Psi \in \mathrm{SHX}(CR-\mathrm{h}, [q,r], \mathbf{R}_I^+)$ *and* $\Psi \in \mathbf{IR}_{[q,r]}$, *then*

$$\begin{aligned}\frac{1}{2h(1/2)}\Psi\left(\frac{2qr}{q+r}\right) &\preceq_{CR} \frac{qr}{r-q}\int_q^r \frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega \\ &\preceq_{CR} [\Psi(q)+\Psi(r)]\int_0^1 h(s)\mathrm{d}s.\end{aligned} \quad (2)$$

The set of all harmonically CR-h-convex functions over $[q,r]$ is denoted by $\mathrm{SHX}(CR-h, [q,r], \mathbf{R}_I^+)$, In addition, Jensen-type inequality was also established using the notion of harmonical h-convexity via center-radius order relation.

Theorem 2 (See [25]). *Let* $c_i \in \mathbf{R}^+$, $j_i \in [q,r]$. *If* h *is non-negative super multiplicative function and* $\Psi \in \mathrm{SHX}(CR-\mathrm{h}, [q,r], \mathbf{R}_I^+)$ *then this holds:*

$$\Psi\left(\frac{1}{1/C_k\sum_{i=1}^k c_i j_i}\right) \preceq_{CR} \sum_{i=1}^k h\left(\frac{c_i}{C_k}\right)\Psi(j_i). \quad (3)$$

There is novelty and significance in this study because for the first time, harmonical (h_1, h_2)-convexity is connected with center-radius order relations. Furthermore, this class is more generalized since different choices of h result in different classes of harmonic convex functions. There are a variety of partial order relations, but CR-order is distinct from them. The center and radius concepts can be calculated by using the endpoints of intervals such as: $q_C = \underline{q}+\overline{q}/2$ and $q_R = \underline{q}-\overline{q}/2$, respectively, where $q = [\underline{q}, \overline{q}]$.

We get our research ideas from the extensive literature and specific articles, see references [21, 25]. Using the notions of harmonical convexity and center-radius order, we introduce a novel class of convexity called harmonical CR-(h_1, h_2)-convex functions. By utilizing this new idea, we developed $\mathscr{H}.\mathscr{H}$ and Jensen-type inequalities. Furthermore, the study provides relevant examples to back up its findings.

2. Preliminaries

This section summarizes some fundamental concepts, results, and definitions. Several terms were mentioned but not explained, see references [16, 25]. As you proceed through the paper, it will prove very helpful to have a basic understanding of interval analysis arithmetic

$$
\begin{aligned}
[q] &= [\underline{q}, \overline{q}]\left(s \in \mathbf{R}, \underline{q} \leqq s \leqq \overline{q};\ s \in \mathbf{R}\right),\\
[r] &= [\underline{r}, \overline{r}]\left(s \in \mathbf{R}, \underline{r} \leqq s \leqq \overline{r};\ s \in \mathbf{R}\right),\\
[q] + [r] &= [\underline{q}, \overline{q}] + [\underline{r}, \overline{r}]\\
&= [\underline{q} + \underline{r}, \overline{q} + \overline{r}],\\
\nu q = \nu[\underline{q}, \overline{q}] &= \begin{cases} [\nu \underline{q}, \nu \overline{q}] & (\nu > 0),\\ \{0\} & (\nu = 0),\\ [\nu \overline{q}, \nu \underline{q}] & (\nu < 0), \end{cases}
\end{aligned} \tag{4}
$$

where $\nu \in \mathbf{R}$. Suppose $\mathbf{R}_I$ and $\mathbf{R}_I^+$ be the collection of all closed and positive intervals of $\mathbf{R}$, respectively. We will now talk about certain interval arithmetic algebraic properties.

Let $q = [\underline{q}, \overline{q}] \in \mathbf{R}_I$, then $q_c = \overline{q} + \underline{q}/2$ and $q_r = \overline{q} - \underline{q}/2$ are the center and radius of interval q, respectively. The center-radius form of interval q can be represented as follows:

$$
\begin{aligned}
q &= \langle q_c, q_r \rangle\\
&= \%\langle \frac{\overline{q} + \underline{q}}{2}, \frac{\overline{q} - \underline{q}}{2} \%\rangle.
\end{aligned} \tag{5}
$$

Definition 1. The CR-order relation for $q = [\underline{q}, \overline{q}] = \langle q_c, q_r \rangle$ and $r = [\underline{r}, \overline{r}] = \langle r_c, r_r \rangle \in \mathbf{R}_I$ can be represented as follows:

$$
q \preceq_{CR} r \Longleftrightarrow \begin{cases} q_c < r_c, & \text{if} \quad q_c \neq r_c,\\ q_r \leq r_r, & \text{if} \quad q_c = r_c, \end{cases} \tag{6}
$$

NOTE: For any arbitrary two intervals $q, r \in \mathbf{R}_I$, this holds either $q \preceq_{CR} r$ or $r \preceq_{CR} q$. Riemann integral operators for $\mathcal{IVFS}$ are represented as follows:

Definition 2 (See [25]). Let $\mathcal{P}$: $[q, r]$ be an interval-valued function ($\mathcal{IVF}$) such that $\mathcal{P} = [\underline{\mathcal{P}}, \overline{\mathcal{P}}]$. Then, $\mathcal{P}$ is Riemann integrable (**IR**) on $[q, r]$ iff $\underline{\mathcal{P}}$ and $\overline{\mathcal{P}}$ are **IR** on $[q, r]$, that is as follows:

$$
\int_q^r \mathcal{P}(x)\mathrm{d}x = \left[\int_q^r \underline{\mathcal{P}}(x)\mathrm{d}x \int_q^r \overline{\mathcal{P}}(x)\mathrm{d}x\right]. \tag{7}
$$

The bundle of all (**IR**)$\mathcal{IVFS}$ on $[q, r]$ is represented by $\mathbf{IR}_{[q,r]}$.

Shi et al. [25] demonstrated that the integral retains order on the basis of CR-order relations.

Theorem 3. *Let $\mathcal{P}, \mathcal{Q}$: $[q, r]$ be $\mathcal{IVFS}$ given by $\mathcal{P} = [\underline{\mathcal{P}}, \overline{\mathcal{P}}]$ and $\mathcal{Q} = [\underline{\mathcal{Q}}, \overline{\mathcal{Q}}]$. If $\mathcal{P}(x) \preceq_{CR} \mathcal{Q}(x)$ and $\forall x \in [q, r]$, then*

$$
\int_q^r \mathcal{P}(x)\mathrm{d}x \preceq_{\mathscr{CR}} \int_q^r \mathcal{Q}(x)\mathrm{d}x. \tag{8}
$$

The following example will help to prove the above-mentioned theorem.

Example 1. Let $\mathcal{P} = [x, 2x]$, $\mathcal{Q} = [x^2, x^2 + 2]$, and for $x \in [0, 1]$, we have

$$
\begin{aligned}
\mathcal{P}_{\mathscr{C}} &= \frac{3x}{2},\\
\mathcal{P}_{\mathscr{R}} &= \frac{x}{2},\\
\mathcal{Q}_{\mathscr{C}} &= x^2 + 1,\\
\mathcal{Q}_{\mathscr{R}} &= 1.
\end{aligned} \tag{9}
$$

From Definition 1, we have $\mathcal{P}(x) \preceq_{CR} \mathcal{Q}(x)$ and $x \in [0, 1]$ (see Figures 1–3).

