Hindawi
Journal of Mathematics
Volume 2023, Article ID 9111626, 12 pages
https://doi.org/10.1155/2023/9111626

Research Article

A New Method of Kronecker Product Decomposition

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Received 30 March 2023; Revised 24 July 2023; Accepted 9 October 2023; Published 21 October 2023

Academic Editor: Ali Jaballah

Kronecker product decomposition is often applied in various fields such as particle physics, signal processing, image processing, semidefinite programming, quantum computing, and matrix time series analysis. In the paper, a new method of Kronecker product decomposition is proposed. Theoretical results ensure that the new method is convergent and stable. The simulation results show that the new method is far faster than the known method. In fact, the new method is very applicable for exact decomposition, fast decomposition, big matrix decomposition, and online decomposition of Kronecker products. At last, the extension direction of the new method is discussed.

1. Introduction

Kronecker product, as a special case of tensor product, is a concept having its origin in group theory and has been successfully applied in various fields such as particle physics, signal processing, image processing, semidefinite programming, and quantum computing et al. [1]. Ford and Tyrtyshnikov [2] combined the discrete wavelet transform approximation and the approximation with a sum of Kronecker products to enable the solution of very large dense linear systems by an iterative technique using a Kronecker product approximation represented in a wavelet basis. Yang et al. [3] researched the generalized Kronecker product linear system associated with a class of consecutive-rank-descending matrices arising from bivariate interpolation problems. Muñoz-Matute et al. [4] introduced an algorithm to speed up the computation of the φ-function action over vectors for two-dimensional (2D) matrices expressed as a Kronecker sum using Kronecker products of one-dimensional matrices. More literature studies can refer to Rifa and Zinoviev [5], Enríquez and Rosas-Ortiz [6], Hao et al. [7], Marco et al. [8], Chen and Kressner [9], and the reference cited in.

Definition 1 (see [10]). Assume matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$, then the $m \times n$ block matrix $(a_{ij}B)_{m\times n}$ is called the Kronecker product of A and B, denoted by $A \otimes B$, that is

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}. \quad (1)$$

The Kronecker product decomposition of a matrix C is the factorization of C into the Kronecker product of two matrices $C = A \otimes B$ where the dimensions of A and B are given.

As the theory of matrix time series analysis develops, we often need to deal with the Kronecker product decomposition, see Chen et al. [11] and Wu and Hua [12]. For example, consider the 1− order autoregressive model for centralized matrix time series in bilinear form

$$X_t = \Phi X_{t-1}\Psi + \varepsilon_t, \quad t = 1, 2, \ldots, \quad (2)$$

where $\{X_t, t \in \mathbb{Z}\}$ is a matrix time series, $\{\varepsilon_t, t \in \mathbb{Z}\}$ is a matrix white noise, and Φ and Ψ are two constant matrices.

According to the moment estimation method, it is easy to obtain that

$$\Psi^T \otimes \Phi = \left(\sum_{t=2}^{N} \operatorname{vec}(X_t)\operatorname{vec}^T(X_{t-1}) \right) \left(\sum_{t=2}^{N} \operatorname{vec}(X_{t-1})\operatorname{vec}^T(X_{t-1}) \right)^{-1}, \tag{3}$$

where $\operatorname{vec}(\cdot)$ is the vectorization of matrix by columns, and N is the length of observation sequence. Then, we need to solve Φ and Ψ, which is a problem on Kronecker product decomposition. Denote

$$C = \left(\sum_{t=2}^{N} \operatorname{vec}(X_t)\operatorname{vec}^T(X_{t-1}) \right) \left(\sum_{t=2}^{N} \operatorname{vec}(X_{t-1})\operatorname{vec}^T(X_{t-1}) \right)^{-1}, \tag{4}$$

that we need to solve Φ and Ψ such that

$$\Psi^T \otimes \Phi = C. \tag{5}$$

As far as we know, there is a method to solve (5), which is the optimization method [11]. That is, (5) is transformed into the following minimum problem on matrices

$$(\Psi, \Phi) = \underset{\Psi,\Phi}{\operatorname{argmin}} \left\| C - \Psi^T \otimes \Phi \right\|_F^2, \tag{6}$$

where $\|\cdot\|_F$ is the Frobenius norm of a matrix. However, it is very slow to solve (6) when C is a big matrix.

In the paper, we will propose a new method of Kronecker product decomposition. The method is convergent and stable. Also, the new method is far easier and faster than the optimization method (6).

2. The New Method of Kronecker Product Decomposition

In the section, we will propose a new method to solve $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ satisfying

$$A \otimes B = C, \tag{7}$$

where $C = (c_{ij})_{mp\times nq}$ is given.

For (7) and any constant $\ell \neq 0$, it follows that

$$(\ell A) \otimes \left(\frac{1}{\ell} B\right) = C. \tag{8}$$

That is, the Kronecker product decomposition of (7) is not unique. Thus, we add the constraint condition that

$$\underset{b_{ij}}{\arg\max} \left\{ \left| b_{ij} \right| \mid i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q \right\} = 1. \tag{9}$$

For the sake of convenience, we denote

$$\operatorname{long}(B) \triangleq \underset{b_{ij}}{\arg\max} \left\{ \left| b_{ij} \right| \mid i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q \right\}. \tag{10}$$

For example, let $B = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$, then $\operatorname{long}(B) = -3$.

For any $mp \times nq$ dimensional matrix C, if $C = O_{mp\times nq}$, then, we can take $A = O_{m\times n}$ and $B = \operatorname{ones}(p, q)$, where $\operatorname{ones}(p, q)$ is the $p \times q$ dimensional matrix each element of which is one. Thus, we always assume $C \neq O_{mp\times nq}$ for Kronecker product decomposition.

(i) Steps of Kronecker product decomposition with the constraint condition (9):

(1) Block matrix C into $C = (C_{ij})_{m\times n}$ and denote $C_{(i-1)n+j} = C_{ij}$, where C_{ij} has the same dimensions as B for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

(2) Denote

$$S_1 = C_1,$$
$$S_k = \begin{cases} S_{k-1} + C_k, & \left\| S_{k-1} + C_k \right\|_M \geq \left\| S_{k-1} - C_k \right\|_M, \\ S_{k-1} - C_k, & \left\| S_{k-1} + C_k \right\|_M < \left\| S_{k-1} - C_k \right\|_M, \end{cases} \quad k = 2, 3, \ldots, mn, \tag{11}$$

where $\|\cdot\|_M$ means taking the maximum value of the absolute value of each element of the matrix.

