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Research Article

Approximate Solution of a Class of Highly Oscillatory Integral Equations Using an Exponential Fitting Collocation Method

S. Khudhair Abbas, S. Sohrabi, and H. Ranjbar

Department of Mathematics, Faculty of Science, Urmia University, Urmia 57561-51818, Iran

Correspondence should be addressed to S. Sohrabi; s.sohrabi@urmia.ac.ir

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This paper deals with the numerical solution of a class of highly oscillatory Volterra integral equations by collocation methods based on the exponential fitting technique. By reviewing the oscillatory structures of solutions of these problems, we construct an exponential fitting collocation method which is best tuned to capture the qualitative behaviour of the solution of these equations. We also investigate the convergence properties of the proposed collocation solution based on the interpolation remainder. Some numerical examples are provided which illustrate the efficiency and accuracy of the proposed method and confirm its superiority over the polynomial collocation methods.

1. Introduction

Volterra integral equations (VIEs) usually arise in mathematical modelling of many applied problems in sciences and engineering [1–3]. The numerical solution of VIEs have been extensively studied by a variety of numerical methods such as collocation methods [1, 4], spectral methods [5, 6], Galerkin methods [7, 8], and Runge–Kutta methods [9, 10]. On the other hand, modelling a variety of wave phenomena leads to problems which usually contain an oscillatory character. For example, determining the transmission and reflection coefficients of the direct scattering problem (as an initial scattering data) for the initial value problem associated with the Korteweg–de Vries (KdV) equation leads to problems which have oscillatory operators [11]. Furthermore, it is easy to check that the solution of the following differential equation:

$$\begin{cases} -u''(x) + q(x)u(x) = \omega^2 u(x), & x \in \mathbb{R}, \\ u(0) = \alpha, \\ u'(0) = \beta, \end{cases} \tag{1}$$

satisfies the following integral equation:

$$u(x) - \int_0^x \frac{\sin(\omega(x-s))}{\omega} q(s)u(s)\mathrm{d}s = \alpha\cos(\omega x) + \beta\frac{\sin(\omega x)}{\omega}, \tag{2}$$

which for large values of ω is an integral equation of oscillatory type. Preliminary studies indicate that conventional methods cannot provide an accurate solution for such problems due to the oscillatory behaviour of the underlying problem. In order to overcome this difficulty, initial studies by researchers showed that integral operators with high oscillation can be efficiently approximated by Filon-type method [12, 13], Levin-type method [14–17], steepest descent method [18, 19], exponential fitting (EF) quadrature rule [20–23], and Gaussian integration rule [18, 24, 25]. For more information about these methods, we refer the reader to see the books [22, 26].

In recent years, some authors also introduced numerical methods for VIEs including highly oscillatory kernels or rapidly input functions which are usually known as highly oscillatory Volterra integral equations (HOVIEs). For VIEs involving Bessel oscillators, Xiang et al. [27, 28] represented the solution of the first kind VIEs in terms of Bessel transforms and then computed the resulting integrals by a Filon-type method. Furthermore, Li et al. [29] applied an improved-Levin quadrature scheme to solve a special Fredholm integral equation whose solution is much less oscillatory than the kernel function. Other novel methods can be found in [15, 30] and the references therein. For VIEs with trigonometric kernels, the theoretical aspects of these equations, such as the existence and uniqueness of solutions and the high-oscillation properties of their solution, have been studied by Brunner et al. [31, 32]. In addition, several numerical methods have been proposed to approximate the solution of mentioned equations such as collocation method, Filon-type method, generalized multistep method, EF method, meshless collocation methods, and radial basis functions method [33–43]. The accuracy of the proposed methods depends heavily on the evaluation of the highly oscillatory integrals in the discretization method and the approximation of the solution of the underlying problem by a suitable function. These schemes usually share the property that the higher the oscillation, the better the approximation i.e., their error behaves as $\omega^{-\alpha}$ $(\alpha > 0)$ when ω goes to infinity.

Among these methods, an EF-type method possesses a number of important advantages: it can be used to reach user-requested accuracy to approximate oscillatory operations, and it is a procedure which can be used to approximate a large variety of oscillatory operations (such as interpolation, quadrature, and numerical solution of ordinary differential) because this method is chosen according to the problem to be solved. Due to these properties, Zhao and Huang [41] used an EF collocation method to construct a scheme for solving a class of VIEs with periodic or oscillatory solutions. They also obtained the global convergence order of the proposed method and proved that EF collocation methods always perform much better in comparison with polynomial collocation methods with the same order. Hence, this inspires us to apply EF methods for approximating the solution of VIEs with highly oscillatory trigonometric kernels. More precisely, we try to construct an EF collocation method for the numerical computation of the following second-kind HOVIE:

$$u(t) = f(t) + \int_0^t K(t,s)e^{i\omega(t-s)}u(s)\mathrm{d}s, \quad t \in I \coloneqq [0,T], \tag{3}$$

where $\omega \gg 1$ denotes the oscillation parameter and $u(s)$ is the function to be determined. Here, the functions $K(t,s)$ and $f(t)$ are sufficiently smooth on $D{:}\ = \{(t,s){:}\ 0 \le s \le t \le T\}$ and I, respectively.

The outline of the paper is as follows. In Section 2, we briefly review piecewise polynomial collocation method for VIE (3). In addition, we introduce EF version of that method to approximate the solution of VIE (3). In Section 3, we investigate the convergence order of the proposed method. Some numerical examples are presented to show the superiority of the proposed method in comparison with classical methods in Section 4. Finally, we conclude the paper in Section 5.

2. The Numerical Schemes for Solving HOVIE (3)

The aim of this section is twofold. First, we review piecewise polynomial collocation method for HOVIE (3) which provides some insight into various aspects of the construction of efficient collocation methods for solving HOVIE (3). Second, we introduce a novel collocation method for solving HOVIE (3) by EF technique.