Since

$$
\begin{aligned}
\int_0^1 [x, 2x]\mathrm{d}x &= \left[\frac{1}{2}, 1\right],\\
\int_0^1 [x^2, x^2 + 2]\mathrm{d}x &= \left[\frac{1}{3}, \frac{7}{3}\right].
\end{aligned} \tag{10}
$$

Now, again using the Definition 1, we have

$$
\int_0^1 \mathcal{P}(x)\mathrm{d}x \preceq_{CR} \int_0^1 \mathcal{Q}(x)\mathrm{d}x. \tag{11}
$$

2.1. Some Novel Definitions Pertaining to Total Order Relations

Definition 3 (See [25]). Let Ψ: $[q, r] \longrightarrow \mathbf{R}^+$ and h: $[0, 1] \longrightarrow \mathbf{R}^+$ be two non-negative functions. Then, Ψ is said to be a harmonically h-convex function or that $\Psi \in \mathrm{SHX}(h, [q, r], \mathbf{R}^+)$, if for all $q_1, r_1 \in [q, r]$, and $\omega \in [0, 1]$, we have

$$
\Psi\left(\frac{q_1 r_1}{\omega q_1 + (1 - \omega) r_1}\right) \leq h(\omega)\Psi(q_1) + h(1 - \omega)\Psi(r_1). \tag{12}
$$

If in (12) $\leq$ replaced with $\geq$ it is called harmonical h-concave function or $\Psi \in SHV(h, [q, r], \mathbf{R}^+)$.

Definition 4 (See [25]). Let Ψ: $[q, r] \longrightarrow \mathbf{R}^+$ and h_1, h_2: $[0, 1] \longrightarrow \mathbf{R}^+$ be non-negative functions. Then, Ψ is said to be a harmonical (h_1, h_2)-convex function, or that $\Psi \in \mathrm{SHX}((h_1, h_2), [q, r], \mathbf{R}^+)$, if for all $q_1, r_1 \in [q, r]$ and $\omega \in [0, 1]$, we have

$$
\begin{aligned}
\Psi\left(\frac{q_1 r_1}{\omega q_1 + (1 - \omega) r_1}\right) &\leq h_1(\omega) h_2(1 - \omega)\Psi(q_1)\\
&\quad + h_1(1 - \omega) h_2(\omega)\Psi(r_1).
\end{aligned} \tag{13}
$$

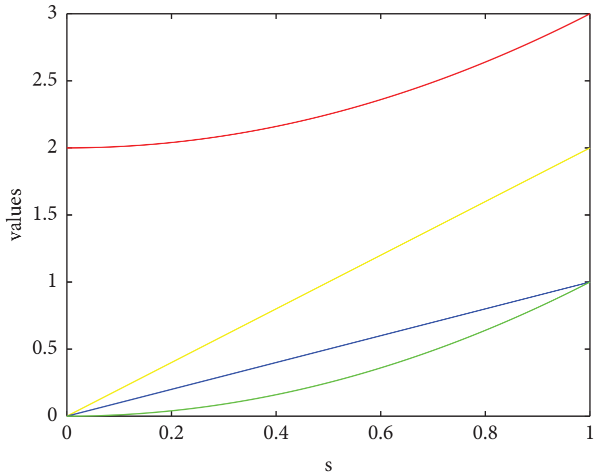

FIGURE 1: x^2+2 is shown as a red, $2x$ is shown as a yellow, x is shown as blue, and x^2 as a green line, respectively. A clear indication of the validity of the CR-order relationship can be seen in the graph.

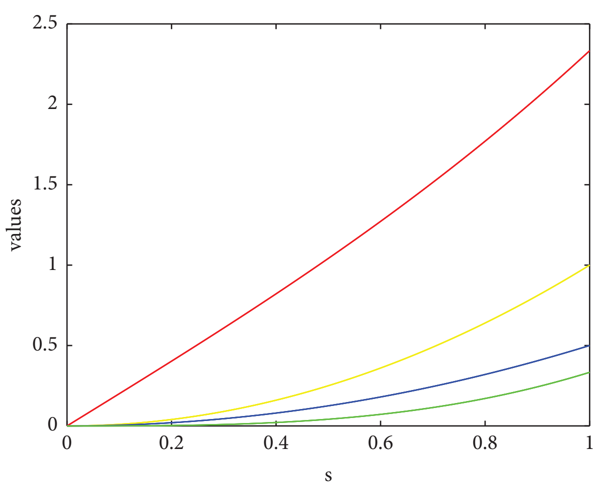

FIGURE 2: $2x + x^3/3$ is shown as a red, x^2 is shown as a yellow, $x^2/2$ is shown as blue, and $x^3/3$ as a green line, respectively. As can be seen from the graph, Theorem 3 is valid.

If in (13)" $\leq$ " replaced with" $\geq$ " it is called harmonical (h_1, h_2)-concave function or $\Psi \in \mathrm{SHV}((h_1, h_2), [q, r], \mathbf{R}^+)$.

Now let us introduce the notion of harmonically CR-convexity.

Definition 5 (See [25]). Let $\Psi\colon [q, r] \longrightarrow \mathbf{R}_\mathbf{I}^+$ be non-negative interval-valued function given by $\Psi = [\underline{\Psi}, \overline{\Psi}]$ and $h\colon [0,1] \longrightarrow \mathbf{R}^+$ be a non-negative function. Then, Ψ is said to be a harmonical CR-h-convex function, or that $\Psi \in \mathrm{SHX}(CR - h, [q, r], \mathbf{R}_\mathbf{I}^+)$, if for all $q_1, r_1 \in [q, r]$ and $\omega \in [0, 1]$, we have

$$\Psi\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) \preceq_{CR} h(\omega)\Psi(q_1) + h(1-\omega)\Psi(r_1). \quad (14)$$

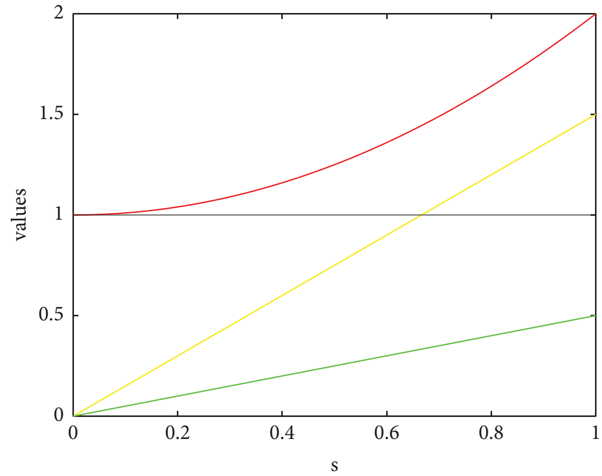

FIGURE 3: $\mathcal{Q}_\mathcal{C} = x^2 + 1$ is shown as a red, $\mathcal{P}_\mathcal{C} = 3x/2$ is shown as a yellow, $\mathcal{Q}_\mathcal{R} = 1$ is shown as black, and $\mathcal{P}_\mathcal{R} = x/2$ as a green line, respectively.

If in (14)" $\preceq_{CR}$" replaced with" $\succcurlyeq_{CR}$" it is called harmonical CR-h-concave function or $\Psi \in \mathrm{SHV}(CR - h, [q, r], \mathbf{R}_\mathbf{I}^+)$

Our next step will be to define a novel definition for harmonically CR-(h_1, h_2)-convex functions.

Definition 6. Let $\Psi\colon [q, r] \longrightarrow \mathbf{R}_\mathbf{I}^+$ be non-negative interval-valued function given by $\Psi = [\underline{\Psi}, \overline{\Psi}]$ and $h_1, h_2\colon [0,1] \longrightarrow \mathbf{R}^+$ be non-negative functions. Then Ψ is said to be harmonical CR-(h_1, h_2)-convex function, or that $\Psi \in \mathrm{SHX}(CR - (h_1, h_2), [q, r], \mathbf{R}_\mathbf{I}^+)$, if for all $q_1, r_1 \in [q, r]$ and $\omega \in [0, 1]$, we have

$$\Psi\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) \preceq_{CR} h_1(\omega) h_2(1-\omega)\Psi(q_1) + h_1(1-\omega) h_2(\omega)\Psi(r_1). \quad (15)$$

If in (15)" $\preceq_{CR}$" replaced with" $\succcurlyeq_{CR}$" it is called harmonical CR-(h_1, h_2)-concave function or $\Psi \in \mathrm{SHV}(CR - (h_1, h_2), [q, r], \mathbf{R}_\mathbf{I}^+)$.