(3) Take

$$\widehat{B} = \frac{1}{\mathrm{long}(S_{\mathrm{mn}})} S_{\mathrm{mn}}. \tag{12}$$

(4) Denote $\widehat{A} = (a_{\mathrm{ij}})_{m\times n}$, then

$$a_{\mathrm{ij}} = \mathrm{mean}\left\{ \frac{C_{\mathrm{ij}}(u,v)}{b_{\mathrm{uv}}} \middle| b_{\mathrm{uv}} \neq 0, u = 1,2,\ldots,p;\ v = 1,2,\ldots,q \right\}, \tag{13}$$

where mean$\{\cdot\}$ means taking the average value, and $C_{\mathrm{ij}}(u,v)$ and b_{uv} are the (u,v) elements of C_{ij} and $\widehat{B}$,respectively, for each $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$.

For example, consider to decompose C into Kronecker product of A and B, where

$$C = \begin{bmatrix} 1 & 2 & -1 & -2 & 2 & 4 \\ 2 & 0 & -2 & 0 & 4 & 0 \\ 1 & 2 & 0 & 0 & 3 & 6 \\ 2 & 0 & 0 & 0 & 6 & 0 \end{bmatrix}, \tag{14}$$

and B is a 2×2 dimensional matrix. It yields from the steps of Kronecker product decomposition with the constraint condition (9) that

$$\begin{gathered} C_1 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1 & -2 \\ -2 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix}, \\ C_4 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_6 = \begin{bmatrix} 3 & 6 \\ 6 & 0 \end{bmatrix}, \\ S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 4 & 8 \\ 8 & 0 \end{bmatrix}, \\ S_4 = \begin{bmatrix} 5 & 10 \\ 10 & 0 \end{bmatrix}, \quad S_5 = \begin{bmatrix} 5 & 10 \\ 10 & 0 \end{bmatrix}, \quad S_6 = \begin{bmatrix} 8 & 16 \\ 16 & 0 \end{bmatrix}. \end{gathered} \tag{15}$$

Then,

$$\begin{gathered} \widehat{B} = \frac{1}{16} S_6 = \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix}, \\ a_{11} = \mathrm{mean}\left\{ \frac{1}{0.5}, \frac{2}{1}, \frac{2}{1} \right\} = 2, a_{12} = \mathrm{mean}\left\{ \frac{-1}{0.5}, \frac{-2}{1}, \frac{-2}{1} \right\} = -2, \\ a_{13} = \mathrm{mean}\left\{ \frac{2}{0.5}, \frac{4}{1}, \frac{4}{1} \right\} = 4, a_{21} = \mathrm{mean}\left\{ \frac{1}{0.5}, \frac{2}{1}, \frac{2}{1} \right\} = 2, \\ a_{22} = \mathrm{mean}\left\{ \frac{0}{0.5}, \frac{0}{1}, \frac{0}{1} \right\} = 0, a_{23} = \mathrm{mean}\left\{ \frac{3}{0.5}, \frac{6}{1}, \frac{6}{1} \right\} = 6. \end{gathered} \tag{16}$$

Thus,

$$\widehat{A} = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 0 & 6 \end{bmatrix}. \tag{17}$$

3. Theoretical Properties of the New Method

As for the new method of Kronecker product decomposition, we present some of its properties in the section. First, we will show the new method is always applicable.

Proposition 2. *The new method of Kronecker product decomposition is well defined. That is,* $\mathrm{long}(S_{mn}) \neq 0$ *in Step (3) holds, and* $\{b_{uv} \neq 0, u = 1,2,\ldots,p;\ v = 1,2,\ldots,q\}$ *is not empty in Step (4).*

The proof of the proposition is presented in Appendix A.

Theorem 3. *If* $C = A \otimes B$ *is not null matrix, where A and B are* $m \times n$ *and* $p \times q$ *dimensional matrices, respectively, it yields from the new method of Kronecker product decomposition that C is decomposed into the Kronecker product of* $\widehat{A}$ *and* $\widehat{B}$, *where*

$$\begin{aligned} \widehat{A} &= \mathrm{long}(B) A, \\ \widehat{B} &= \frac{1}{\mathrm{long}(B)} B. \end{aligned} \tag{18}$$

The proof of the theorem is presented in Appendix B.

Corollary 4. *If* $C = A \otimes B$ *is not null matrix, where B is a* $p \times q$ *dimensional matrix satisfying* $\mathrm{long}(B) = 1$, *it yields from the new method of Kronecker product decomposition that C is decomposed into the Kronecker product of* $\widehat{A}$ *and* $\widehat{B}$, *then*

$$\begin{aligned} \widehat{A} &= A, \\ \widehat{B} &= B. \end{aligned} \tag{19}$$

Theorem 3 shows the new method of Kronecker product decomposition can obtain the exact solution if C exactly equals a Kronecker product of A and B. However, C is obtained by some estimation method in practice and it will be affected by some random disturbance. That is,

$$C = A \otimes B + \varepsilon, \tag{20}$$

where $\varepsilon = (\varepsilon_{uv})_{mp\times nq}$ is a matrix-valued white noise. That ε is a matrix-valued white noise means that $\mathrm{vec}(\varepsilon)$ is a vector-valued white noise.

Theorem 5. *If $C = A \otimes B + \varepsilon$ is not null matrix, where A and B are $m\times n$ and $p\times q$ dimensional matrices, respectively, A is a nonzero matrix and long$(B) = 1$, it yields from the new method of Kronecker product decomposition that C is decomposed into the Kronecker product of $\widehat{A}$ and $\widehat{B}$, then*

$$\lim_{\|\varepsilon\|_M \downarrow 0} \widehat{B} \overset{L^2}{=} B, \qquad \lim_{\|\varepsilon\|_M \downarrow 0} \widehat{A} \overset{L^2}{=} A. \tag{21}$$

The proof of the theorem is presented in Appendix C.