2.1. Classical Collocation Method. Let $I_h{:}\ = \{t_n{:}\ = nh, n = 0, 1, \ldots, N, Nh{:}\ = T\}$ be a uniform mesh for $I = [0,T]$ and set $\sigma_n = (t_n, t_{n+1}]$, $= 0, 1, \ldots, N-1$. Also, let $t_{n,j}{:}\ = t_n + c_j h$ denote the collocation points with m fixed collocation parameters $0 \le c_1 < c_2 < \ldots < c_m \le 1$. In piecewise polynomial collocation method, we approximate the solution u of the HOVIE (3) by $u_h \in S_{m-1}^{(-1)}(I_h)$, where

$$S_{m-1}^{(-1)}(I_h) = \left\{v{:}\ v|_{\sigma_n} \in \pi_{m-1}, 0 \le n \le N-1\right\}, \tag{4}$$

and π_{m-1} denotes the space of all polynomials of total degree $\le m-1$. Since $u_h|_{\sigma_n} \in \pi_{m-1}$, it can be expressed in the following form:

$$u_h(t_n + \mathrm{sh}) = \sum_{j=1}^{m} L_j(s)U_{n,j}, \tag{5}$$

where $L_j(s)$ is Lagrange functions with respect to the collocation parameters, which are defined by the following equations:

$$L_j(s) = \prod_{\substack{i=1\\ i\neq j}}^{m} \frac{s-c_i}{c_j-c_i}, \quad j = 1, 2, \ldots, m. \tag{6}$$

Therefore, the collocation solution u_h for equation (3) can be defined by the following collocation equation:

$$u_h(t_{n,i}) = f(t_{n,i}) + \int_0^{t_{n,i}} K(t_{n,i}, s)e^{i\omega(t_{n,i}-s)}u_h(s)\mathrm{d}s, \qquad n = 0, 1, \ldots, N-1, i = 1, \ldots, m. \tag{7}$$

Inserting equation (5) into equation (7), we obtain the following equation:

$$U_{n,i} = f(t_{n,i}) + e^{i\omega t_{n,i}} \int_0^{t_{n,i}} K(t_{n,i}, s) e^{-i\omega s} u_h(s) \mathrm{d}s$$
$$= f(t_{n,i}) + h e^{i\omega t_{n,i}} \sum_{l=0}^{n-1} \sum_{j=1}^{m} e^{-i\omega t_l} \cdot \left(\int_0^1 K(t_{n,i}, t_l + \mathrm{sh}) L_j(s) e^{-i\omega sh} \mathrm{d}s \right) U_{l,j}$$
$$+ \mathrm{h} e^{i\omega t_{n,i}} \sum_{j=1}^{m} e^{-i\omega t_n} \left(\int_0^{c_i} K(t_{n,i}, t_n + \mathrm{sh}) L_j(s) e^{-i\omega sh} \mathrm{d}s \right) U_{n,j},$$
$$n = 0, 1, \ldots, N-1, i = 1, \ldots, m, \tag{8}$$

where $U_{n,i}: = u_h(t_{n,i})$. In general, highly oscillatory integrals in system (8) cannot be found analytically. So, we have to approximate them by suitable existing quadrature rules such as Filon-type method, Levin-type method, steepest descent method, and EF method. As we know, for highly oscillatory equations involving the big oscillation parameter, polynomial collocation methods require a large number of nodes to achieve an acceptable approximate solution. Therefore, it seems that polynomial-based collocation methods give a useless estimation of the solution of highly oscillatory problems unless the oscillation parameter ω is small. The above discussion implies that efficient collocation methods can only be constructed by taking into account not only suitable evaluation of the highly oscillatory integrals in collocation equation but also an accurate approximation of the solution of the underlying problem. Constructing such methods is the main purpose of this paper and we do this in the next subsection.

2.2. EF Collocation Method. EF is an efficient approach to interpolate and integrate periodic or oscillatory functions. Since this method is constructed by the structure of the function to be solved, we first need to study the oscillatory structure of the solution of HOVIE (3). We commence by reviewing the concept of oscillatory order from [44].

Definition 1. Let $C[a,b]$ be the space of all continuous functions on the interval $[a,b]$ endowed with the usual maximum norm and $n \in \mathbb{N}_0: = \{0, 1, 2, \ldots\}$. The oscillatory function $\varphi \in C[a,b]$ is called ω-oscillatory of order n if there exists a positive constant C independent of ω such that

$$\omega^{-n} \|\varphi\|_\infty \leq C, \quad \forall \omega > 1. \tag{9}$$

If φ satisfies the above inequality with $n = 0$, it is called a non-ω-oscillatory function.

Now, let us introduce some spaces that are used in the next theorem.

Definition 2. For complex-valued functions defined on the interval $X = [a,b]$, the spaces $C^m(X)$ and $C^{p,q}(X)$ are defined as follows:

$$C^m(X) = \{v: v^{(m)} \in C(X)\}, \quad m \in \mathbb{N}_0,$$
$$C^{p,q}(X) = \left\{v: \frac{\partial^p v}{\partial x}, \frac{\partial^q v}{\partial y} \in C(X^2)\right\}, \quad p, q \in \mathbb{N}_0, \tag{10}$$

where

$$\forall v \in C^m(X): \|v\|_{C^m(X)} := \sum_{j \in \{0,1,\ldots,m\}} \left\|u^{(j)}\right\|_{C(X)},$$
$$\left\|v^{(n)}\right\|_{C(X)} := \max_{x \in X} \left|v^{(n)}(x)\right|, \tag{11}$$
$$\forall v \in C^{p,q}(X): \|v\| := \left(\int_{X^2} |v(x,y)|^2 \mathrm{d}x\,\mathrm{d}y \right)^{1/2}.$$

Definition 3. The non-ω-oscillatory structured space $C^m_{\omega,0}(X)$ is defined as follows:

$$C^m_{\omega,0}(X) := \{v_\omega(x) \in C^m(X): \|v_\omega(.)\|_m \leq \rho\}, \tag{12}$$

where the constant $\rho > 0$ depends on the function v but is independent of ω.

Finally, the following theorem (see [45]) gives the structure of the solution of HOVIE (3).

Theorem 4. *Assume that $K \in C^{m,m}(I^2)$. Thus, for any $f \in C^m(I)$ with the structure*

$$f(x) = f_1(x) + f_2(x) e^{i\omega x} + f_3(x) e^{-i\omega x}, f_i \in C^m_{\omega,0}(I), \quad i = 1, 2, 3, \tag{13}$$

the solution u of HOVIE (3) lies in $u \in C^m(I)$ and has the following form:

$$u(x) = v_1(x) + v_2(x) e^{i\omega x} + v_3(x) e^{-i\omega x}, v_i \in C^m_{\omega,0}(I), \quad i = 1, 2, 3. \tag{14}$$