Remark 1

(1) If $h_1 = h_2 = 1$, Definition 6 becomes a harmonical CR-P-function [25].

(2) If $h_1(\omega) = 1/h_1(\omega)$, $h_2 = 1$, Definition 6 becomes a harmonical CR-GL-convex function [25].

(3) If $h_1(\omega) = h_1(\omega)$, $h_2 = 1$, Definition 6 becomes a harmonical CR-h-convex function [25].

(4) If $h_1(\omega) = \omega^s$, $h_2 = 1$, Definition 6 becomes a harmonical CR-s-convex function [25].

3. Main Results

Proposition 1. *Consider* $\Psi: [q, r] \longrightarrow \mathscr{R}_{\mathscr{I}}$ *given by* $[\underline{\Psi}, \overline{\Psi}] = (\Psi_C, \Psi_R)$. *If* Ψ_C *and* Ψ_R *are harmonical* (h_1, h_2)*-convex over* $[q, r]$*, then* Ψ *is a harmonical CR-*(h_1, h_2)*-convex function over* $[q, r]$.

Proof. Since Ψ_C and Ψ_R are harmonical (h_1, h_2)-convex over $[q, r]$, for each $\omega \in (0, 1)$ and for all $q_1, r_1 \in [q, r]$, we have

$$\begin{aligned}\Psi_C\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) &\leq h_1(\omega) h_2(1-\omega) \Psi_C(q_1) \\ &\quad + h_1(1-\omega) h_2(\omega) \Psi_C(r_1), \\ \Psi_R\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) &\leq h_1(\omega) h_2(1-\omega) \Psi_R(q_1) \\ &\quad + h_1(1-\omega) h_2(\omega) \Psi_R(r_1),\end{aligned} \tag{16}$$

Now, if

$$\begin{aligned}\Psi_C\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) &\neq h_1(\omega) h_2(1-\omega) \Psi_C(q_1) \\ &\quad + h_1(1-\omega) h_2(\omega) \Psi_C(r_1),\end{aligned} \tag{17}$$

for each $\omega \in (0, 1)$ and for all $q_1, r_1 \in [q, r]$

$$\begin{aligned}\Psi_C\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) &< h_1(\omega) h_2(1-\omega) \Psi_C(q_1) \\ &\quad + h_1(1-\omega) h_2(\omega) \Psi_C(r_1),\end{aligned} \tag{18}$$

accordingly

$$\begin{aligned}\Psi_C\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) &\preceq_{CR} h_1(\omega) h_2(1-\omega) \Psi_C(q_1) \\ &\quad + h_1(1-\omega) h_2(\omega) \Psi_C(r_1).\end{aligned} \tag{19}$$

Otherwise, for each $\omega \in (0, 1)$ and for all $q_1, r_1 \in [q, r]$

$$\begin{aligned}\Psi_R\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) &\leq h_1(\omega) h_2(1-\omega) \Psi_R(q_1) \\ &\quad + h_1(1-\omega) h_2(\omega) \Psi_R(r_1) \\ \Longrightarrow \Psi\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) &\preceq_{CR} h_1(\omega) h_2(1-\omega) \Psi(q_1) \\ &\quad + h_1(1-\omega) h_2(\omega) \Psi(r_1).\end{aligned} \tag{20}$$

Combining all the above, from Definition 6, it can be written as

$$\begin{aligned}\Psi\left(\frac{q_1 r_1}{\omega q_1 + (1-\omega) r_1}\right) &\preceq_{CR} h_1(\omega) h_2(1-\omega) \Psi(q_1) \\ &\quad + h_1(1-\omega) h_2(\omega) \Psi(r_1),\end{aligned} \tag{21}$$

for each $\omega \in (0, 1)$ and for all $q_1, r_1 \in [q, r]$.

This completes the proof. □

Example 2. Consider $[q, r] = [1, 2]$, $h_1(s) = s$, $h_2(s) = 1$ and $\forall s \in [0, 1]$. $\Psi: [q, r] \longrightarrow \mathbf{R}_{\mathbf{I}}^{+}$ are defined as follows:

$$\Psi(\omega) = \left[\frac{-1}{\omega^2} + 4, \frac{1}{\omega^2} + 5\right], \omega \in [1, 2]. \tag{22}$$

Then,

$$\Psi_C(\omega) = \frac{9}{2}, \Psi_R(\omega) = \frac{1}{\omega^2} + \frac{1}{2}, \quad \omega \in [1, 2]. \tag{23}$$

It is obvious that $\Psi_C(\omega), \Psi_R(\omega)$ are harmonical (h_1, h_2) convex functions over [0, 1] (see Figure 4). This implies that from Proposition 1, Ψ is also a harmonical CR-(h_1, h_2) convex function on [0,1].

3.1. Some Variants of Hermite–Hadamard Inequalities for Harmonical-Convex Mappings Using Total Order Relations

Theorem 4. *Let* $h_1, h_2: (0, 1) \longrightarrow \mathbf{R}^{+}$ *and* $h_1(1/2) h_2(1/2) \neq 0$. *Let* $\Psi: [q, r] \longrightarrow \mathbf{R}_{\mathbf{I}}^{+}$, *if* $\Psi \in \mathrm{SHX}(\mathrm{CR} - (h_1, h_2), [q, r], \mathbf{R}_{\mathbf{I}}^{+})$*, and* $\Psi \in \mathbf{IR}_{[\mathbf{q},\mathbf{r}]}$*, we have*

$$\begin{aligned}\frac{1}{2[H(1/2, 1/2)]} \Psi\left(\frac{2qr}{q + r}\right) &\preceq_{CR} \frac{qr}{r - q} \int_q^r \frac{\Psi(\omega)}{\omega^2} d\omega \\ &\preceq_{CR} [\Psi(q) + \Psi(r)] \int_0^1 H(s, 1 - s) \mathrm{d}s.\end{aligned} \tag{24}$$

Proof. Since $\Psi \in \mathrm{SHX}(CR - (h_1, h_2), [q, r], \mathbf{R}_{\mathbf{I}}^{+})$, we have

$$\frac{1}{[H(1/2, 1/2)]} \Psi\left(\frac{2qr}{q + r}\right) \preceq_{CR} \Psi\left(\frac{qr}{sq + (1 - s)r}\right) + \Psi\left(\frac{qr}{(1 - s)q + sr}\right). \tag{25}$$

By integrating of the above inequality over (0,1), we have

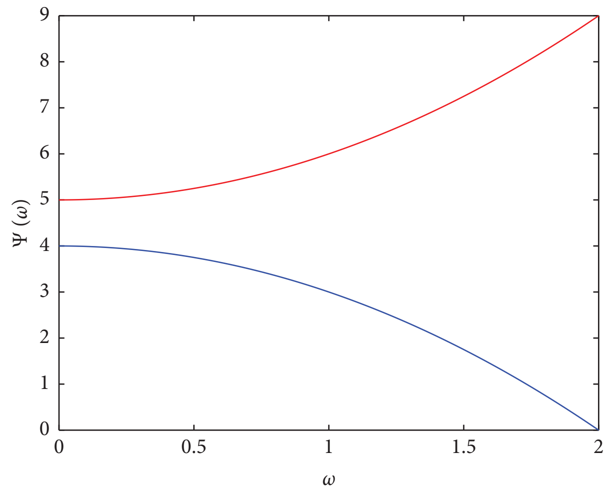

Figure 4: Ψ is shown as a blue and $\overline{\Psi}$ as a red line, respectively.