Theorem 5 shows the new method of Kronecker product decomposition is effective, that is, the results of decomposition are close to the original matrices as long as the disturbance ε is not too large.

4. Simulation

For sake of convenience, we have compiled a MATLAB program for the new method of Kronecker product decomposition in the appendix, named by "KronDecomposition.m," which is based on MATLAB R2020b version.

4.1. Simulation for Convergence. Consider

$$A = \begin{bmatrix} 10 & 10 & -20 \\ 20 & -30 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0.5 \\ 0.4 & 1 \end{bmatrix}, \quad C = A\otimes B = \begin{bmatrix} -10 & 5 & -10 & 5 & 20 & -10 \\ 4 & 10 & 4 & 10 & -8 & -20 \\ -20 & 10 & 30 & -15 & -10 & 5 \\ 8 & 20 & -12 & -30 & 4 & 10 \end{bmatrix}, \tag{22}$$

and C is decomposed into Kronecker product by our "KronDecomposition.m" that

$$\widehat{A} = \begin{bmatrix} 10 & 10 & -20 \\ 20 & -30 & 10 \end{bmatrix}, \widehat{B} = \begin{bmatrix} -1 & 0.5 \\ 0.4 & 1 \end{bmatrix}, \tag{23}$$

that is, at this time the Kronecker product decomposition has no error.

In the following, we consider the Kronecker product decomposition of C with random disturbance as follows:

$$C = A \otimes B + \lambda\varepsilon, \tag{24}$$

where ε follows the uniform distribution on the interval $[-1, 1]$ or the standard normal distribution, i.e., $\varepsilon \sim U(-1, 1)$ or $\varepsilon \sim N(0, 1)$. For each λ, we simulate N times ε, and compute the mean μ, standard deviation σ, maximum value of the absolute value of maximum error maxError and running time of the decomposition t in which the Kronecker product decomposition is by our "KronDecomposition.m," see Table 1. Also, the corresponding results in which the Kronecker product decomposition is by the optimization method (6) are presented in Table 2.

Table 1 shows that the mean μ, standard deviation σ, and maximum value maxError of the absolute value of maximum error by the new method decreases as the disturbance ε decreases whether ε obeys a uniform distribution or a normal distribution, which is consistent with Theorem 5.

Comparing Tables 1 and 2, it shows that the absolute value of maximum errors by the new method is a little greater than those by the optimization method (6). However, the computing time of the new method is far less than that by the optimization method (6).

4.2. Simulation for Computing Speed. In the subsection, we will present a comparison of the new method and the optimization method (6) in terms of computing speed. We consider the Kronecker product decomposition of C with different dimensions as follows:

$$C_{mp\times nq} = A_{m\times n} \otimes B_{p\times q}. \tag{25}$$

For the sake of simplicity, we set $n = 4$, $q = 2$, and $p = m$, in which it is only to make the optimization method (6) easier that we take $p = m$. Also, each row of $A_{m\times 4}$ is [1, 2, 3, 4] and that of $B_{m\times 2}$ is [1, 1], where $m = 1, 2, 3, \cdots$. For example,

$$C_{4\times 8} = A_{2\times 4} \otimes B_{2\times 2} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{bmatrix}. \tag{26}$$

We decompose $C_{m^2\times 8}$ into a Kronecker product of $m\times 4$ and $m\times 2$ matrices by the new method "KronDecomposition.m" and the optimization method (6) for $m = 2, 4, 6, \ldots, 50$, and present the maximum error and running time of the decomposition in Table 3, where Time1 is the running time of the new method "KronDecomposition.m," Time2 is the running time of the optimization method (6), maxError1 is the maximum error of the new method "KronDecomposition.m," and maxError2 is the maximum error of the optimization method (6). Then draw the running time of the new method "KronDecomposition.m" and the optimization method (6) in Figure 1.

Table 3 shows there is no error using the new method "KronDecomposition.m" to decompose $C_{m^2\times 8}$ into Kronecker product for all $m = 2, 4, 6, \ldots, 50$, and the error using the optimization method (6) is also very small. And then, Table 3 and Figure 1 show the running time by using the new method "KronDecomposition.m" to decompose $C_{m^2\times 8}$ into

Table 1: Kronecker product decomposition for different random disturbances by the new method.

ε	λ	N	μ	σ	maxError	t
$U(-1,1)$	1	1000	1.4972	0.4257	3.3119	0.0985
	1	10000	1.4877	0.4485	4.189	0.7142
	1	20000	1.4873	0.4432	4.5674	1.3077
	0.1	1000	0.146	0.0427	0.3142	0.0723
	0.1	10000	0.1479	0.0429	0.3792	0.7277
	0.1	20000	0.1484	0.0437	0.3935	1.322
	0.01	1000	0.0146	0.0043	0.0374	0.0654
	0.01	10000	0.0148	0.0043	0.0373	0.6497
	0.01	20000	0.0148	0.0043	0.0394	1.29
$N(0,1)$	1	1000	2.6451	0.864	6.908	0.0824
	1	10000	2.5984	0.8374	8.3167	0.6626
	1	20000	2.5964	0.8378	8.1575	1.4022
	0.1	1000	0.2543	0.0776	0.7356	0.0662
	0.1	10000	0.2566	0.0791	0.7045	0.6721
	0.1	20000	0.2573	0.0796	0.7861	1.3305
	0.01	1000	0.0255	0.008	0.0586	0.066
	0.01	10000	0.0258	0.008	0.0679	0.6537
	0.01	20000	0.0257	0.008	0.0759	1.3183

Table 2: Kronecker product decomposition for different random disturbances by (6).