We now consider introducing the EF collocation method for solving HOVIE (3). To construct EF interpolation formula, which approximate u in each subinterval σ_n, we use the following formula:

$$u(t_n + \mathrm{sh}) \simeq \sum_{j=1}^{m} b_j(s) u(t_n + c_j h), \quad n = 0, 1, \ldots, N-1, \tag{15}$$

and try to find functions $b_j(s)$ such that the operator

$$\mathscr{L}[h, \mathbf{b}] u(t) = u(t_n + \mathrm{sh}) - \sum_{j=1}^{m} b_j(s) u(t_n + c_j h), \tag{16}$$

vanishes for any elements of a suitable fitting space $\mathscr{B}$ where $\mathbf{b} = [b_1(s), b_2(s), \ldots, b_m(s)]^T$. The space $\mathscr{B}$ must be selected such that formula (15) captures the oscillatory structure of the solutions of equation (3). According to Theorem 4, the fitting space $\mathscr{B}$ should cover both oscillatory and polynomial components of the solution of equation (3). Therefore, the space

$$\mathscr{B} = \left\{1, x, x^2, \ldots, x^p, e^{\pm i\omega x}, xe^{\pm i\omega x}, x^2 e^{\pm i\omega x}, \ldots, x^q e^{\pm i\omega x}\right\},$$
$$p + 2q = m - 3, \tag{17}$$

is appropriate for the obtained form of $u(x)$ which is a hybrid set of polynomial and exponential functions. It is crucial to note that for equations with a purely trigonometric or polynomial solution, we have to chose $p = -1$ or $q = -1$, respectively. For simplicity of the explanation, we restrict our attention to the case of $m = 4$ and chose the fitting space as follows:

$$\mathscr{B} = \left\{1, x, e^{\pm i\omega x}\right\}, \tag{18}$$

but our analysis of the procedure can be easily extended to all $m > 4$.

In this case, $\mathscr{L}[h, \mathbf{b}]u(t)$ is required to be identically zero for any element of $\mathscr{B}$. That is, the unknown coefficients $b_i(s)$ can be obtained by solving the following linear system:

$$\begin{cases} b_1(s) + b_2(s) + b_3(s) + b_4(s) = 1, \\ b_1(s)c_1 + b_2(s)c_2 + b_3(s)c_3 + b_4(s)c_4 = s, \\ b_1(s)\sin(c_1 z) + b_2(s)\sin(c_2 z) + b_3(s)\sin(c_3 z) + b_4(s)\sin(c_4 z) = \sin(sz), \\ b_1(s)\cos(c_1 z) + b_2(s)\cos(c_2 z) + b_3(s)\cos(c_3 z) + b_4(s)\cos(c_4 z) = \cos(sz), \end{cases} \tag{19}$$

where $z = \omega h$. System (19) can be solved by some well-known numerical solvers such as Mathematica's routine LinearSolve (see the appendix).

In this position, we carry out the error of the proposed EF interpolation formula using the ideas used by Ixaru and Vanden Berghe [22] to obtain EF interpolation and its error formula. The details are as follows.

Theorem 5. *If the function $u(t)$ is differentiable indefinitely many times, then the expression of the error for the proposed EF interpolation formula is given by the following equation:*

$$\mathscr{L}[h, \mathbf{b}]u(t) = \sum_{j=0}^{\infty} h^{4+j} T_j D^{j+2}\left(D^2 + \omega^2\right)^2 u(t), \tag{20}$$

where

$$T_0 = \frac{s^2 - \left(b_1(s)c_1^2 + b_2(s)c_2^2 + b_3(s)c_3^2 + b_4(s)c_4^2\right)}{2z^2}. \tag{21}$$

Proof. It is easy to see that each element of the fitting space $\mathscr{B}$ is also an independent solution of the following ODE:

$$D^2\left(D^2 + \omega^2\right)u(t) = 0, \tag{22}$$

which usually called the reference differential equation. Following the EF theory [22], it follows that the error of the proposed EF interpolation formula can be expressed in the form (20) and the leading term of the error is given by the following equation:

$$\text{lte} = h^4 \frac{L^*[h, \mathbf{b}]}{2z^2} D^2\left(D^2 + \omega^2\right)u(t), \tag{23}$$

where

$$\begin{aligned} L^*[h, \mathbf{b}] &= \frac{\mathscr{L}[h, \mathbf{b}]t^2\big|_{t=0}}{h^2} \\ &= s^2 - \left(b_1(s)c_1^2 + b_2(s)c_2^2 + b_3(s)c_3^2 + b_4(s)c_4^2\right). \end{aligned} \tag{24}$$

This completes the proof. □

Remark 6. It is interesting to note that the limit of the functions $b_i(s), i = 1, \ldots, 4$, and T_0 as z approaches 0 is equal to the Lagrange fundamental polynomials (with respect to the collocation parameters c_i) and the error of the Lagrange interpolation, respectively. More precisely, we have the following equation:

$$\begin{aligned} &\lim_{z \longrightarrow 0} b_i(s) = L_i(s), \quad i = 1, \ldots, 4, \\ &\lim_{z \longrightarrow 0} T_0 = \frac{(s - c_1)(s - c_2)(s - c_3)(s - c_4)}{4!}. \end{aligned} \tag{25}$$

The above relation shows that the polynomial interpolation is a special case of EF interpolation formula as $z \longrightarrow 0$.

We now turn our attention to HOVIE (3) and try to solve it by an EF collocation method. To do this, instead of polynomial interpolation, we approximate $u(t)$ by the EF interpolation formula (15). Hence, we can rewrite equation (8) as follows:

$$
\begin{aligned}
U_{n,i} = {} & f\left(t_{n,i}\right) + he^{i\omega t_{n,i}} \sum_{l=0}^{n-1} \sum_{j=1}^{4} e^{-i\omega t_l} \\
& \cdot \left(\int_0^1 K\left(t_{n,i}, t_l + \mathrm{sh}\right) b_j(s) e^{-i\omega sh} \mathrm{d}s \right) U_{l,j} \\
& + he^{\mathrm{i}\omega t_{n,i}} \sum_{j=1}^{4} e^{-i\omega t_n} \left(\int_0^{c_i} K\left(t_{n,i}, t_n + \mathrm{sh}\right) b_j(s) e^{-i\omega sh} \mathrm{d}s \right) U_{n,j}, \\
& n = 0, 1, \ldots, N-1, i = 1, \ldots, 4.
\end{aligned} \tag{26}
$$

System (26) can be written in the following matrix form:

$$
\left[I_4 - hB_n\right] \mathbf{U}_n = \mathbf{F}_n + \sum_{l=0}^{n-1} hB_n^{(l)} \mathbf{U}_l, \tag{27}
$$

where

$$
\begin{aligned}
B_n &:= \left(e^{i\omega\left(t_{n,i} - t_n\right)} \int_0^{c_i} K\left(t_{n,i}, t_n + \mathrm{sh}\right) b_j(s) e^{-i\omega sh} \mathrm{d}s \right)_{i,j=1,\ldots,4}, \mathbf{U}_n := \left[U_{n,1}, \ldots, U_{n,4}\right]^T, \\
B_n^{(l)} &:= \left(e^{i\omega\left(t_{n,i} - t_l\right)} \int_0^1 K\left(t_{n,i}, t_l + \mathrm{sh}\right) b_j(s) e^{-i\omega sh} \mathrm{d}s \right)_{i,j=1,\ldots,4}, \mathbf{F}_n := \left[f\left(t_{n,1}\right), \ldots, f\left(t_{n,4}\right)\right]^T,
\end{aligned} \tag{28}
$$

and I_4 is the identity matrix of dimension 4. To ensure the existence and uniqueness of the solution to the system (27), we first remark that all the elements of the matrices B_n are bounded due to the continuity of the kernel function K. This together with Neumann Lemma implies that the matrix $I_4 - hB_n$ is invertible for sufficiently small h [1]. Hence, the unique EF collocation solution for the HOVIE (3) on the subinterval σ_n is given by equation (15).