$$\begin{aligned}
\frac{1}{[H(1/2,1/2)]}\Psi\left(\frac{2qr}{q+r}\right) &\preceq_{CR}\left[\int_0^1\Psi\left(\frac{qr}{sq+(1-s)r}\right)\mathrm{d}s+\int_0^1\Psi\left(\frac{qr}{(1-s)q+sr}\right)\mathrm{d}s\right]\\
&=\left[\int_0^1\underline{\Psi}\left(\frac{qr}{sq+(1-s)r}\right)\mathrm{d}s+\int_0^1\underline{\Psi}\left(\frac{qr}{(1-s)q+sr}\right)\mathrm{d}s,\int_0^1\overline{\Psi}\left(\frac{qr}{sq+(1-s)r}\right)\mathrm{d}s\right.\\
&\quad\left.+\int_0^1\overline{\Psi}\left(\frac{qr}{(1-s)q+(sr}\right)\mathrm{d}s\right]\\
&=\left[\frac{2qr}{r-q}\int_q^r\frac{\underline{\Psi}(\omega)}{\varnothing}\mathrm{d}\omega,\frac{2qr}{r-q}\int_q^r\frac{\overline{\Psi}(\omega)}{\omega^2}\mathrm{d}\omega\right]\\
&=\frac{2qr}{r-q}\int_q^r\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega.
\end{aligned}\tag{26}$$

By Definition 6, we have

$$\begin{aligned}
\Psi\left(\frac{qr}{sq+(1-s)r}\right)&\preceq_{CR}h_1(s)h_2(1-s)\Psi(q)\\
&\quad+h_1(1-s)h_2(s)\Psi(r).
\end{aligned}\tag{27}$$

By integrating of the abovementioned inequality over (0, 1), we have

$$\begin{aligned}
\int_0^1\Psi\left(\frac{qr}{sq+(1-s)r}\right)\mathrm{d}s&\preceq_{CR}\Psi(q)\int_0^1 h_1(s)h_2(1-s)\mathrm{d}s\\
&\quad+\Psi(r)\int_0^1 h_1(1-s)h_2(s)\mathrm{d}s.
\end{aligned}\tag{28}$$

Accordingly

$$\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)}{\omega^2}d\omega\preceq_{CR}[\Psi(q)+\Psi(r)]\int_0^1 H(s,1-s)\mathrm{d}s.\tag{29}$$

Now, combining (26) and (29), we get required result

$$\begin{aligned}
\frac{1}{2[H(1/2,1/2)]}\Psi\left(\frac{2qr}{q+r}\right)&\preceq_{CR}\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega\\
&\preceq_{CR}[\Psi(q)+\Psi(r)]\int_0^1 H(s,1-s)\mathrm{d}s.
\end{aligned}\tag{30}$$

□

Remark 2

(1) If $h_1(s)=h_2(s)=1$, then Theorem 4 becomes result for harmonical CR- P-function:

$$\frac{1}{2}\Psi\left(\frac{2qr}{q+r}\right)\preceq_{CR}\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega\preceq_{CR}[\Psi(q)+\Psi(r)]. \tag{31}$$

(2) If $h_1(s)=1/h(s)$, $h_2(s)=1$, then Theorem 4 becomes result for harmonical CR-h-Godunova–Levin-function

$$\frac{h\left(\int_q^r\Psi(\omega)/\omega^2\mathrm{d}\omega\right)}{2}\Psi\left(\frac{2qr}{q+r}\right)\preceq_{CR}\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega \preceq_{CR}\int_0^1\frac{\mathrm{d}s}{h(s)}. \tag{32}$$

(3) If $h_1(s)=h(s)$, $h_2(s)=1$, then Theorem 4 becomes result for harmonical CR-h-convex function

$$\frac{1}{2h(1/2)}\Psi\left(\frac{2qr}{q+r}\right)\preceq_{CR}\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega\preceq_{CR}\int_0^1 h(s)\mathrm{d}s. \tag{33}$$

(4) If $h_1(s)=1/h_1(s)$, $h_2(s)=1/h_2(s)$, then Theorem 4 becomes result for harmonical CR-(h_1,h_2)-Godunova–Levin function

$$\frac{[H(1/2,1/2)]}{2}\Psi\left(\frac{2qr}{q+r}\right)\preceq_{CR}\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega \preceq_{CR}\int_0^1\frac{\mathrm{d}s}{H(s,1-s)}. \tag{34}$$

Example 3. Further by Example 2, we have

$$\begin{aligned}
&\frac{1}{2[H(1/2,1/2)]}\Psi\left(\frac{2qr}{q+r}\right)=\Psi\left(\frac{4}{3}\right)=\left[\frac{55}{16},\frac{89}{16}\right],\\
&\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega=2\left[\int_1^2\left(\frac{4\omega^2-1}{\omega^4}\right)\mathrm{d}\omega,\int_1^2\left(\frac{5\omega^2+1}{\omega^4}\right)\mathrm{d}\omega\right]\\
&=\left[\frac{82}{24},\frac{134}{24}\right],\\
&[\Psi(q)+\Psi(r)]\int_1^2 H(s,1-s)\mathrm{d}s=\left[\frac{27}{8},\frac{45}{8}\right].
\end{aligned} \tag{35}$$

As a result

$$\left[\frac{55}{16},\frac{89}{16}\right]\preceq_{CR}\left[\frac{82}{24},\frac{134}{24}\right]\preceq_{CR}\left[\frac{27}{8},\frac{45}{8}\right]. \tag{36}$$

Consequently, Theorem 4 is verified.

Theorem 5. *Let $h_1,h_2\colon(0,1)\longrightarrow\mathbf{R}^+$ and $h_1(1/2)h_2(1/2)\neq 0$. Let $\Psi\colon[q,r]\longrightarrow\mathbf{R}_\mathbf{I}^+$, if $\Psi\in \mathrm{SHX}(\mathrm{CR}-(h_1,h_2),[q,r],\mathbf{R}_\mathbf{I}^+)$ and $\Psi\in IR_{[q,r]}$, we have*

$$\begin{aligned}
&\frac{1}{4[H(1/2,1/2)]^2}\Psi\left(\frac{2qr}{q+r}\right)\preceq_{CR}\Delta_1\preceq_{CR}\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega\preceq_{CR}\Delta_2,\\
&\preceq_{CR}\left\{[\Psi(q)+\Psi(r)]\left[\frac{1}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)\right]\right\}\\
&\cdot\int_0^1 H(s,1-s)\mathrm{d}s,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\Delta_1&=\frac{1}{4H(1/2,1/2)}\left[\Psi\left(\frac{4qr}{q+3r}\right)+\Psi\left(\frac{4qr}{3q+r}\right)\right],\\
\Delta_2&=\left[\Psi\left(\frac{2qr}{q+r}\right)+\frac{\Psi(q)+\Psi(r)}{2}\right]\int_0^1 H(s,1-s)\mathrm{d}s.
\end{aligned} \tag{38}$$

Proof. Take $[q,2qr/q+r]$, we have

$$\begin{aligned}
&\Psi\left(\frac{4qr}{q+3r}\right)\\
&\preceq_{CR}H\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{q(2qr/q+r)}{xq+(1-x)2qr/q+r}\right)\\
&+H\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{q(2qr/q+r)}{(1-x)q+x(2qr/q+r)}\right).
\end{aligned} \tag{39}$$