ε	λ	N	μ	σ	maxError	t
$U(-1,1)$	1	1000	0.9483	0.1453	1.427	8.3599
	1	10000	0.9461	0.1473	1.6125	105.3545
	1	20000	0.9452	0.1496	1.6342	214.1129
	0.1	1000	0.0948	0.0145	0.1425	9.9417
	0.1	10000	0.0945	0.0147	0.1566	99.9426
	0.1	20000	0.0946	0.0148	0.1573	203.773
	0.01	1000	0.0095	0.0015	0.0146	8.5368
	0.01	10000	0.0095	0.0015	0.0161	85.4399
	0.01	20000	0.0095	0.0015	0.0157	171.651
$N(0,1)$	1	1000	1.7623	0.4189	3.1831	11.1421
	1	10000	1.7778	0.4328	3.9279	110.6974
	1	20000	1.7827	0.43	4.0511	215.7052
	0.1	1000	0.1779	0.0431	0.3359	10.8529
	0.1	10000	0.1786	0.0436	0.4126	108.5869
	0.1	20000	0.1778	0.0432	0.4523	217.6272
	0.01	1000	0.0177	0.0042	0.0382	9.3313
	0.01	10000	0.0178	0.0043	0.0415	92.6919
	0.01	20000	0.0178	0.0043	0.0455	176.1501

Table 3: Comparison of running time and maximum error for Kronecker product decomposition.

m	n	p	q	Time1	Time2	maxError1	maxError2
2	4	2	2	0.0004	0.0063	0	6.98×10^{-8}
4	4	4	2	0.0003	0.0093	0	2.27×10^{-7}
6	4	6	2	0.0003	0.0164	0	1.84×10^{-6}
8	4	8	2	0.0004	0.0330	0	7.20×10^{-7}
10	4	10	2	0.0005	0.0398	0	4.88×10^{-8}
12	4	12	2	0.0007	0.0545	0	1.31×10^{-7}
14	4	14	2	0.0006	0.0804	0	1.94×10^{-6}
16	4	16	2	0.0007	0.1193	0	7.08×10^{-8}
18	4	18	2	0.0007	0.1232	0	7.08×10^{-7}
20	4	20	2	0.0009	0.9106	0	7.65×10^{-7}
22	4	22	2	0.0009	1.1837	0	7.66×10^{-7}
24	4	24	2	0.0013	1.8662	0	6.71×10^{-7}
26	4	26	2	0.0011	2.9251	0	2.00×10^{-7}
28	4	28	2	0.0013	6.0990	0	3.33×10^{-8}

Table 3: Continued.

m	n	p	q	Time1	Time2	maxError1	maxError2
30	4	30	2	0.0014	6.7932	0	1.39×10^{-6}
32	4	32	2	0.0016	12.6489	0	9.85×10^{-8}
34	4	34	2	0.0017	14.4383	0	4.06×10^{-8}
36	4	36	2	0.0018	20.3946	0	9.05×10^{-8}
38	4	38	2	0.0021	29.8060	0	3.74×10^{-8}
40	4	40	2	0.0019	38.2979	0	2.34×10^{-8}
42	4	42	2	0.0020	47.0558	0	8.53×10^{-8}
44	4	44	2	0.0020	55.4895	0	7.35×10^{-8}
46	4	46	2	0.0024	72.7245	0	5.78×10^{-8}
48	4	48	2	0.0029	82.6001	0	2.36×10^{-8}
50	4	50	2	0.0026	87.4809	0	1.49×10^{-6}

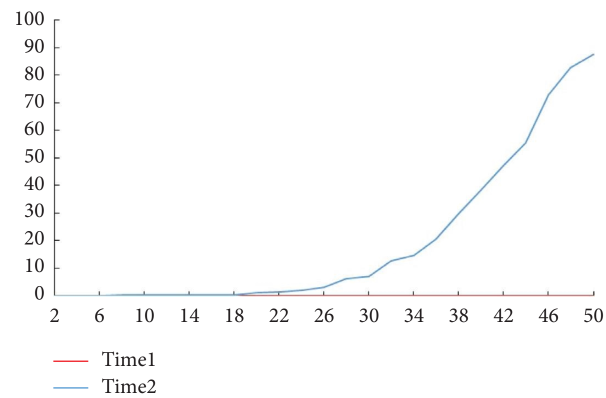

Figure 1: Time1 of the new method and Time2 of optimization method.

Kronecker product is far less than that by using the optimization method (6) for all $m = 2, 4, 6, \ldots, 50$. In summary, as to decompose $C_{m^2\times 8}$ into Kronecker product, the new method "KronDecomposition.m" is much better than the optimization method (6).

5. Applications of Kronecker Product Decomposition

In this section, we consider the daily closing prices and the daily volumes of China Overseas Holdings Group Limited (Stock code: 000046), Shaanxi International Trust Company Limited (Stock code: 000563), and CNPC Capital Company Limited (Stock code: 000617), abbreviated as Stock 000046, Stock 000563, and Stock 000617, respectively. The data are downloaded from the China Stock Market and Accounting Research Database (CSMAR), and the time window is from July 6, 2018 to July 5, 2023, which includes 1205 complete records.

For the sake of clarity, we denote the time series by

$$\left\{\begin{bmatrix} P_1(t) & V_1(t) \\ P_2(t) & V_2(t) \\ P_3(t) & V_3(t) \end{bmatrix}, t = 1, 2, 3, \cdots\right\}, \quad (27)$$

where $P_1(t)$ and $V_1(t)$ are the daily closing price and daily volume of Stock 000046, $P_2(t)$ and $V_2(t)$ are the daily closing price and daily volume of Stock 000563, and $P_3(t)$ and $V_3(t)$ are the daily closing price and daily volume of Stock 000617.