Remark 7. As we have already said, highly oscillatory integrals arising in the matrices B_n and $B_n^{(l)}$ require to be approximated by suitable quadrature rules. The approach taken in this paper to the accurate computation of such integrals is the exponentially fitted Gaussian quadrature rule [22]. In this method, the researchers constructed a ν-point quadrature formula of the following form:

$$
\int_{-1}^{1} f(x) \mathrm{d}x \simeq \sum_{j=1}^{\nu} w_k f\left(x_k\right), \tag{29}
$$

where the quadrature points and weights can be found by the fact that this rule is exact on the fitting space.

$$
\mathscr{B} = \left\{x^k e^{\pm i\omega x}, k = 0, 1, \ldots, \nu - 1\right\}. \tag{30}
$$

For increasing values of ν and ω, the nonlinear system, including the nodes and weights of the quadrature rule, becomes increasingly ill-conditioned. Therefore, the authors constructed an efficient subroutine to compute them, see [22]. In addition, they showed that their proposed rule is convergent and has the asymptotic order $\mathcal{O}(\omega^{\overline{\nu} - \nu})$ where $\overline{\nu} = [(\nu - 1)/2]$.

3. Convergence Analysis

The aim of this section is to study the convergence property of the proposed EF collocation method. We carry out this by using the ideas used in the proof of Theorem 2.2.3 in [1]. The details are as follows.

Inserting collocation points $t_{n,i}$ into equation (3) yields the following equation:

$$
\begin{aligned}
u\left(t_{n,i}\right) &= f\left(t_{n,i}\right) + \int_0^{t_{n,i}} K\left(t_{n,i}, s\right) e^{i\omega\left(t_{n,i} - s\right)} u(s) \mathrm{d}s \\
&= f\left(t_{n,i}\right) + he^{i\omega t_{n,i}} \sum_{l=0}^{n-1} e^{-i\omega t_l} \\
&\quad \cdot \left(\int_0^1 K\left(t_{n,i}, t_l + \mathrm{sh}\right) u\left(t_l + \mathrm{sh}\right) e^{-i\omega sh} \mathrm{d}s \right) \\
&\quad + he^{i\omega t_{n,i}} e^{-i\omega t_n} \left(\int_0^{c_i} K\left(t_{n,i}, t_n + \mathrm{sh}\right) u\left(t_n + \mathrm{sh}\right) e^{-i\omega sh} \mathrm{d}s \right), \\
&\quad n = 0, 1, \ldots, N-1, i = 1, \ldots, 4.
\end{aligned} \tag{31}
$$

Subtracting equation (31) from equation (26), we obtain the following equation:

$$
\begin{aligned}
\varepsilon_{n,i} = {} & he^{i\omega t_{n,i}} \sum_{l=0}^{n-1} e^{-i\omega t_l} \left(\int_0^1 K\left(t_{n,i}, t_l + \mathrm{sh}\right) e^{-i\omega sh} \left(u\left(t_l + \mathrm{sh}\right) - u_h\left(t_l + \mathrm{sh}\right)\right) \mathrm{d}s \right) \\
& + he^{i\omega t_{n,i}} e^{-i\omega t_n} \left(\int_0^{c_i} K\left(t_{n,i}, t_n + \mathrm{sh}\right) \mathrm{e}^{-i\omega sh} \left(u\left(t_n + \mathrm{sh}\right) - u_h\left(t_n + \mathrm{sh}\right)\right) \mathrm{d}s \right),
\end{aligned} \tag{32}
$$

where $\varepsilon_{n,i} := u(t_{n,i}) - U_{n,i}$. Let $e_h = u - u_h$ denote the EF collocation error. From Theorem 5, it follows that

$$\begin{aligned} e_h(t_n + \mathrm{sh}) &= u(t_n + \mathrm{sh}) - u_h(t_n + \mathrm{sh}) \\ &= \sum_{j=1}^{4} b_j(s)\left(u\left(t_{n,j}\right) - U_{n,j}\right) + \mathcal{O}\left(h^4\right) \\ &= \sum_{j=1}^{4} b_j(s)\varepsilon_{n,j} + \mathcal{O}\left(h^4\right). \end{aligned} \tag{33}$$

Now, inserting equation (33) into equation (32) and some manipulations, we get the following equation:

$$\begin{aligned} \varepsilon_{n,i} = {} & he^{i\omega t_{n,i}} \sum_{l=0}^{n-1}\sum_{j=1}^{4} e^{-i\omega t_l}\left(\int_0^1 K\left(t_{n,i}, t_l + \mathrm{sh}\right)b_j(s)e^{-i\omega sh}\mathrm{d}s\right)\varepsilon_{l,j} \\ & + he^{i\omega t_{n,i}} \sum_{j=1}^{4} e^{-i\omega t_n}\left(\int_0^{c_i} K\left(t_{n,i}, t_n + \mathrm{sh}\right)b_j(s)e^{-i\omega sh}\mathrm{d}s\right) \\ & \cdot \varepsilon_{n,j} + \eta_{n,i}, \end{aligned} \tag{34}$$

where

$$\begin{aligned} \eta_{n,i} := {} & he^{i\omega t_{n,i}} \sum_{l=0}^{n-1}\sum_{j=1}^{4} e^{-i\omega t_l}\left(\int_0^1 K\left(t_{n,i}, t_l + \mathrm{sh}\right)e^{-i\omega sh}\mathcal{O}\left(h^4\right)\mathrm{d}s\right) \\ & + he^{i\omega t_{n,i}} \sum_{j=1}^{4} e^{-i\omega t_n}\left(\int_0^{c_i} K\left(t_{n,i}, t_n + \mathrm{sh}\right)e^{-i\omega sh}\mathcal{O}\left(h^4\right)\mathrm{d}s\right) \\ = {} & \mathcal{O}\left(h^4\right). \end{aligned} \tag{35}$$