By integrating of the abovementioned inequality over (0,1), we have

$$\begin{aligned}
&\Psi\left(\frac{4qr}{q+3r}\right)\\
&\preceq_{CR}H\left(\frac{1}{2},\frac{1}{2}\right)\left[\int_0^1\Psi\left(\frac{q(2qr/q+r)}{xq+(1-x)(2qr/q+r)}\right)\mathrm{d}s\right.\\
&\left.+\int_0^1\Psi\left(\frac{q(2qr/q+r)}{(1-x)q+x(2qr/q+r)}\right)\mathrm{d}s\right]\\
&=H\left(\frac{1}{2},\frac{1}{2}\right)\left[\frac{2qr}{r-q}\int_q^{(2qr/q+r)}\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega+\frac{2qr}{r-q}\int_q^{(2qr/q+r)}\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega\right]\\
&=H\left(\frac{1}{2},\frac{1}{2}\right)\left[\frac{4qr}{r-q}\int_q^{(2qr/q+r)}\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega\right].
\end{aligned} \tag{40}$$

Accordingly

$$\frac{1}{4H(1/2,1/2)}\Psi\left(\frac{4qr}{q+3r}\right)\preceq_{CR}\frac{qr}{r-q}\int_{q}^{2qr/q+r}\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega. \quad (41)$$

Similarly for interval $[2qr/q+r, r]$, we have

$$\frac{1}{4H(1/2,1/2)}\Psi\left(\frac{4qr}{3q+r}\right)\preceq_{CR}\frac{qr}{r-q}\int_{2qr/q+r}^{r}\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega. \quad (42)$$

Adding inequalities (41) and (42), we get

$$\begin{aligned}\Delta_1 &= \frac{1}{4H(1/2,1/2)}\left[\Psi\left(\frac{4qr}{q+3r}\right)+\Psi\left(\frac{4qr}{3q+r}\right)\right]\\ &\preceq_{CR}\left[\frac{qr}{r-q}\int_{q}^{r}\frac{\Psi(\omega)}{\omega^2}\mathrm{d}\omega\right].\end{aligned} \quad (43)$$

Now

$$\begin{aligned}&\frac{1}{4[H(1/2,1/2)]^2}\Psi\left(\frac{2qr}{q+r}\right)\\ &=\frac{1}{4[H(1/2,1/2)]^2}\Psi\left(\frac{1}{2}\left(\frac{4qr}{q+3r}\right)+\frac{1}{2}\left(\frac{4qr}{3q+r}\right)\right)\\ &\preceq_{CR}\frac{1}{4[H(1/2,1/2)]^2}\left[H\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{4qr}{q+3r}\right)+H\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{4qr}{3q+r}\right)\right]\\ &=\frac{1}{4H(1/2,1/2)}\left[\Psi\left(\frac{4qr}{q+3r}\right)+\Psi\left(\frac{4qr}{3q+r}\right)\right]\\ &=\Delta_1\\ &\preceq_{CR}\frac{1}{4H(1/2,1/2)}\left\{H\left(\frac{1}{2},\frac{1}{2}\right)\left[\Psi(q)+\Psi\left(\frac{2qr}{q+r}\right)\right]+H\left(\frac{1}{2},\frac{1}{2}\right)\left[\Psi(r)+\Psi\left(\frac{2qr}{q+r}\right)\right]\right\}\\ &=\frac{1}{2}\left[\frac{\Psi(q)+\Psi(r)}{2}+\Psi\left(\frac{2qr}{q+r}\right)\right]\\ &\preceq_{CR}\left[\frac{\Psi(q)+\Psi(r)}{2}+\Psi\left(\frac{2qr}{q+r}\right)\right]\int_0^1 H(s,1-s)\mathrm{d}s\\ &=\Delta_2\\ &\preceq_{CR}\left[\frac{\Psi(q)+\Psi(r)}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)\Psi(q)+H\left(\frac{1}{2},\frac{1}{2}\right)\Psi(r)\right]\int_0^1 H(s,1-s)\mathrm{d}s\\ &\preceq_{CR}\left[\frac{\Psi(q)+\Psi(r)}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)[\Psi(q)+\Psi(r)]\right]\int_0^1 H(s,1-s)\mathrm{d}s\\ &\preceq_{CR}\left\{[\Psi(q)+\Psi(r)]\left[\frac{1}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)\right]\right\}\int_0^1 H(s,1-s)\mathrm{d}s.\end{aligned} \quad (44)$$

□

Example 4. Further by Example 3, we have

$$\frac{1}{4[H(1/2,1/2)]^2}\Psi\left(\frac{2qr}{q+r}\right)=\Psi\left(\frac{4}{3}\right)=\left[\frac{55}{16},\frac{89}{16}\right],$$

$$\begin{aligned}\Delta_1&=\frac{1}{2}\left[\Psi\left(\frac{8}{5}\right)+\Psi\left(\frac{8}{7}\right)\right]=\left[\frac{219}{64},\frac{357}{64}\right],\\ \Delta_2&=\left[\frac{\Psi(1)+\Psi(2)}{2}+\Psi\left(\frac{4}{3}\right)\right]\int_0^1 H(s,1-s)\mathrm{d}s,\\ \Delta_2&=\frac{1}{2}\left(\left[\frac{27}{8},\frac{45}{8}\right]+\left[\frac{55}{16},\frac{89}{16}\right]\right),\\ \Delta_2&=\left[\frac{109}{32},\frac{179}{32}\right],\end{aligned} \tag{45}$$

$$\left\{[\Psi(q)+\Psi(r)]\left[\frac{1}{2}+H\left(\frac{1}{2},\frac{1}{2}\right)\right]\right\}\int_0^1 H(s,1-s)\mathrm{d}s=\left[\frac{27}{8},\frac{45}{8}\right].$$

Thus, we obtain

$$\begin{aligned}\left[\frac{55}{16},\frac{89}{16}\right]&\preceq_{CR}\left[\frac{219}{64},\frac{357}{64}\right]\preceq_{CR}\left[\frac{82}{24},\frac{134}{24}\right]\\ &\preceq_{CR}\left[\frac{109}{32},\frac{179}{32}\right]\preceq_{CR}\left[\frac{27}{8},\frac{45}{8}\right].\end{aligned} \tag{46}$$

Consequently, Theorem 5 is verified.

Theorem 6. *Let* $\Psi,\phi\colon [q,r]\longrightarrow \mathbf{R}_{\mathbf{I}}^{+}, h_1, h_2\colon (0,1)\longrightarrow \mathbf{R}^{+}$ *such that* $h_1,h_2\neq 0$. *If* $\Psi\in \mathrm{SHX}(\mathrm{CR}-h_2,[q,r],\mathbb{R}_{\mathbb{I}}^{+})$, $\phi\in \mathrm{SHX}(\mathrm{CR}-h_2,[q,r],\mathbf{R}_{\mathbf{I}}^{+})$, *and* $\Psi,\phi\in \mathbf{IR}_{[\mathbf{q},\mathbf{r}]}$ *then, we have*

$$\begin{aligned}&\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)\phi(\omega)}{\omega^2}\mathrm{d}\omega\\ &\preceq_{CR}M(q,r)\int_0^1 H^2(s,1-s)\mathrm{d}s+N(q,r)\\ &\int_0^1 H(s,s)H(1-s,1-s)\mathrm{d}s,\end{aligned} \tag{47}$$

where

$$\begin{aligned}M(q,r)&=\Psi(q)\phi(q)+\Psi(r)\phi(r),N(q,r)\\ &=\Psi(q)\phi(r)+\Psi(r)\phi(q).\end{aligned} \tag{48}$$

Proof. Consider $\Psi\in \mathrm{SHX}(CR-h_1,[q,r],\mathbf{R}_{\mathbf{I}}^{+})$, $\phi\in \mathrm{SHX}(CR-h_1,[q,r],\mathbf{R}_{\mathbf{I}}^{+})$ then, we have

$$\begin{aligned}\Psi\left(\frac{qr}{qs+(1-s)r}\right)&\preceq_{CR}h_1(s)h_2(1-s)\Psi(q)+h_1(1-s)h_2(s)\Psi(r),\\ \phi\left(\frac{qr}{qs+(1-s)r}\right)&\preceq_{CR}h_1(s)h_2(1-s)\Psi(q)+h_1(1-s)h_2(s)\Psi(r).\end{aligned} \tag{49}$$