In the following, we will consider the logarithmic rates (log rate) of daily closing prices and daily volumes of the three stocks. Denote

$$\left\{R_t = \begin{bmatrix} R_{11}(t) & R_{12}(t) \\ R_{21}(t) & R_{22}(t) \\ R_{31}(t) & R_{32}(t) \end{bmatrix}, t = 2, 3, 4, \cdots\right\}, \quad (28)$$

where

$$\begin{aligned} R_{k1}(t) &= \ln\left(\frac{P_k(t)}{P_k(t-1)}\right), \\ R_{k2}(t) &= \ln\left(\frac{V_k(t)}{V_k(t-1)}\right), \quad k = 1, 2, 3. \end{aligned} \quad (29)$$

In order to obtain the first-order matrix autoregressive model, MAR (1), following as

$$R_t = C + \Phi R_{t-1}\Psi + \varepsilon_t, \tag{30}$$

and using the conditional least square method in Chen et al. [11], we obtain that

$$C = \begin{bmatrix} 7.4784\times 10^{-4} & 1.1913\times 10^{-2} \\ 6.2472\times 10^{-4} & 1.2221\times 10^{-2} \\ 7.2695\times 10^{-4} & 1.1984\times 10^{-2} \end{bmatrix}, \tag{31}$$

$$\Psi' \otimes \Phi = \begin{bmatrix} 34.096 & 1.8981 & 46.098 & 2.1493 & 36.609 & 2.1070 \\ 543.12 & 30.236 & 734.31 & 34.237 & 583.16 & 33.564 \\ 28.482 & 1.5856 & 38.509 & 1.7954 & 30.582 & 1.7602 \\ 557.18 & 31.018 & 753.32 & 35.123 & 598.25 & 34.432 \\ 33.143 & 1.8451 & 44.810 & 2.0892 & 35.586 & 2.0482 \\ 546.37 & 30.417 & 738.71 & 34.442 & 586.64 & 33.764 \end{bmatrix}\times 10^{-3}, \tag{32}$$

where $\{\varepsilon_t, t\geq 2\}$ is a 3×2-dimensional matrix white noise series.

Using the new method of Kronecker product decomposition, it yields from (32) that

$$\Phi = \begin{bmatrix} 0.58080 & 0.71970 & 0.63446 \\ 0.53838 & 0.66714 & 0.58812 \\ 0.57405 & 0.71133 & 0.62708 \end{bmatrix},$$
$$\Psi' = \begin{bmatrix} 5.8130\times 10^{-2} & 3.0629\times 10^{-3} \\ 1 & 5.2691\times 10^{-2} \end{bmatrix}. \tag{33}$$

Thus, the MAR (1) model (32) follows as

$$R_t = \begin{bmatrix} 7.4784 & 119.13 \\ 6.2472 & 122.21 \\ 7.2695 & 119.84 \end{bmatrix}\times 10^{-4} + \begin{bmatrix} 0.58080 & 0.71970 & 0.63446 \\ 0.53838 & 0.66714 & 0.58812 \\ 0.57405 & 0.71133 & 0.62708 \end{bmatrix} R_{t-1} \begin{bmatrix} 58.130 & 1000 \\ 3.0629 & 52.691 \end{bmatrix}\times 10^{-3} + \varepsilon_t, \tag{34}$$

where $\{\varepsilon_t, t\geq 2\}$ is a 3×2-dimensional matrix white noise series.

6. Conclusion

A new method of Kronecker product decomposition is proposed, which is easy, convergent, stable, and fast. The new method is very applicable for exact decomposition, fast decomposition, big matrix decomposition, and online decomposition of Kronecker products.

Comparing with the known method of Kronecker product decomposition, i.e., optimization method, the computing speed of the new method is very faster than that of the known method. If the matrix to be decomposed into a Kronecker product just equals a Kronecker product of two matrices, the new method can fast obtain its exact solution, but the known method has a little error. If the matrix to be decomposed into a Kronecker product does not equal a Kronecker product of two matrices, the error of the new method is a little bigger than that of the known method, but the computing speed of the new method is very faster than that of the known method.

There are many directions to extend the scope of the new method. It is a possible extension of the new method that using weighted average instead of arithmetic average in Step (4). Furthermore, the method can be applied in many fields such as group theory, particle physics, matrix time series analysis, and dynamic complex network modeling.

Appendix

A. Proof of Proposition 2

Assume $\text{long}(S_{\text{mn}}) = 0$ in Step (3), that is, $S_{\text{mn}} = O_{p\times q}$. It yields from Step (2) that

$$\begin{aligned} S_{\text{mm}-1} + C_{\text{mm}} &= O_{p\times q,} \\ S_{\text{mm}-1} - C_{\text{mm}} &= O_{p\times q}, \end{aligned} \tag{A.1}$$

and then

$$\begin{aligned} S_{\text{mm}-1} &= O_{p\times q}, \\ C_{\text{mm}} &= O_{p\times q}. \end{aligned} \tag{A.2}$$

By the recursive method, we can obtain that

$$\begin{aligned} S_1 = S_2 = \cdots = S_{\text{mn}} &= O_{p\times q}, \\ C_2 = C_3 = \cdots = C_{\text{mn}} &= O_{p\times q}. \end{aligned} \tag{A.3}$$

Then,

$$C_1 = C_2 = C_3 \cdots = C_{\text{mn}} = O_{p\times q}, \tag{A.4}$$

thus $C = O_{\text{mp}\times\text{nq}}$, which contradicts the assumption $C \neq O_{\text{mp}\times\text{nq}}$. That is, $\text{long}(S_{\text{mn}}) \neq 0$ in Step (3) holds.

Furthermore, it yields from $\text{long}(S_{\text{mn}}) \neq 0$ that $S_{\text{mn}} \neq O_{p\times q}$, then $B \neq O_{p\times q}$, so $\{b_{\text{uv}} \neq 0, u = 1, 2, \ldots, p;\ v = 1, 2, \ldots, q\}$ is not empty in Step (4).

B. Proof of Theorem 3

It follows from Step (1) that

$$C_{(i-1)n+j} = C_{\text{ij}} = a_{\text{ij}}B, \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n, \tag{B.1}$$

where a_{ij} is the (i, j) element of A. Without loss of generality, we assume $a_{11} \neq 0$, otherwise, $S_1 = O_{p\times q}$ and we consider whether a_{12} equals zero, and so on. Owning to $a_{11} \neq 0$, it obtains from Step (2) that

$$\begin{aligned} S_1 &= a_{11}B, \\ S_k &= S_{k-1} + \text{sign}\left(\frac{a_k}{a_{11}}\right)a_k B, \quad k = 2, 3, \ldots, \text{mn}, \end{aligned} \tag{B.2}$$

where $a_{(i-1)n+j} = a_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and $\text{sign}(\cdot)$ is the sign function, i.e.,

$$\text{sign}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1 & x < 0. \end{cases} \tag{B.3}$$