The obtained equations can be represented in matrix notation as follows:

$$\left[I_4 - hB_n\right]\varepsilon_n = \sum_{l=0}^{n-1} hB_n^{(l)}\varepsilon_l + \eta_n, \tag{36}$$

where

$$\varepsilon_n := \left[\varepsilon_{n,1}, \ldots, \varepsilon_{n,4}\right]^T, \eta_n := \left[\eta_{n,1}, \ldots, \eta_{n,4}\right]^T. \tag{37}$$

From the boundedness of the matrices B_n and Neumann Lemma, we can deduce that $I_4 - hB_n$ has a bounded inverse for sufficiently small h. Assuming $\|(I_4 - hB_n)^{-1}\|_1 \leq D_0$ and using equation (36), we obtain the following equation:

$$\left\|\varepsilon_n\right\|_1 \leq D_0 D_1 \sum_{l=0}^{n-1} h\left\|\varepsilon_l\right\|_1 + \mathcal{O}\left(h^4\right), \tag{38}$$

where we have assumed that $\|B_n^{(l)}\|_1 \leq D_1$. Now using the Gronwall inequality, we get the following equation:

$$\left\|\varepsilon_n\right\|_1 = \mathcal{O}\left(h^4\right). \tag{39}$$

Thus, the above equation together with the boundedness of the functions $b_j(s), s \in (0, 1]$, implies that

$$\left|e_h\left(t_n + \mathrm{sh}\right)\right| = \left|\sum_{j=1}^{4} b_j(s)\varepsilon_{n,j} + \mathcal{O}\left(h^4\right)\right| = \mathcal{O}\left(h^4\right), \quad n = 0, 1, \ldots, N-1. \tag{40}$$

Finally, estimate (39) yields the convergence property of the presented scheme, as is made precise in the following theorem.

Theorem 8. *Assume that both of functions $f(t)$ and $K(t,s)$ in HOVIE (3) are sufficiently smooth. Let u_h be the EF collocation approximation of u, which is defined by equation (15). Then, for any choice of the collocation parameters $\{c_i\} \subseteq [0, 1]$, the following estimate hold.*

$$\left\|u - u_h\right\|_\infty := \sup_{t\in I}\left|u(t) - u_h(t)\right| = \mathcal{O}\left(h^4\right). \tag{41}$$

Remark 9. It is easy to check that the determinant D of system (19) may vanish for some values of z. This implies that the coefficients $b_i(s)$ display a pole-like behaviour around these zeros. To illustrate this phenomenon, we plot in Figure 1 the function $D(z)$ for $0 \leq z \leq 50$ and four typical collocation parameters. Therefore, in practice, h must be chosen such that $z = wh$ is not too close to the zeros of the function $D(z)$.

4. Numerical Results

In this section, we give some numerical examples to illustrate the efficiency and accuracy of the proposed EF collocation method. Furthermore, we choose the equidistant points $c_i = i - 1/3, i = 1, \ldots, 4$ as collocation parameters. In this case, the function $D(z)$ is equal to $1/3\,(-32)\sin^5(z/6)\cos(z/6)$ and its zeros are given by the following equation:

$$z_k = 12k\pi, 6\left(2k\pi - \frac{\pi}{2}\right), 6\left(2k\pi + \frac{\pi}{2}\right), 12k\pi + 6\pi. \tag{42}$$

Therefore, to avoid dealing with singular matrices in the system (27), we select the values of ω and h such that $\omega h \neq z_k$. For numerical comparison, we report the difference between exact and numerical solutions for both classical and EF collocation methods in each example. We also obtain the order of convergence of the proposed method for each example to confirm the expected convergence orders of Theorem 8. All numerical computations were performed by Mathematica software.

Example 1. Consider the following HOVIE [46]:

$$u(t) = 1 + \int_0^t K(t,s)e^{i\omega(t-s)}u(s)\mathrm{d}s, \quad t \in [0, 1], \tag{43}$$

where $K(t,s) = \lambda \neq 0$. The exact solution of this integral equation is as follows:

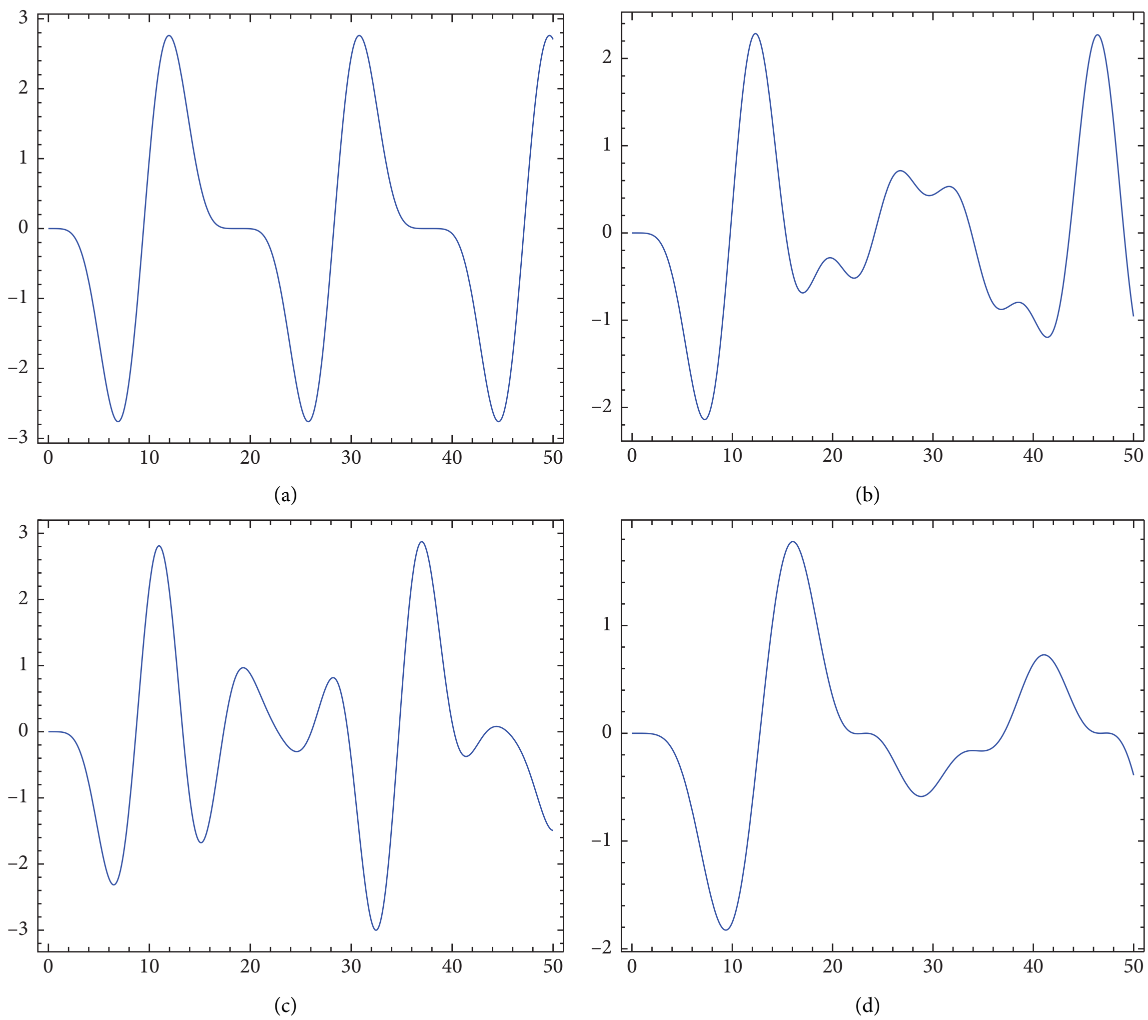

Figure 1: The graph of $D(z)$ for $0 \le z \le 50$ and four typical collocation parameters. (a) Equidistant points. (b) Radau points. (c) Lobatto points. (d) Gauss points.

Table 1: The global errors for $\lambda = 1$ and a range of increasing values of N in Example 1.

ω	Method	$N = 2^4$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
50	Classical	$3.49e-04$	$4.20e-05$	$3.00e-06$	$1.93e-07$	$1.22e-08$	$7.63e-10$	$4.73e-11$
	EF	$1.45e-05$	$1.71e-06$	$1.20e-07$	$7.74e-09$	$4.87e-10$	$3.05e-11$	$1.91e-12$
100	Classical	$2.32e-03$	$1.74e-04$	$2.10e-05$	$1.50e-06$	$9.67e-08$	$6.09e-09$	$3.81e-10$
	EF	$1.03e-04$	$3.62e-06$	$4.27e-07$	$3.01e-08$	$1.94e-09$	$1.22e-10$	$7.62e-12$
200	Classical	$8.50e-03$	$1.16e-03$	$8.72e-05$	$1.05e-05$	$7.49e-07$	$4.83e-08$	$3.04e-09$
	EF	$1.41e-04$	$2.58e-05$	$9.05e-07$	$1.07e-07$	$7.53e-09$	$4.83e-10$	$3.05e-11$
400	Classical	$2.11e-02$	$4.25e-03$	$5.80e-04$	$4.36e-05$	$5.25e-06$	$3.75e-07$	$2.42e-08$
	EF	$1.59e-04$	$3.52e-05$	$6.45e-06$	$2.62e-07$	$2.67e-08$	$1.88e-09$	$1.21e-10$

$$u(t) = 1 - \frac{\lambda}{\lambda + i\omega} + \frac{\lambda e^{\lambda t}}{\lambda + i\omega} e^{i\omega t}. \tag{44}$$

The global errors of both classical and EF collocation methods for different values of N are listed in Table 1. From this table, we can observe that the errors decay as N increases, i.e., both methods converge as $h \longrightarrow 0$. However, the results of this table imply that the EF collocation methods always perform significantly better than the classical collocation methods. Figure 2 displays the errors obtained in Table 1. This figure again shows the accuracy of the proposed methods and verifies the theoretical results.

Also, the orders of convergence of the proposed collocation methods have been reported in Table 2. From Table 2,