Then

$$\begin{aligned}&\Psi\left(\frac{qr}{qs+(1-s)r}\right)\phi(ts+(1-s)u)\\ &\preceq_{CR}H^2(s,1-s)\Psi(q)\phi(f)+H^2(1-s,s)\\ &\quad[\Psi(q)\phi(g)+\Psi(r)\phi(h)]\\ &\quad+H(s,s)H(1-s,1-s)\Psi(r)\phi(g).\end{aligned} \tag{50}$$

By integrating of the above inequality over (0,1), we have

$$\begin{aligned}
&\int_0^1 \Psi\left(\frac{qr}{qs+(1-s)r}\right)\phi\left(\frac{qr}{qs+(1-s)r}\right)\mathrm{d}s \\
&= \left[\int_0^1 \underline{\Psi}\left(\frac{qr}{qs+(1-s)r}\right)\underline{\phi}\left(\frac{qr}{qs+(1-s)r}\right)\mathrm{d}s,\right. \\
&\quad \left.\int_0^1 \overline{\Psi}\left(\frac{qr}{qs+(1-s)r}\right)\overline{\phi}\left(\frac{qr}{qs+(1-s)r}\right)\mathrm{d}s\right] \\
&= \left[\frac{qr}{r-q}\int_q^r \frac{\underline{\Psi}(\omega)\underline{\phi}(\omega)}{\omega^2}\mathrm{d}\omega, \frac{qr}{r-q}\int_q^r \frac{\overline{\Psi}(\omega)\overline{\phi}(\omega}{\omega^2}\mathrm{d}\omega\right] \\
&= \frac{qr}{r-q}\int_q^r \frac{\Psi(\omega)\phi(\omega)}{\omega^2}d\omega \\
&\preceq_{CR} \int_0^1 [\Psi(q)\phi(f)+\Psi(r)\phi(g)]H^2(s,1-s)\mathrm{d}s \\
&\quad + \int_0^1 [\Psi(q)\phi(g)+\Psi(r)\phi(f)]H(s,s)H(1-s,1-s)\mathrm{d}s.
\end{aligned} \tag{51}$$

It follows that

$$\begin{aligned}
&\frac{qr}{r-q}\int_q^r \frac{\Psi(\omega)}{\omega^2}\phi(\omega)\mathrm{d}\omega \\
&\preceq_{CR} M(q,r)\int_0^1 H^2(s,1-s)\mathrm{d}s + N(q,r)\int_0^1 H(s,s)H(1-s,1-s)\mathrm{d}s.
\end{aligned} \tag{52}$$

□

Example 5. Let $[q,r]=[1,2]$, $h_1(s)=h_2(s)=s$, $\forall s \in (0,1)$, *and* Ψ,ϕ: $[q,r] \longrightarrow \mathbf{R}_{\mathbf{I}}^+$ be defined as follows:

$$\begin{aligned}
\Psi(\omega) &= \left[\frac{-1}{\omega^2}+3, \frac{1}{\omega^2}+4\right], \\
\phi(\omega) &= \left[\frac{-1}{\omega}+1, \frac{1}{\omega}+2\right].
\end{aligned} \tag{53}$$

Then

$$\begin{aligned}
\frac{qr}{r-q}\int_f^g \frac{\Psi(\omega)\phi(\omega)}{\omega^2}\mathrm{d}\omega &= \left[\frac{122}{192}, \frac{2426}{192}\right], \\
M(q,r)\int_0^1 H^2(s,1-s)\mathrm{d}s = M(1,2)\int_0^1 s^2\mathrm{d}s &= \left[\frac{11}{24}, \frac{205}{24}\right], \\
N(q,r)\int_0^1 H(s,s)H(1-s,1-s)\mathrm{d}s = N(1,2)\int_0^1 s(1-s)\mathrm{d}s &= \left[\frac{1}{6}, \frac{101}{24}\right].
\end{aligned} \tag{54}$$

It follows that

$$\left[\frac{122}{192}, \frac{2426}{192}\right] \preceq_{CR} \left[\frac{11}{24}, \frac{205}{24}\right] + \left[\frac{1}{6}, \frac{101}{24}\right] = \left[\frac{5}{8}, \frac{51}{24}\right]. \tag{55}$$

Consequently, Theorem 6 is verified.

Theorem 7. *Let* $\phi, \Psi\colon [q, r] \longrightarrow \mathbf{R}_{\mathbf{I}}^{+}, h_1, h_2\colon (0,1) \longrightarrow \mathbf{R}^{+}$ *such that* $h_1, h_2 \neq 0$. *If* $\Psi \in \mathrm{SHX}(CR - h_1, [q, r], \mathbf{R}_{\mathbf{I}}^{+})$, $\phi \in \mathrm{SHX}(\mathrm{CR} - h_2, [q, r], \mathbf{R}_{\mathbf{I}}^{+})$, *and* $\Psi, \phi \in \mathbf{IR}_{[\mathbf{q},\mathbf{r}]}$ *then, we have*

$$\begin{aligned}&\frac{1}{2[H(1/2,1/2)]^2}\Psi\left(\frac{2qr}{q+r}\right)\phi\left(\frac{2qr}{q+r}\right)\\&\preceq_{CR}\frac{qr}{r-q}\int_q^r\frac{\Psi(\omega)\phi(\omega)}{\omega^2}\mathrm{d}\omega+M(q,r)\int_0^1H(s,s)H(1-s,1-s)\mathrm{d}s+N(q,r)\int_0^1H^2(s,1-s)\mathrm{d}s.\end{aligned}\tag{56}$$

Proof. Since $\Psi \in \mathrm{SHX}(CR - h_1, [q, r], \mathbf{R}_{\mathbf{I}}^{+})$ and $\phi \in \mathrm{SHX}(CR - h_1, [q, r], \mathbf{R}_{\mathbf{I}}^{+})$, we have

$$\begin{aligned}&\Psi\left(\frac{2qr}{q+r}\right)\preceq_{CR}H\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{qr}{qs+(1-s)r}\right)+H\left(\frac{1}{2},\frac{1}{2}\right)\Psi\left(\frac{qr}{q(1-s)+sr}\right),\\&\phi\left(\frac{2qr}{q+r}\right)\preceq_{CR}H\left(\frac{1}{2},\frac{1}{2}\right)\phi\left(\frac{qr}{qs+(1-s)r}\right)+H\left(\frac{1}{2},\frac{1}{2}\right)\phi\left(\frac{qr}{q(1-s)+sr}\right).\end{aligned}\tag{57}$$

Then

$$\begin{aligned}&\Psi\left(\frac{2qr}{q+r}\right)\phi\left(\frac{2qr}{q+r}\right)\\&\preceq_{CR}\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[\Psi\left(\frac{qr}{qs+(1-s)r}\right)\phi\left(\frac{qr}{qs+(1-s)r}\right)+\Psi\left(\frac{qr}{q(1-s)+sr}\right)\phi\left(\frac{qr}{q(1-s)+sr}\right)\right]\\&\quad+\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[\Psi\left(\frac{qr}{qs+(1-s)r}\right)\phi\left(\frac{qr}{q(1-s)+sr}\right)+\Psi\left(\frac{qr}{q(1-s)+sr}\right)\phi\left(\frac{qr}{qs+(1-s)r}\right)\right]\\&\quad+\preceq_{CR}\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[\Psi\left(\frac{qr}{qs+(1-s)r}\right)\phi\left(\frac{qr}{qs+(1-s)r}\right)+\Psi\left(\frac{qr}{q(1-s)+sr}\right)\phi\left(\frac{qr}{q(1-s)+sr}\right)\right]\\&\quad+\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2[H(s,1-s)\Psi(q)+H(1-s,s)\Psi(r)(H(1-s,s)\phi(q)+H(s,1-s)\phi(r))]\\&\quad+\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2[H(s,1-s)\phi(q)+H(s,1-s)\phi(r)(H(s,1-s)\phi(q)+H(1-s,s)\phi(r))]\\&\preceq_{CR}\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[\Psi\left(\frac{qr}{qs+(1-s)r}\right)\phi\left(\frac{qr}{qs+(1-s)r}\right)+\Psi\left(\frac{qr}{q(1-s)+sr}\right)\phi\left(\frac{qr}{q(1-s)+sr}\right)\right]\\&\quad+\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[(2H(s,s)H(1-s,1-s))M(q,r)+\left(H^2(s,1-s)+H^2(1-s,s)\right)N(q,r)\right].\end{aligned}\tag{58}$$