Thus,

$$S_{\text{mn}} = \sum_{i=1}^{p}\sum_{j=1}^{q} \text{sign}\left(\frac{a_{\text{ij}}}{a_{11}}\right)a_{\text{ij}}B = \text{sign}(a_{11}) \sum_{i=1}^{p}\sum_{j=1}^{q} \left|a_{\text{ij}}\right|B. \tag{B.4}$$

At this time,

$$\text{long}(S_{\text{mn}}) = \text{sign}(a_{11}) \sum_{i=1}^{p}\sum_{j=1}^{q} \left|a_{\text{ij}}\right| \cdot \text{long}(B), \tag{B.5}$$

so it yields from Step (3) that

$$\widehat{B} = \frac{1}{\text{long}(B)}B. \tag{B.6}$$

And then,

$$\widehat{A} = \text{long}(B)A. \tag{B.7}$$

C. Proof of Theorem 5

First, we block the matrix-valued white noise ε into $\varepsilon = (E_{\text{ij}})_{m\times n}$ and denote $E_{(i-1)n+j} = E_{\text{ij}}$, where E_{ij} is a $p \times q$ dimensional matrix for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. It follows from Step (1) that

$$C_{(i-1)n+j} \stackrel{\Delta}{=} C_{\text{ij}} = a_{(i-1)n+j}B + E_{(i-1)n+j}, \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n, \tag{C.1}$$

where $a_{(i-1)n+j} = a_{\text{ij}}$ and a_{ij} is the (i, j) element of A.

Case C.1. The minimum of all elements of B is greater than -1.

Noting that A is a nonzero matrix, denote

$$\begin{aligned}\delta &= \min\left\{\left|a_{\mathrm{ij}}\right| \,\middle|\, a_{\mathrm{ij}} \neq 0, i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n\right\},\\ \delta_B^- &= \min\left\{b_{\mathrm{uv}} \,\middle|\, u = 1, 2, \ldots, p;\ v = 1, 2, \ldots, q\right\},\end{aligned} \tag{C.2}$$

then

$$\begin{aligned}&\delta > 0,\\ &-1 < \delta_B^- \leq 1.\end{aligned} \tag{C.3}$$

When $\|\varepsilon\|_M < (\delta(1+\delta_B^-))/(2(\mathrm{mn}+1))$, for any $a_{ij} \neq 0$ it follows that

$$\begin{aligned}\max_{\substack{u=1,2,\ldots p,\\ v=1,2,\ldots q,}} \left\{b_{\mathrm{uv}} + \frac{1}{a_{\mathrm{ij}}} E_{\mathrm{ij}}(u,v)\right\} &> 1 - \frac{1}{\delta} \cdot \frac{\delta(1+\delta_B^-)}{2(\mathrm{mn}+1)} = 1 - \frac{1+\delta_B^-}{2(\mathrm{mn}+1)} > 0,\\ \min_{\substack{u=1,2,\ldots p,\\ v=1,2,\ldots q,}} \left\{b_{\mathrm{uv}} + \frac{1}{a_{\mathrm{ij}}} E_{\mathrm{ij}}(u,v)\right\} &> \delta_B^- - \frac{1}{\delta} \cdot \frac{\delta(1+\delta_B^-)}{2(\mathrm{mn}+1)} = \delta_B^- - \frac{1+\delta_B^-}{2(\mathrm{mn}+1)},\end{aligned} \tag{C.4}$$

where $E_{\mathrm{ij}}(u,v)$ is the (u,v) element of E_{ij}. Thus,

$$\max_{\substack{u=1,2,\ldots p,\\ v=1,2,\ldots q,}} \left\{b_{uv} + \frac{1}{a_{\mathrm{ij}}} E_{\mathrm{ij}}(u,v)\right\} > \left|\min_{\substack{u=1,2,\ldots p,\\ v=1,2,\ldots q,}} \left\{b_{\mathrm{uv}} + \frac{1}{a_{\mathrm{ij}}} E_{\mathrm{ij}}(u,v)\right\}\right|, \tag{C.5}$$

and then

$$\left\|C_{\mathrm{ij}}\right\|_M = \left\|a_{\mathrm{ij}} B + E_{\mathrm{ij}}\right\|_M = \left|a_{\mathrm{ij}}\right| \cdot \max_{\substack{u=1,2,\ldots p,\\ v=1,2,\ldots q,}} \left\{b_{\mathrm{uv}} + \frac{1}{a_{\mathrm{ij}}} E_{\mathrm{ij}}(u,v)\right\}. \tag{C.6}$$

In the following, we will show the determining of addition or subtraction to compute S_k in Step (2).

Denote

$$\ell_0 = \arg\min_{\ell}\left\{\ell \,\middle|\, a_\ell \neq 0, \ell = 1, 2, \ldots, \mathrm{mn}\right\}, \tag{C.7}$$

then

$$a_\ell = 0, \quad \ell = 1, 2, \ldots, \ell_0 - 1. \tag{C.8}$$

It yields from Step (2) that

$$S_{\mathrm{mn}} = \sum_{\ell=1}^{\mathrm{mn}} (-1)^{I_\ell} C_\ell = S_{\ell_0 - 1} + \sum_{\ell=\ell_0}^{\mathrm{mn}} (-1)^{I_\ell} C_\ell, \tag{C.9}$$

where $S_{\ell_0-1} = \sum_{\ell=1}^{\ell_0-1} (-1)^{I_\ell} E_\ell$, and we stipulate $I_1 = 0$, and

$$I_\ell = \begin{cases} 0, & \left\|S_{\ell-1} + C_\ell\right\|_M \geq \left\|S_{\ell-1} - C_\ell\right\|_M,\\ 1, & \left\|S_{\ell-1} + C_\ell\right\|_M < \left\|S_{\ell-1} - C_\ell\right\|_M,\end{cases} \quad \ell = 2, 3, \ldots, \mathrm{mn}. \tag{C.10}$$