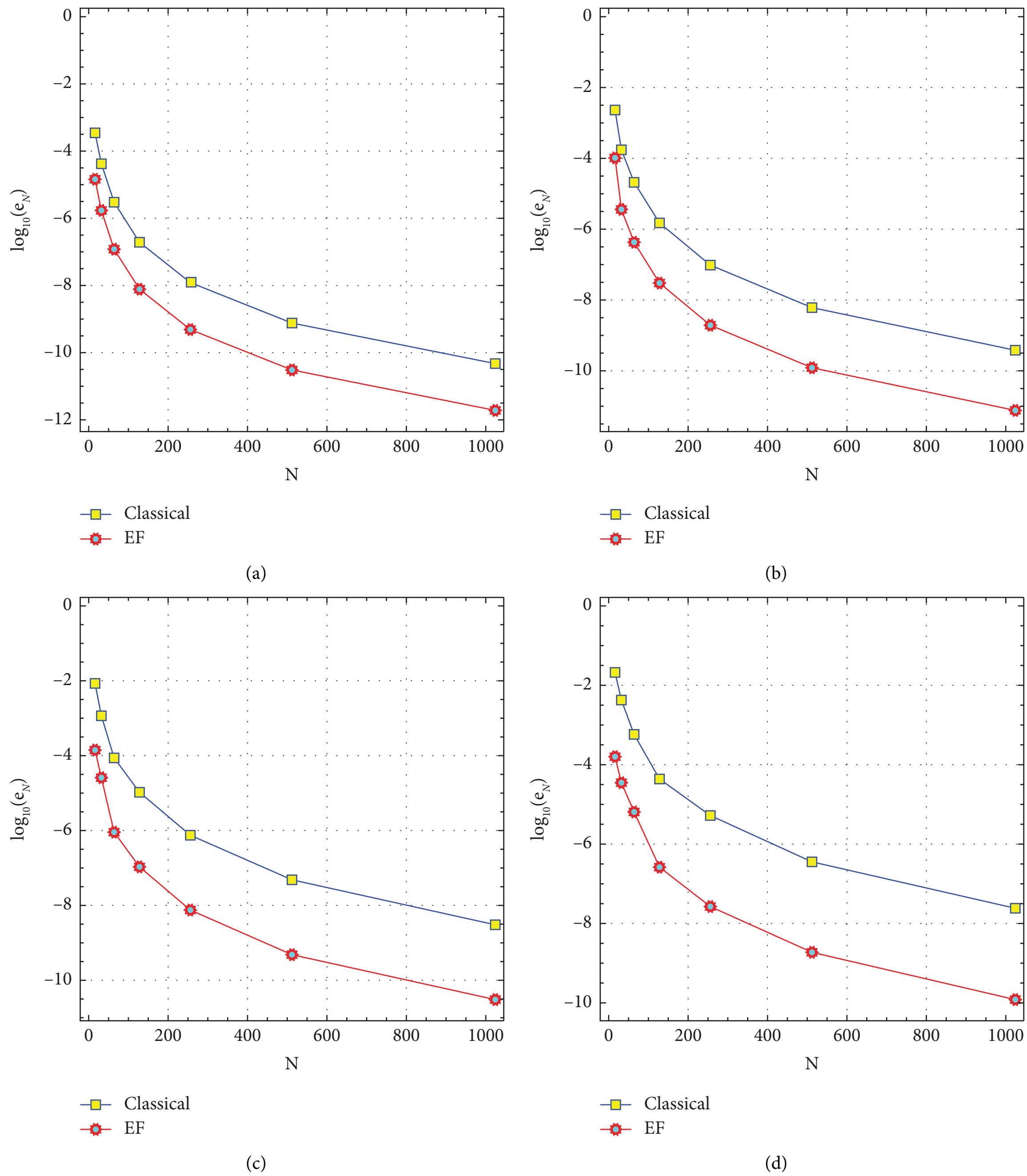

Figure 2: Graph of the global errors versus N for Example 1. (a) $\omega = 50$. (b) $\omega = 100$. (c) $\omega = 200$. (d) $\omega = 400$.

Table 2: Orders of convergence for Example 1.

ω	Method	$N = 2^4$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
50	Classical	—	3.05	3.81	3.96	3.98	4.00	4.01
	EF	—	3.08	3.83	3.95	3.99	4.00	4.00
100	Classical	—	3.74	3.05	3.81	3.96	3.99	4.00
	EF	—	4.83	3.08	3.83	3.96	3.99	4.00
200	Classical	—	2.31	2.87	3.73	3.81	3.95	3.99
	EF	—	2.87	3.73	3.05	3.83	3.96	3.99
400	Classical	—	2.31	2.87	3.73	3.05	3.81	3.95
	EF	—	2.18	2.45	4.62	3.29	3.82	3.96

Table 3: The global errors for a range of increasing values of N in Example 2.

ω	Method	$N = 2^4$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
50	Classical	$4.86e-05$	$5.75e-06$	$4.08e-07$	$2.62e-08$	$1.65e-09$	$1.03e-10$	$6.46e-12$
	EF	$2.73e-06$	$2.53e-07$	$1.76e-08$	$1.13e-09$	$7.08e-11$	$4.40e-12$	$2.66e-13$
100	Classical	$3.47e-04$	$2.39e-05$	$2.86e-06$	$2.03e-07$	$1.31e-08$	$8.25e-10$	$5.16e-11$
	EF	$6.1e-05$	$5.87e-07$	$6.09e-08$	$4.25e-09$	$2.72e-10$	$1.71e-11$	$1.06e-12$
200	Classical	$8.50e-03$	$1.16e-03$	$8.72e-05$	$1.05e-05$	$1.02e-07$	$6.54e-09$	$4.12e-10$
	EF	$1.41e-04$	$2.58e-05$	$9.05e-07$	$1.07e-07$	$1.04e-09$	$6.68e-11$	$4.20e-12$
400	Classical	$2.11e-02$	$4.25e-03$	$5.80e-04$	$4.36e-05$	$7.11e-07$	$5.07e-08$	$3.27e-09$
	EF	$1.59e-04$	$3.52e-05$	$6.45e-06$	$2.62e-07$	$3.66e-09$	$2.57e-10$	$1.65e-11$

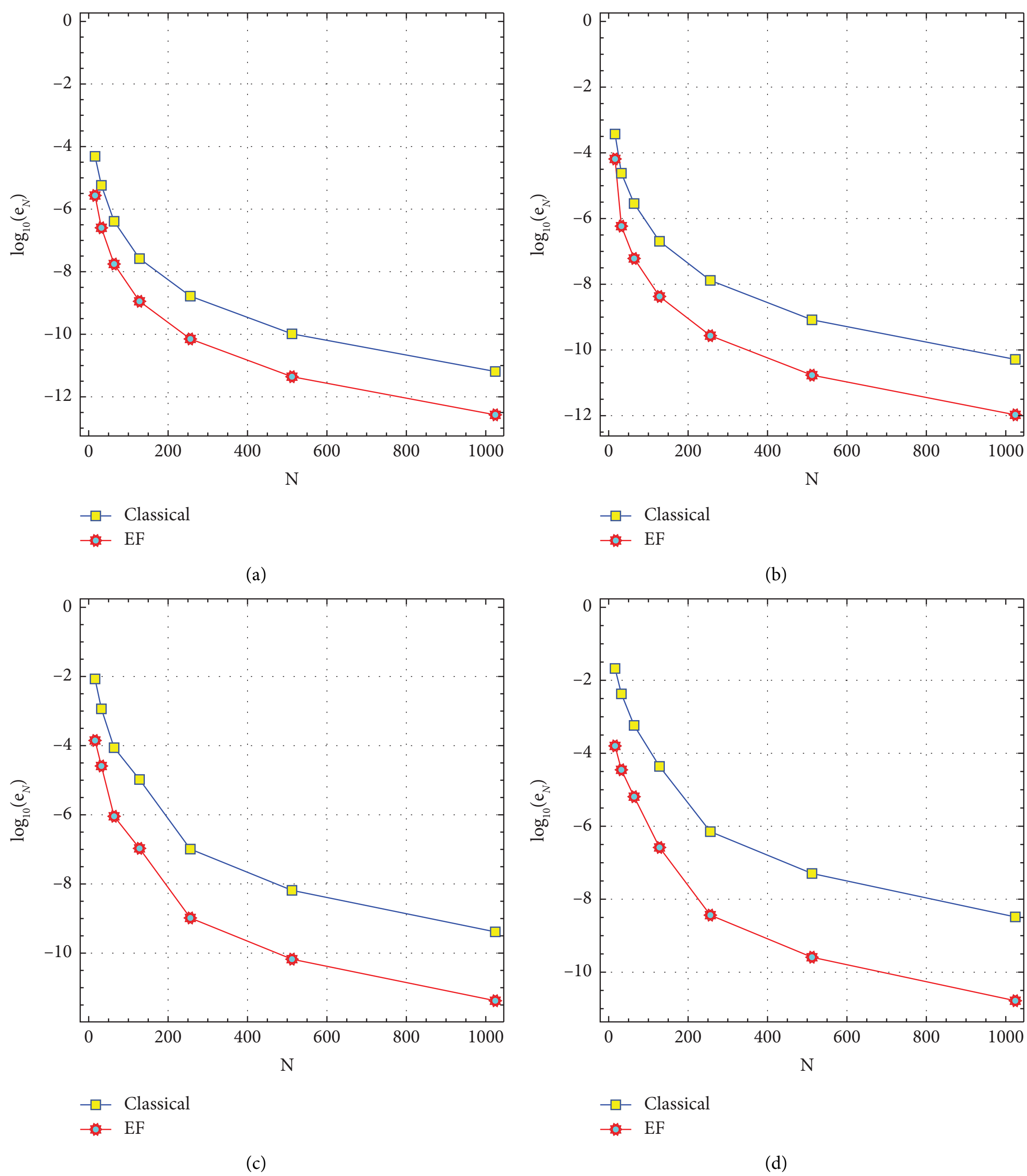

Figure 3: Graph of the global errors versus N for Example 2. (a) $\omega = 50$. (b) $\omega = 100$. (c) $\omega = 200$. (d) $\omega = 400$.