By integrating of the above inequality over (0,1), we have

$$\begin{aligned}\int_0^1 \Psi\left(\frac{2qr}{q+r}\right)\phi\left(\frac{2qr}{q+r}\right)\mathrm{d}s &= \left[\int_0^1 \underline{\Psi}\left(\frac{2qr}{q+r}\right)\underline{\phi}\left(\frac{2qr}{q+r}\right)\mathrm{d}s, \int_0^1 \overline{\Psi}\left(\frac{2qr}{q+r}\right)\overline{\phi}\left(\frac{2qr}{q+r}\right)\mathrm{d}s\right] \\ &= \Psi\left(\frac{2qr}{q+r}\right)\phi\left(\frac{2qr}{q+r}\right)\mathrm{d}s \\ &\preceq_{CR} 2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[\frac{qr}{r-q}\int_q^r \frac{\Psi(\omega)\phi(\omega)}{\omega^2}\mathrm{d}\omega\right] \\ &+ 2\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2\left[M(q,r)\int_0^1 H(s,s)H(1-s,1-s)\mathrm{d}s + N(q,r)\int_0^1 H^2(s,1-s)\mathrm{d}s\right]\end{aligned} \tag{59}$$

Divide both sides by $1/2[H(1/2,1/2)]^2$ above equation, we get the required result

$$\begin{aligned}&\frac{1}{2[H(1/2,1/2)]^2}\Psi\left(\frac{2qr}{q+r}\right)\phi\left(\frac{2qr}{q+r}\right) \\ &\preceq_{CR}\frac{qr}{r-q}\int_q^r \frac{\Psi(\omega)}{\omega^2}\phi(\omega)\mathrm{d}\omega + M(q,r)\int_0^1 H(s,s)H(1-s,1-s)\mathrm{d}s + N(q,r)\int_0^1 H^2(s,1-s)\mathrm{d}s.\end{aligned} \tag{60}$$

The abovementioned theorem is proved. □

Example 6. Further by Example 5, we have

$$\begin{aligned}\frac{1}{2[H(1/2,1/2)]^2}\Psi\left(\frac{2qr}{q+r}\right)\phi\left(\frac{2qr}{q+r}\right) &= \frac{1}{2}\Psi\left(\frac{4}{3}\right)\phi\left(\frac{4}{3}\right) = \left[\frac{39}{128},\frac{803}{128}\right], \\ \frac{qr}{r-q}\int_w^r \frac{\Psi(\omega)\phi(\omega)}{\omega^2}\mathrm{d}\omega &= \left[\frac{5}{12},\frac{227}{12}\right], \\ M(q,r)\int_0^1 H(s,s)H(1-s,1-s)\mathrm{d}s &= M(1,2)\int_0^1 s(1-s)\mathrm{d}s = \left[\frac{11}{48},\frac{205}{48}\right], \\ N(q,r)\int_0^1 H^2(s,1-s)\mathrm{d}s &= N(1,2)\int_0^1 s^2\mathrm{d}s = \left[\frac{1}{3},\frac{101}{12}\right].\end{aligned} \tag{61}$$

It follows that

$$\begin{aligned}\left[\frac{39}{128},\frac{803}{128}\right]\preceq_{CR}\left[\frac{122}{192},\frac{2426}{192}\right] + \left[\frac{11}{48},\frac{205}{48}\right] \\ + \left[\frac{1}{3},\frac{101}{12}\right] = \left[\frac{115}{96},\frac{2431}{96}\right].\end{aligned} \tag{62}$$

Consequently, Theorem 7 is verified.

Next, we will establish Jensen inequality for harmonical CR-(h_1,h_2)-convex mapping.

4. Jensen-Type Inequality for Harmonical CR-(h_1,h_2)-Convex Mappings

Theorem 8 (See [25]). *Let $c_i \in \mathbf{R}^+$, $j_i \in [q,r]$. If h_1, h_2 is super multiplicative non-negative functions and if*

$\Psi \in \mathrm{SHX}(CR-(h_1,h_2),[q,r],\mathbf{R}_{\mathrm{I}}^{+})$. *Inequality then becomes as follows:*

$$\Psi\left(\frac{1}{1/C_k\sum_{i=1}^{k}c_i j_i}\right)\preceq_{CR}\sum_{i=1}^{k}H\left(\frac{c_i}{C_k},\frac{C_{k-1}}{C_k}\right)\Psi(j_i), \tag{63}$$

where $C_k=\sum_{i=1}^{k}c_i$

Proof. When $k=2$, then (63) holds. Suppose that (63) is also valid for $k-1$, then

$$\begin{aligned}\Psi\left(\frac{1}{1/C_k\sum_{i=1}^{k}c_i j_i}\right)&=\Psi\left(\frac{1}{c_k/C_k v_k+\sum_{i=1}^{k-1}c_i/C_k j_i}\right)\\&\preceq_{CR}h_1\left(\frac{c_k}{C_k}\right)h_2\left(\frac{C_{k-1}}{C_k}\right)\Psi(j_k)+h_1\left(\frac{C_{k-1}}{C_k}\right)h_2\left(\frac{c_k}{C_k}\right)\Psi\left(\sum_{i=1}^{k-1}\frac{c_i}{C_k}j_i\right)\\&\preceq_{CR}h_1\left(\frac{c_k}{C_k}\right)h_2\left(\frac{C_{k-1}}{C_k}\right)\Psi(j_k)+h_1\left(\frac{C_{k-1}}{C_k}\right)h_2\left(\frac{c_k}{C_k}\right)\sum_{i=1}^{k-1}\left[H\left(\frac{c_i}{C_k},\frac{C_{k-2}}{C_{k-1}}\right)\Psi(j_i)\right]\\&\preceq_{CR}h_1\left(\frac{c_k}{C_k}\right)h_2\left(\frac{C_{k-1}}{C_k}\right)\Psi(j_k)+\sum_{i=1}^{k-1}H\left(\frac{c_i}{C_k},\frac{C_{k-2}}{C_{k-1}}\right)\Psi(j_i)\\&\preceq_{CR}\sum_{i=1}^{k}H\left(\frac{c_i}{C_k},\frac{C_{k-1}}{C_k}\right)\Psi(j_i).\end{aligned} \tag{64}$$

It follows from mathematical induction that the conclusion is correct. □

Remark 3

(1) If $h_1(s)=h_2(s)=1$, Theorem 8 becomes result for harmonical CR- P-function

$$\Psi\left(\frac{1}{1/C_k\sum_{i=1}^{k}c_i j_i}\right)\preceq_{CR}\sum_{i=1}^{k}\Psi(j_i). \tag{65}$$

(2) If $h_1(s)=s$, $h_2(s)=1$ Theorem 8 becomes result for harmonical CR-convex function

$$\Psi\left(\frac{1}{1/C_k\sum_{i=1}^{k}c_i j_i}\right)\preceq_{CR}\sum_{i=1}^{k}\frac{c_i}{C_k}\Psi(j_i). \tag{66}$$