Denote the sign of the first element whose absolute value equals $\|S_{\ell_0-1}\|_M$ by "κ," then

$$\kappa = \begin{cases} +1, & \operatorname{long}\left(S_{\ell_0-1}\right) \geq 0, \\ -1, & \operatorname{long}\left(S_{\ell_0-1}\right) < 0, \end{cases} \tag{C.11}$$

then it follows from (C.6) that

$$\begin{aligned} S_{\ell_0} &= S_{\ell_0-1} + \kappa\left|a_{\ell_0}\right|\left(B + \frac{1}{a_{\ell_0}}E_{\ell_0}\right) \\ &= S_{\ell_0-1} + \kappa \cdot \operatorname{sign}\left(a_{\ell_0}\right)C_{\ell_0}, \end{aligned} \tag{C.12}$$

and the sign of the first element with the largest absolute value in S_{ℓ_0} is also κ. Furthermore, using a series of complex calculations, we can obtain that

$$\begin{aligned} S_{\mathrm{mn}} &= S_{\ell_0-1} + \kappa \sum_{\substack{\ell=\ell_0 \\ a_\ell \neq 0}}^{\mathrm{mn}} \operatorname{sign}(a_\ell)C_\ell + \sum_{\substack{\ell=\ell_0+1 \\ a_\ell=0}}^{\mathrm{mn}} (-1)^{I_\ell}E_\ell \\ &= \sum_{\substack{\ell=1 \\ a_\ell=0}}^{\mathrm{mn}} (-1)^{I_\ell}E_\ell + \kappa \sum_{\substack{\ell=\ell_0 \\ a_\ell \neq 0}}^{\mathrm{mn}} \operatorname{sign}(a_\ell)C_\ell \\ &= \sum_{\substack{\ell=1 \\ a_\ell=0}}^{\mathrm{mn}} (-1)^{I_\ell}E_\ell + \kappa \sum_{\ell=1}^{\mathrm{mn}} \operatorname{sign}(a_\ell)C_\ell \\ &= \sum_{\ell=1}^{\mathrm{mn}} (-1)^{I_\ell}E_\ell + \kappa \sum_{\ell=1}^{\mathrm{mn}} \left|a_\ell\right|B, \end{aligned} \tag{C.13}$$

where the penultimate equation comes from $\operatorname{sign}(a_\ell) = 0$ as $a_\ell = 0$. Thus, it yields from (C.13) and Step (3) that

$$\widehat{B} = \frac{S_{\mathrm{mn}}}{\operatorname{long}(S_{\mathrm{mn}})} = \frac{\kappa\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|B + \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{I_{(i,j)}}E_{\mathrm{ij}}}{\operatorname{long}(S_{\mathrm{mn}})}. \tag{C.14}$$

When $\|\varepsilon\|_M < (\delta(1+\delta_B^-))/(2(\mathrm{mn}+1))$ and $\|\varepsilon\|_M \downarrow 0$, it is easy to show that

$$\begin{aligned} &\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{I_{(i,j)}}E_{\mathrm{ij}} \xrightarrow{(\cdot)_M} O_{p\times q}, \\ &\operatorname*{arg\,max}_{s_{\mathrm{uv}}}\left\{\left|s_{\mathrm{uv}}\right| \,\middle|\, u = 1,\ldots,p;\ v = 1,\ldots,q\right\} \longrightarrow \kappa\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{\mathrm{ij}}\right|, \end{aligned} \tag{C.15}$$

thus

$$\widehat{B} \overset{(\cdot)M}{\rightarrow} \|\cdot\|_M B. \tag{C.16}$$

And then,

$$\begin{aligned} &\lim_{\|\varepsilon\|_M \downarrow 0} \widehat{B} \overset{L^2}{=} B, \\ &\lim_{\|\varepsilon\|_M \downarrow 0} \widehat{A} \overset{L^2}{=} A. \end{aligned} \tag{C.17}$$

Case C.2. The minimum of all elements of B equals −1.

Similar to Case C.1, we can obtain from Step (2) that

$$\begin{aligned} S_{\mathrm{mn}} &= S_{\ell_0-1} + \sum_{\ell=\ell_0}^{\mathrm{mn}} (-1)^{I_\ell}C_\ell \\ &= \sum_{\ell=1}^{\mathrm{mn}} (-1)^{I_\ell}E_\ell + \sum_{\ell=\ell_0}^{\mathrm{mn}} (-1)^{I_\ell}a_\ell B. \end{aligned} \tag{C.18}$$

In the following, we will investigate the more explicit form of I_ℓ in (C.18).

First, it follows from Step (2) that

$$S_{\ell_0} = S_{\ell_0 - 1} + (-1)^{I_{\ell_0}} C_{\ell_0}, \quad (C.19)$$

then

$$S_{\ell_0} = \begin{cases} S_{\ell_0-1} + \kappa C_{\ell_0}, & \text{long}\left(C_{\ell_0}\right) \geq 0, \\ S_{\ell_0-1} - \kappa C_{\ell_0}, & \text{long}\left(C_{\ell_0}\right) < 0. \end{cases} \quad (C.20)$$

That is,

$$S_{\ell_0} = S_{\ell_0-1} + \kappa \cdot \text{sign}\left(\text{long}\left(C_{\ell_0}\right)\right) C_{\ell_0}. \quad (C.21)$$

Denote

$$\ell_1 = \arg\max_{\ell} \{\ell \mid a_\ell \neq 0, \ell > \ell_0, \ell = 1, 2, \ldots, \text{mn}\}, \quad (C.22)$$

when $\|\varepsilon\|_M < (\delta(1 + \delta_B^-))/(2(\text{mn} + 1))$, we have

$$S_{\ell_1} = \sum_{\ell=1}^{\ell_1} (-1)^{I_\ell} E_\ell + \kappa \cdot \text{sign}\left(\text{long}\left(C_{\ell_0}\right)\right) a_{\ell_0} B + \kappa \cdot \text{sign}\left(\text{long}\left(C_{\ell_0}\right)\right) \frac{\text{sign}\left(a_{\ell_1}\right)}{\text{sign}\left(a_{\ell_0}\right)} a_{\ell_1} B. \quad (C.23)$$