Table 4: Orders of convergence for Example 2.

ω	Method	$N=2^4$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
50	Classical	—	3.07	3.82	3.96	3.99	4.00	3.99
	EF	—	3.43	3.84	3.96	4.00	4.01	4.05
100	Classical	—	3.97	3.06	3.82	3.95	3.99	4.00
	EF	—	6.79	3.63	3.84	3.97	3.99	4.01
200	Classical	—	2.31	2.87	3.73	6.69	3.96	3.99
	EF	—	2.87	3.73	3.05	6.68	3.96	3.99
400	Classical	—	2.31	2.87	3.73	5.94	3.81	3.95
	EF	—	2.18	2.45	4.62	6.17	3.83	3.96

we can see that both methods have the same convergence order, which confirms the classical and the EF expected order of Theorem 8.

Example 2. As a final example, consider the following HOVIE:

$$u(t) = e^t - \int_0^t e^{i\omega(t-s)} u(s)\mathrm{d}s, \quad t \in [0,1]. \tag{45}$$

The exact solution is as follows:

$$\begin{aligned} u(t) &= \left(\int_0^t F(s) e^{-cs} \mathrm{d}s + f(0) \right) e^{ct}, \\ c &= i\omega - 1, F(s) = f'(s) - i\omega f(s). \end{aligned} \tag{46}$$

We employ the proposed collocation methods for solving HOVIE (45) and report the errors for several values of N in Table 3. We also plot (in logarithmic scale) the errors embedded in Table 3 for both methods in Figure 3. The results are the same as the previous example: both collocation methods converge as $h \longrightarrow 0$, while the errors of the new EF methods are much better than the classical polynomial collocation methods.

Similar to the previous example, we list the orders of convergence of both proposed methods in Table 4. The reported results of this table confirm the expected orders of the presented methods.

5. Conclusions

We studied the construction of an EF collocation method for VIEs with highly oscillatory trigonometric kernels. We also carried out the global error of the presented method and illustrated the efficiency and accuracy of our scheme with some numerical examples. Furthermore, we showed the superiority of the proposed method in comparison with classical collocation methods.

As we have seen, the design of efficient numerical methods for solving HOVIE (3) requires the approximation of its solution and the discretization of the highly oscillatory integrals arising in the numerical scheme. With this in mind, we will introduce higher-order numerical methods for solving HOVIE (3) in our future work.

Appendix

In this section, we derive the explicit solution of the linear system (19). Assume that M denotes the coefficient matrix associated with system (19) and $M_i(s)$ be a matrix which formed by replacing the i-th column of M by the column vector $\mathbf{k}$ where

$$\mathbf{k} = [1, s, \sin(sz), \cos(sz)]^T. \tag{A.1}$$

Then, the solution of system (19) using Cramer's rule is given by the following equation:

$$\begin{aligned} b_1(s) &= \frac{M_1(s)}{D}, b_2(s) = \frac{M_2(s)}{D}, \\ b_3(s) &= \frac{M_3(s)}{D}, b_4(s) = \frac{M_4(s)}{D}, \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} M_1(s) = \frac{1}{D} &(-s(\sin(z(c_2-c_3)) - \sin(z(c_2-c_4)) + \sin(z(c_3-c_4))) \\ &+ (\sin(z(s-c_3)) - \sin(z(s-c_4)) + \sin(z(c_3-c_4)))c_2 \\ &+ (\sin(z(s-c_4)) - \sin(z(c_2-c_4)))c_3 \\ &+ (\sin(z(c_2-c_3)) - \sin(z(s-c_3)))c_4 + \sin(z(s-c_2))(c_4-c_3)), \\ M_2(s) = \frac{1}{D} &(-s(\sin(z(c_1-c_3)) - \sin(z(c_1-c_4)) + \sin(z(c_3-c_4))) \\ &+ (\sin(z(s-c_3)) - \sin(z(s-c_4)) + \sin(z(c_3-c_4)))c_1 \end{aligned}$$

$$
\begin{aligned}
&+ (\sin(z(s-c_4)) - \sin(z(c_1-c_4)))c_3 \\
&+ (\sin(z(c_1-c_3)) - \sin(z(s-c_3)))c_4 + \sin(z(s-c_1))(c_4-c_3)), \\
M_3(s) = \frac{1}{D}&(-s(\sin(z(c_1-c_2)) - \sin(z(c_1-c_4)) + \sin(z(c_2-c_4))) \\
&+ (\sin(z(s-c_2)) - \sin(z(s-c_4)) + \sin(z(c_2-c_4)))c_1 \\
&+ (\sin(z(s-c_4)) - \sin(z(c_1-c_4)))c_2 \\
&+ (\sin(z(c_1-c_2)) - \sin(z(s-c_2)))c_4 + \sin(z(s-c_1))(c_4-c_2)), \\
M_4(s) = \frac{1}{D}&(-s(\sin(z(c_1-c_2)) - \sin(z(c_1-c_3)) + \sin(z(c_2-c_3))) \\
&+ (\sin(z(s-c_2)) - \sin(z(s-c_3)) + \sin(z(c_2-c_3)))c_1 \\
&+ (\sin(z(s-c_3)) - \sin(z(c_1-c_3)))c_2 \\
&+ (\sin(z(c_1-c_2)) - \sin(z(s-c_2)))c_3 + \sin(z(s-c_1))(c_3-c_2)), \\
D = c_1&(-(\sin((c_2-c_3)z) - \sin((c_2-c_4)z) + \sin((c_3-c_4)z))) \\
&+ c_2(\sin((c_1-c_3)z) - \sin((c_1-c_4)z) + \sin((c_3-c_4)z)) \\
&- c_3\sin((c_1-c_2)z) + c_3\sin((c_1-c_4)z) - c_3\sin((c_2-c_4)z) \\
&+ c_4\sin((c_1-c_2)z) - c_4\sin((c_1-c_3)z) + c_4\sin((c_2-c_3)z).
\end{aligned}
\tag{A.3}
$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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