(3) If $h_1(s)=h(s)$, $h_2(s)=1$ Theorem 8 becomes result for harmonical CR-h-convex function

$$\Psi\left(\frac{1}{1/C_k\sum_{i=1}^{k}c_i j_i}\right)\preceq_{CR}\sum_{i=1}^{k}h\left(\frac{c_i}{C_k}\right)\Psi(j_i). \tag{67}$$

(4) If $h_1(s)=1/h(s)$, $h_2(s)=1$ Theorem 8 becomes result for harmonical CR-h-GL-function

$$\Psi\left(\frac{1}{1/C_k\sum_{i=1}^{k}c_i j_i}\right)\preceq_{CR}\sum_{i=1}^{k}\left[\frac{\Psi(j_i)}{h(c_i/C_k)}\right]. \tag{68}$$

(5) If $h_1(s)=1/(s)^s$, $h_2(s)=1$ Theorem 8 becomes result for harmonical CR-s-convex function

$$\omega\left(\frac{1}{1/C_k\sum_{i=1}^{k}c_i j_i}\right)\preceq_{CR}\sum_{i=1}^{k}\left(\frac{c_i}{C_k}\right)^{s}\Psi(j_i). \tag{69}$$

5. Conclusions

In this article, we developed the notion of harmonically center-radius order (h_1,h_2)-convex mappings. By using these notions, we developed $\mathscr{H}.\mathscr{H}$ and Jensen-type inequalities. In comparison with other order relations, this order produces much better results. Moreover, we generalize some recently developed results, see reference [25]. Furthermore, the study provides relevant examples to back up its findings. These ideas can be used to take convex optimization in a new direction. Interval integral operators and integral inequalities studied in our study will expand the potential applications of integral inequalities in practice due to the widespread use of integral operators in engineering and other applied sciences, including different kinds of mathematical modeling. Various integral operators are appropriate for different practical problems. It is anticipated that this concept will be beneficial to other researchers working in a range of scientific disciplines.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Conceptualization was done by W.A., K.S., and M.A.; J.K.K.A. was responsible for validation; A.M.G., W.A., and K.S. performed the investigation; W.A., K.S., and M.A. wrote the original draft; J.K.K.A. and A.M.G. reviewed the manuscript; K.S was responsible for supervision; W.A. and M.A. were responsible for project administration. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

References

[1] R. E. Moore, *Interval Analysis*, Prentice-Hall, Hoboken, NJ, USA, 1966.

[2] N. A. Gasilov and Ş. Emrah Amrahov, "Solving a nonhomogeneous linear system of interval differential equations," *Soft Computing*, vol. 22, pp. 3817–3828, 2018.

[3] J. Zhu and Z. Qiu, "Interval analysis for uncertain aerodynamic loads with uncertain-but-bounded parameters," *Journal of Fluids and Structures*, vol. 81, pp. 418–436, 2018.

[4] Y. Li and T. Wang, "Interval analysis of the wing divergence," *Aerospace Science and Technology*, vol. 74, pp. 17–21, 2018.

[5] M. S. Rahman, A. A. Shaikh, and A. K. Bhunia, "Necessary and sufficient optimality conditions for non-linear unconstrained and constrained optimization problem with interval valued objective function," *Computers and Industrial Engineering*, vol. 147, Article ID 106634, 2020.

[6] E. J. Rothwell and M. J. Cloud, "Automatic error analysis using intervals," *IEEE Transactions on Education*, vol. 55, no. 1, pp. 9–15, 2012.

[7] S. Faisal, M. A. Khan, and S. Iqbal, "Generalized Hermite-Hadamard-Mercer type inequalities via majorization," *Filomat*, vol. 36, no. 2, pp. 469–483, 2022.

[8] M. B. Khan, J. E. Macías-Díaz, S. Treanţă, M. S. Soliman, and H. G. Zaini, "Hermite–hadamard inequalities in fractional calculus for left and right harmonically convex functions via interval-valued settings," *Fractal Fract*, vol. 6, no. 4, p. 178, 2022.

[9] E. R. Nwaeze, M. A. Khan, and Y. M. Chu, "Fractional inclusions of the Hermite–Hadamard type for m-polynomial convex intervalvalued functions," *Advances in Difference Equations*, vol. 2020, no. 1, p. 507, 2020.

[10] M. Nowicka and A. Witkowski, "Applications of the hermite–hadamard inequality," 2016, https://arxiv.org/abs/1603.07170.

[11] W. W. Breckner, "Continuity of generalized convex and generalized concave set-valued functions," *Rev. D'Anal. Numér. Théor. Approx*, vol. 22, pp. 39–51, 1993.

[12] Y. Chalco-Cano, A. Flores-Franulic, and H. Román-Flores, "Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative," *Computational and Applied Mathematics*, vol. 31, pp. 457–472, 2012.

[13] T. M. Costa, H. Román-Flores, and Y. Chalco-Cano, "Opial-type inequalities for interval-valued functions," *Fuzzy Sets and Systems*, vol. 358, pp. 48–63, 2019.

[14] A. Flores-Franulic and H. Román-Flores, "Some integral inequalities for fuzzy-interval-valued functions," *Information Sciences*, vol. 420, pp. 110–115, 2017.

[15] S. Varosanec, "On h-convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303–311, 2007.

[16] D. Zhao, T. An, G. Ye, and W. Liu, "New Jensen and Hermite–Hadamard type inequalities for h-convex interval-valued functions," *Journal of Inequalities and Applications*, vol. 2018, pp. 302–314, 2018.

[17] X. Zhang, K. Shabbir, W. Afzal, H. Xiao, and D. Lin, "Hermite–hadamard and jensen-type inequalities via Riemann integral operator for a generalized class of godunova–levin functions," *Journal of Mathematics*, vol. 2022, Article ID 3830324, 12 pages, 2022.

[18] W. Afzal, A. Alb Lupaş, and K. Shabbir, "Hermite–hadamard and jensen-type inequalities for harmonical (h1, h2)-godunova–levin interval-valued functionsh1,h2-godunova–levin interval-valued functions," *Mathematics*, vol. 10, no. 16, p. 2970, 2022.

[19] W. Afzal, K. Shabbir, S. Treanţă, and K. Nonlaopon, "Jensen and Hermite-Hadamard type inclusions for harmonical $ h $-Godunova-Levin functionsh-Godunova-Levin functions," *AIMS Mathematics*, vol. 8, no. 2, pp. 3303–3321, 2023.

[20] M. U. Awan, M. A. Noor, K. I. Noor, and A. G. Khan, "Some new classes of convex functions and inequalities," *Miskolc Mathematical Notes*, vol. 19, no. 1, pp. 77–94, 2018.

[21] W. G. Yang, "Hermite-Hadamard type inequalities for (p1,h1)-(p2,h2)-convex functions on the co-ordinatesp1,h1-p2,h2-convex functions on the co-ordinates," *Tamkang Journal of Mathematics*, vol. 47, no. 3, pp. 289–322, 2016.

[22] D. P. Shi, B. Y. si, and F. Qi, "Hermite–Hadamard type inequalities for m,h1,h2-convex functions via Riemann–Liouville fractional integrals," *Turkish J. Anal. Number Theory*, vol. 2, pp. 22–27, 2014.

[23] T. Saeed, W. Afzal, K. Shabbir, S. Treanţă, and M. De La Sen, "Some novel estimates of hermite–hadamard and jensen type inequalities for (h1,h2)-Convex functions pertaining to total order relationh1,h2-convex functions pertaining to total order relation," *Mathematics*, vol. 10, no. 24, p. 4777, 2022.

[24] A. K. Bhunia and S. S. Samanta, "A study of interval metric and its application in multi-objective optimization with interval objectives," *Computers and Industrial Engineering*, vol. 74, pp. 169–178, 2014.

[25] W. Liu, F. Shi, G. Ye, and D. Zhao, "The properties of harmonically CR-h-convex function and its applications," *Mathematics*, vol. 10, no. 12, p. 2089, 2022.