In fact, it is obvious that

$$\begin{aligned} S_{\ell_1 - 1} &= S_{\ell_0} + \sum_{\ell=\ell_0+1}^{\ell_1-1} (-1)^{I_\ell} E_\ell \\ &= \sum_{\substack{\ell=1 \\ \ell \neq \ell_0}}^{\ell_1-1} (-1)^{I_\ell} E_\ell + \kappa \cdot \text{sign}\left(\text{long}\left(C_{\ell_0}\right)\right) C_{\ell_0} \\ &= \sum_{\ell=1}^{\ell_1-1} (-1)^{I_\ell} E_\ell + \kappa \cdot \text{sign}\left(\text{long}\left(C_{\ell_0}\right)\right) a_{\ell_0} B \\ S_{\ell_1} &= \begin{cases} S_{\ell_1-1} + C_{\ell_1}, & \left\|S_{\ell_1-1} + C_{\ell_1}\right\|_M \geq \left\|S_{\ell_1-1} - C_{\ell_1}\right\|_M, \\ S_{\ell_1-1} - C_{\ell_1}, & \left\|S_{\ell_1-1} + C_{\ell_1}\right\|_M < \left\|S_{\ell_1-1} - C_{\ell_1}\right\|_M. \end{cases} \end{aligned} \quad (C.24)$$

Thus,

$$S_{\ell_1} = \sum_{\ell=1}^{\ell_1} (-1)^{I_\ell} E_\ell + \kappa \cdot \text{sign}\left(\text{long}\left(C_{\ell_0}\right)\right) a_{\ell_0} B + (-1)^{I_{\ell_1}} a_{\ell_1} B. \quad (C.25)$$

Noting that $\|E_\ell\|_M < (\delta(1 + \delta_B^-))/(2(\text{mn} + 1))$ for all $\ell = 1, 2, \ldots, \text{mn}$ and

$$\left\|a_{\ell_i} B\right\|_M = \left\|a_{\ell_i}\right\|_M \geq \delta \geq \ell_1 \frac{\delta(1 + \delta_B^-)}{2(\text{mn} + 1)} \geq \left\|\sum_{\ell=1}^{\ell_1} (-1)^{I_\ell} E_\ell\right\|_M, \quad i = 0, 1, \quad (C.26)$$

we know the sign of $(-1)^{I_{\ell_1}} a_{\ell_1}$ must be the same as that of $\kappa \cdot \text{sign}(\text{long}(C_{\ell_0})) a_{\ell_0}$, so

$$(-1)^{I_{\ell_1}} \text{sign}\left(a_{\ell_1}\right) = \kappa \cdot \text{sign}\left(\text{long}\left(C_{\ell_0}\right)\right) \text{sign}\left(a_{\ell_0}\right). \quad (C.27)$$

That is,

$$\begin{aligned} (-1)^{I_{\ell_1}} &= \kappa \cdot \text{sign}\left(\text{long}\left(C_{\ell_0}\right)\right) \frac{\text{sign}\left(a_{\ell_0}\right)}{\text{sign}\left(a_{\ell_1}\right)} \\ &= \kappa \cdot \text{sign}\left(\text{long}\left(C_{\ell_0}\right)\right) \frac{\text{sign}\left(a_{\ell_1}\right)}{\text{sign}\left(a_{\ell_0}\right)}. \end{aligned} \quad (C.28)$$

It yields from (C.25) and (C.28) that (C.23) holds. Analogically, we obtain that

$$\begin{aligned} S_{\mathrm{mn}} &= \sum_{\ell=1}^{\mathrm{mn}} (-1)^{I_\ell} E_\ell + \kappa \cdot \frac{\operatorname{sign}\left(\operatorname{long}\left(C_{\ell_0}\right)\right)}{\operatorname{sign}\left(a_{\ell_0}\right)} \sum_{\substack{\ell=\ell_0 \\ a_\ell \neq 0}}^{\mathrm{mn}} \operatorname{sign}\left(a_\ell\right) a_\ell B \\ &= \sum_{\ell=1}^{\mathrm{mn}} (-1)^{I_\ell} E_\ell + \kappa \cdot \frac{\operatorname{sign}\left(\operatorname{long}\left(C_{\ell_0}\right)\right)}{\operatorname{sign}\left(a_{\ell_0}\right)} \sum_{\ell=1}^{\mathrm{mn}} \left|a_\ell\right| B. \end{aligned} \tag{C.29}$$

Thus, it yields from (C.29) and Step (3) that

$$\widehat{B} = \frac{S_{\mathrm{mn}}}{\operatorname{long}\left(S_{\mathrm{mn}}\right)} = \frac{\sum_{\ell=1}^{\mathrm{mn}} (-1)^{I_\ell} E_\ell + \kappa \cdot \left(\operatorname{sign}\left(\operatorname{long}\left(C_{\ell_0}\right)\right) / \operatorname{sign}\left(a_{\ell_0}\right)\right) \sum_{\ell=1}^{mn} \left|a_\ell\right| B}{\operatorname{long}\left(S_{\mathrm{mn}}\right)}. \tag{C.30}$$

When $\|\varepsilon\|_M < (\delta(1 + \delta_B^-))/(2(\mathrm{mn} + 1))$ and $\|\varepsilon\|_M \downarrow 0$, it is easy to show that

$$\begin{aligned} &\sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{I(i,j)} E_{\mathrm{ij}} \overset{(\cdot)M}{\rightarrow} \| \cdot \|_M O_{p \times q}, \\ &\operatorname{long}\left(S_{\mathrm{mn}}\right) \rightarrow \kappa \cdot \frac{\operatorname{sign}\left(\operatorname{long}\left(C_{\ell_0}\right)\right)}{\operatorname{sign}\left(a_{\ell_0}\right)} \sum_{\ell=1}^{\mathrm{mn}} \left|a_\ell\right|, \end{aligned} \tag{C.31}$$

thus

$$\widehat{B} \overset{(\cdot)_M}{\longrightarrow} B. \tag{C.32}$$

And then,

$$\begin{aligned} &\lim_{\|\varepsilon\|_M \downarrow 0} \widehat{B} \overset{L^2}{=} B, \\ &\lim_{\|\varepsilon\|_M \downarrow 0} \widehat{A} \overset{L^2}{=} A. \end{aligned} \tag{C.33}$$

Data Availability

All data, models, and code generated or used during the study appear in the submitted article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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